Chapter 6. Applications of Definite Integrals

6.1 Volumes Using Cross-Sections.

1) Method of slicing

The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is

$$V = \int_a^b A(x)\, dx \quad \text{if cross-sections} \perp x\text{-axis}$$

The volume of a solid of integrable cross-sectional area $A(x)$ from $y=c$ to $y=d$ is

$$V = \int_c^d A(y)\, dy \quad \text{if cross-sections} \perp y\text{-axis}$$

2) Solids of Revolution: The disk Method, Washer method

* Disk:

Volume by diskes for rotation about the x-axis

$$V = \int_a^b A(X)\, dx = \int_a^b \pi \left[R(x)\right]^2 dx$$

Volume by diskes for rotation about y-axis

$$V = \int_c^d A(y)\, dy = \int_c^d \pi \left[R(y)\right]^2 dy$$

* Washer method:

The volume by washers for rotation about the x-axis $V = \int_a^b \pi \left[R^2(x) - r^2(x) \right] dx$

The volume by washers for rotation about the y-axis $V = \int_c^d \pi \left[R^2(y) - r^2(y) \right] dy$

Question 2
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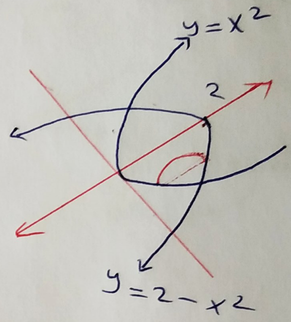
The solid lies between planes perpendicular to the x-axis at $x=-1$ and $x=1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters قطرها run from the parabola $y=x^2$ to the parabola $y=2-x^2$



$$V=\int_a^b A(x)\,dx = \int_{-1}^{1} A(x)\,dx$$

$$A(x) = r^2\pi = \left(\frac{2-x^2-x^2}{2}\right)^2\pi$$

$$= \frac{(2-2x^2)^2}{4}\pi = \left(\frac{4-8x^2+4x^4}{4}\right)\pi$$

$$V=\pi\int_{-1}^{1}\frac{4-8x^2+4x^4}{4}\,dx = \frac{\pi}{4}\left[4x-\frac{8x^3}{3}+\frac{4x^5}{5}\right]_{-1}^{1}$$

$$= \frac{\pi}{4}\left[4(1)-\frac{8(1)^3}{3}+\frac{4}{5}(1)^5-\left(4(-1)-\frac{8(-1)^3}{3}+\frac{4(-1)^5}{5}\right)\right]$$

$$= \frac{\pi}{4}\left[4-\frac{8}{3}+\frac{4}{5}+4-\frac{8}{3}+\frac{4}{5}\right]$$

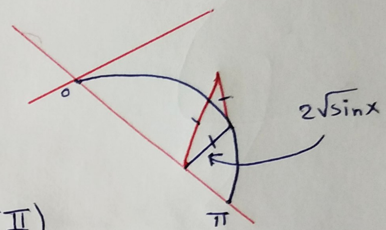
$$= \frac{\pi}{4}\left[8-\frac{16}{3}+\frac{8}{5}\right] = \pi\left[2-\frac{4}{3}+\frac{2}{5}\right]$$

$$= \pi\left[\frac{30-20+6}{15}\right] = \frac{16\pi}{15}$$

Question 5. Page 316

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are:

a) متساوي الاضلاع
equilateral triangles with bases running from the x-axis to the curve as shown



$$V = \int_0^{\pi} A(x)\, dx$$

$$A(x) = \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\sin\left(\frac{\pi}{3}\right)$$

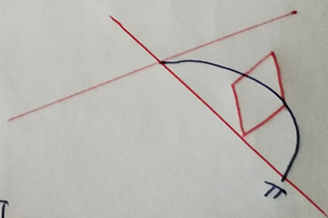
$$= 2\sin x \cdot \frac{\sqrt{3}}{2} = \sqrt{3}\sin x$$

$$V = \int_0^{\pi} \sqrt{3}\sin x\, dx = \sqrt{3} \cdot -\cos x \Big|_0^{\pi} = -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$= -\sqrt{3}\left[-1-1\right] = 2\sqrt{3}$$

b) squares with bases running from the x-axis to the curve

$$V = \int_0^{\pi} A(x)\, dx$$



$$A(x) = \left(2\sqrt{\sin x}\right)^2 = 4\sin x$$

$$V = \int_0^{\pi} 4\sin x\, dx = -4\cos x \Big|_0^{\pi}$$

$$= -4\left[\cos\pi - \cos 0\right] = -4\left[-1-1\right] = 8$$

Question 6. page 316

The solid lies between planes perpendicular to the x-axis at $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

The cross-sections perpendicular to the x-axis are

a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} A(x)\, dx$$

$$A(x) = r^2 \pi = \left(\frac{\sec x - \tan x}{2}\right)^2 \pi$$

$$= \frac{\pi}{4}\left[\sec^2 x - 2\sec x \tan x + \tan^2 x\right]$$

$$= \frac{\pi}{4}\left[\sec^2 x - 2\sec x \tan x + \sec^2 x - 1\right]$$

$$= \frac{\pi}{4}\left[2\sec^2 x - 2\sec x \tan x - 1\right]$$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\pi}{4}\left[2\sec^2 x - 2\sec x \tan x - 1\right] dx$$

$$= \frac{\pi}{4}\left[2\tan x - 2\sec x - x\right]\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{4}\left[2\tan\frac{\pi}{3} - 2\sec\frac{\pi}{3} - \frac{\pi}{3} - \left(2\tan\left(-\frac{\pi}{3}\right) - 2\sec\left(-\frac{\pi}{3}\right) - \left(-\frac{\pi}{3}\right)\right)\right]$$

$$= \frac{\pi}{4}\left[2\sqrt{3} - 2(2) - \frac{\pi}{3} - \left(-2\sqrt{3} - 2(2) + \frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{4}\left[4\sqrt{3} - 2\frac{\pi}{3}\right]$$

b) Squares whose bases run from the curve $y=\tan x$ to the curve $y=\sec x$.

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} A(x)\,dx$$

$$A(x) = (\sec x - \tan x)^2 = \sec^2 x - 2\sec x \tan x + \tan^2 x$$

$$= \sec^2 x - 2\sec x \tan x + \sec^2 x - 1$$

$$= 2\sec^2 x - 2\sec x \tan x - 1$$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(2\sec^2 x - 2\sec x \tan x - 1\right) dx$$

$$= \left[2\tan x - 2\sec x - x\right] \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 2\tan\frac{\pi}{3} - 2\sec\frac{\pi}{3} - \frac{\pi}{3} - \left(2\tan\left(-\frac{\pi}{3}\right) - 2\sec\left(-\frac{\pi}{3}\right) - \left(-\frac{\pi}{3}\right)\right)$$

$$= 2\sqrt{3} - 2(2) - \frac{\pi}{3} - \left(2(-\sqrt{3}) - 2(2) + \frac{\pi}{3}\right)$$

$$= 4\sqrt{3} - \frac{2\pi}{3}$$

Question 10
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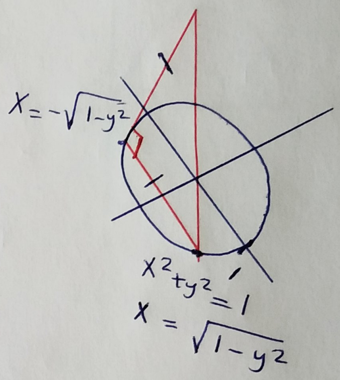
The base of the solid is the disk $x^2+y^2 \leq 1$

The cross-sections by planes perpendicular to the y-axis between $y=-1$, $y=1$ are isosceles right triangles (مثلث قائم متساوي الساقين) with one leg in the disk



$$V = \int_c^d A(y)\,dy = \int_{-1}^{1} A(y)\,dy$$

$$A(y) = \frac{1}{2}(\text{side})(\text{side}) \cdot \sin\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{2}(\text{side})^2 (1)$$

$$= \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2$$

$$= \frac{1}{2}\left[2\sqrt{1-y^2}\right]^2 = \frac{1}{2}(4)(1-y^2) = 2(1-y^2)$$

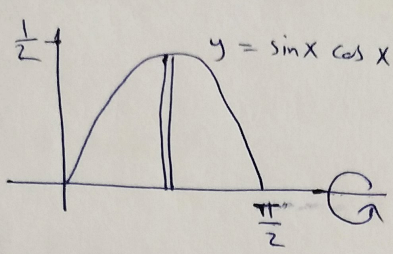
$$V = \int_{-1}^{1} 2(1-y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]\Big|_{-1}^{1}$$

$$= 2\left[1 - \frac{1}{3} - \left((-1) - \frac{(-1)^3}{3}\right)\right]$$

$$= 2\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = 2\left(2 - \frac{2}{3}\right)$$

$$= 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

18 Find the volume of the solid generated by revolving the shaded region about the given axis.



about the x-axis, use the disk method.

$$V = \int_a^b \pi R^2(x)\, dx = \int_0^{\frac{\pi}{2}} \pi (\sin x \cos x)^2 dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} 1 - \cos^2 2x \, dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \left[1 - \left(\frac{1 + \cos(4x)}{2}\right)\right] dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} - \frac{\cos(4x)}{2}\right] dx = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} 1 - \cos 4x \, dx$$

$$= \frac{\pi}{8} \left[x - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}\left[\frac{\pi}{2} - \frac{\sin 4(\frac{\pi}{2})}{4} - 0 + \frac{\sin(}{4}\right.$$

Question 22
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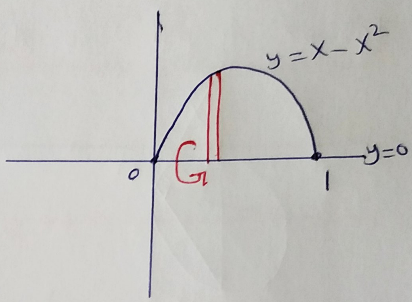
Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$, $y = 0$ about the x-axis.

$$V = \int_0^1 \pi R^2(x)\, dx$$



$$= \int_0^1 \pi \left[x - x^2\right]^2 dx$$

$$= \pi \int_0^1 x^2 - 2x^3 + x^4\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]\Big|_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{2(1)}{4} + \frac{1}{5} - (0 - 0 + 0)\right]$$

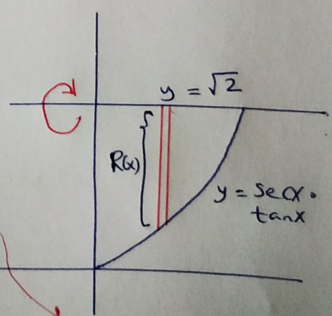
$$= \pi \left[\frac{10 - 15 + 6}{30}\right] = \frac{\pi}{30}$$

Question 25
page 317

The region in the first quadrant bounded above by $y = \sqrt{2}$, below by $y = \sec x \tan x$ and on the left by the y-axis about $y = \sqrt{2}$

$$V = \int_a^b \pi\, R^2(x)\, dx$$

$$\sec x \tan x = \sqrt{2} \rightarrow x = \frac{\pi}{4}$$



$$V = \pi \int_0^{\frac{\pi}{4}} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$$

$$= \pi \int 2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x\, dx$$

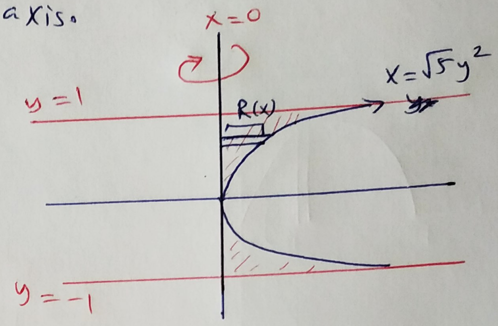
Let $u = \tan x$
$du = \sec^2 x\, dx$

$$= \pi \left[2x - 2\sqrt{2} \sec x + \frac{\tan^3 x}{3}\right]_0^{\frac{\pi}{4}}$$

$$= \pi \left[\frac{2\pi}{4} - 2\sqrt{2} \cdot \sqrt{2} + \frac{1}{3} - \left(0 - 2\sqrt{2}(1) + 0\right)\right] = \pi\left(\frac{\pi}{2} - \frac{\pi}{3} + 2\sqrt{2}\right)$$

Question 27. Page 317

Find the volume generated by revolving the region bounded by $x=\sqrt{5}y^2$, $x=0$, $y=-1$, $y=1$ about the y-axis.



$$V=\int_c^d \pi R^2(y)\,dy$$

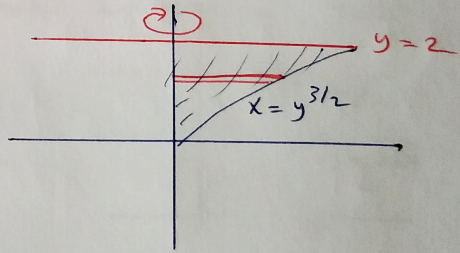
$$=\pi\int_{-1}^{1}\left(\sqrt{5}y^2\right)^2 dy$$

$$=\pi\int_{-1}^{1} 5y^4\,dy$$

$$=\pi\left[5\frac{y^5}{5}\right]\Big|_{-1}^{1}=\pi\left[(1)^5-(-1)^5\right]=2\pi$$

Question 28 Page 317

Find the volume generated by revolving the region bounded by $x=y^{3/2}$, $x=0$, $y=2$ about the y-axis



$$V=\int_c^d \pi R^2(y)\,dy$$

$$=\pi\int_0^2 \left(y^{3/2}\right)^2 dy$$

$$=\pi\int_0^2 y^3\,dy=\pi\,\frac{y^4}{4}\Big|_0^2$$

$$=\frac{\pi}{4}\left[(2)^4-(0)^4\right]=\frac{\pi}{4}(16)=4\pi$$

Question 32
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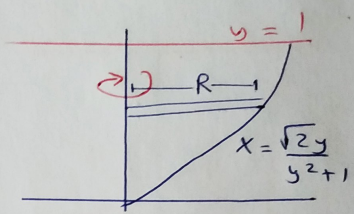
Find the volume of the region bounded by $x = \frac{\sqrt{2y}}{y^2+1}$, $x=0$, $y=1$ about y-axis

$$V = \int_0^1 \pi R^2(y)\, dy$$



$$= \pi \int_0^1 \left(\frac{\sqrt{2y}}{y^2+1}\right)^2 dy$$

$$= \pi \int_0^1 \frac{2y}{(y^2+1)^2}\, dy \qquad \text{Let } u = y^2+1,\ du = 2y\, dy$$

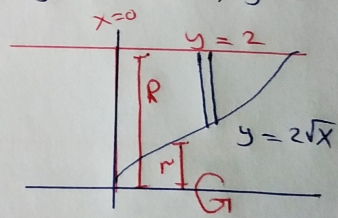
$$= \pi \int \frac{du}{u^2} = \pi \int u^{-2}\, du = \pi \frac{u^{-1}}{-1} = -\frac{\pi}{u}$$

$$= \frac{-\pi}{y^2+1}\Big|_0^1 = -\pi\left[\frac{1}{1+1} - \frac{1}{1+0}\right] = -\pi\left[\frac{1}{2} - 1\right] = \frac{\pi}{2}$$

Question 36
page 317

Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$, $y=2$, $x=0$ about x-axis.

$$V = \pi \int_a^b R^2(x) - r^2(x)\, dx$$



$$2\sqrt{x} = 2 \rightarrow \sqrt{x} = 1 \rightarrow x = 1$$

$$V = \pi \int_0^1 (2)^2 - (2\sqrt{x})^2\, dx$$

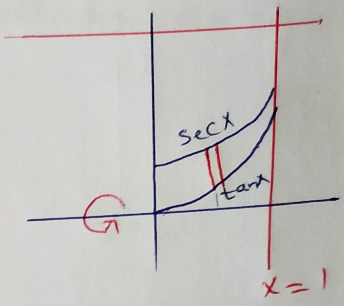
$$= \pi \int_0^1 4 - 4x\, dx = \pi\left[4x - \frac{4x^2}{2}\right]\Big|_0^1$$

$$= \pi\left[4(1) - 2(1) - (4(0) - 2(0))\right]$$

$$= 2\pi$$

Question 40, Find the volume of the solid generated by
page 317 revolving the region bounded by $y = \sec x$

$y = \tan x$, $x = 0$, $x = 1$ about x-axis

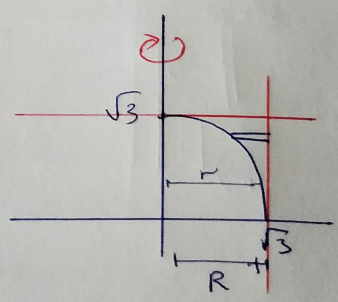


$$V = \int_0^1 \pi \left[R^2(x) - r^2(x)\right] dx$$

$$= \pi \int_0^1 (\sec x)^2 - (\tan x)^2 \, dx$$

$$= \pi \int_0^1 1 \, dx = \pi(1) = \pi$$

Question 44, Find the volume of the solid generated by
page 317 revolving the region bounded in the first quadrant bounded on the left by $x^2 + y^2 = 3$, on the right by $x = \sqrt{3}$ and above the line $y = \sqrt{3}$ about y-axis



$$V = \pi \int_0^{\sqrt{3}} R^2(y) - r^2(y) \, dy$$

$$= \pi \int_0^{\sqrt{3}} (\sqrt{3})^2 - \left(\sqrt{3 - y^2}\right)^2 dy$$

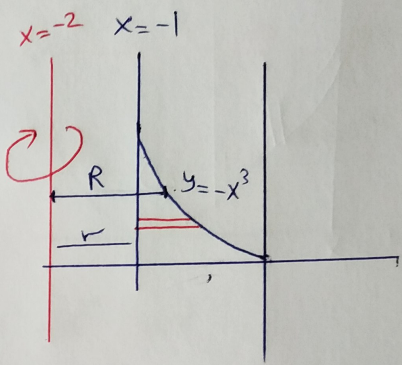
$$= \pi \int_0^{\sqrt{3}} 3 - 3 + y^2 \, dy$$

$$= \pi \left(\frac{y^3}{3}\right) \Big|_0^{\sqrt{3}} = \frac{\pi}{3}\left((\sqrt{3})^3 - 0^3\right)$$

$$= \frac{\pi}{3}(3)^{3/2} = \pi\sqrt{3}$$

Question 46
page 317

Find the volume of the region ~~bo~~ in the second quadrant bounded above by $y=-x^3$ below by the x-axis and on the left by $x=-1$ about the line $x=-2$.



$$V = \pi \int_c^d R^2(y) - r^2(y)\, dy$$

$$y = -x^3 \Rightarrow \cancel{-} x = \sqrt[3]{-y}$$

$$\cancel{x^3 = -1} \rightarrow \cancel{x = -1}$$

$$x = x$$

$$-1 = \sqrt[3]{-y} \rightarrow -y = -1 \rightarrow y = 1$$

$$V = \pi \int_0^1 \left(2 + \sqrt[3]{y}\right)^2 - (1)^2\, dy$$

$$= \pi \int_0^1 4 + 4\sqrt[3]{y} + y^{2/3} - 1 \quad dy$$

$$= \pi \left[3y + 4\frac{y^{4/3}}{4/3} + \frac{y^{5/3}}{5/3} \right]_0^1$$

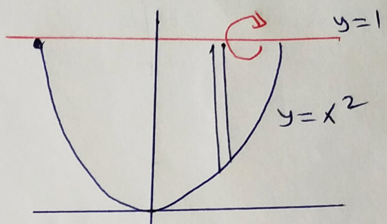
$$= \pi \left[3y + 3y^{4/3} + \frac{3}{5} y^{5/3} \right]_0^1$$

$$= \pi \left[3(1) + 3(1) + \frac{3}{5}(1) - \left(3(0) + 3(0) + \frac{3}{5}(0)\right) \right]$$

$$= \pi \left[6 + \frac{3}{5} \right] = \pi \left(\frac{33}{5} \right)$$

Question 49. page 318 Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about :

a) the line $y = 1$



$$V = \pi \int_a^b R^2(x)\, dx$$

$$x^2 = 1 \rightarrow x = \pm 1$$

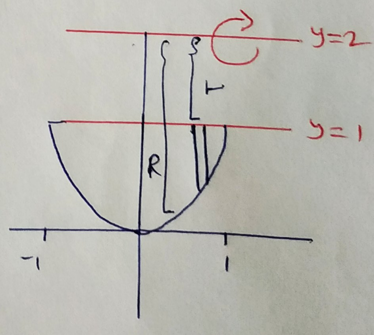
$$V = \pi \int_{-1}^{1} (1 - x^2)^2 dx = \pi \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]\Big|_{-1}^{1} = \pi \left[1 - \frac{2}{3} + \frac{1}{5} - \left(-1 - \frac{2(-1)^3}{3} + \frac{(-1)^5}{5} \right) \right]$$

$$= \pi \left[1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} \right] = \pi \left[2 - \frac{4}{3} + \frac{2}{5} \right]$$

$$= \pi \left[\frac{30 - 20 + 6}{15} \right] = \frac{16\pi}{15}$$

b) the line $y = 2$



$$V = \pi \int_{-1}^{1} R^2(x) - r^2(x)\, dx$$

$$= \pi \int_{-1}^{1} (2 - x^2)^2 - (1)^2 \, dx$$

$$= \pi \int_{-1}^{1} 4 - 4x^2 + x^4 - 1 \, dx$$

$$= \pi \left(3x - \frac{4x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^{1}$$

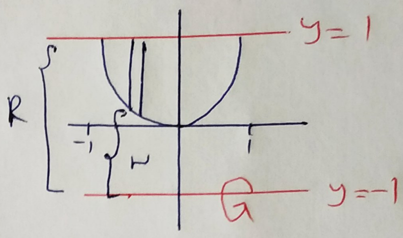
$$= \pi \left[3(1) - \frac{4(1)}{3} + \frac{1}{5} - \left(3(-1) - \frac{4(-1)^3}{3} + \frac{(-1)^5}{5} \right) \right]$$

$$= \pi \left[3 - \frac{4}{3} + \frac{1}{5} + 3 - \frac{4}{3} + \frac{1}{5} \right] = \pi \left[6 - \frac{8}{3} + \frac{2}{5} \right]$$

$$\pi \left[\frac{90 - 40 + 6}{15} \right] = 56\pi$$

c) the line $y=-1$

$$V = \pi \int_{-1}^{1} R^2(x) - r^2(x)\, dx$$



$$= \pi \int_{-1}^{1} (2)^2 - (1+x^2)^2\, dx$$

$$= \pi \int_{-1}^{1} 4 - (1+2x^2+x^4)\, dx$$

$$= \pi \int_{-1}^{1} 3 - 2x^2 - x^4\, dx$$

$$= \pi \left[3x - \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{1}$$

$$= \pi \left[3(1) - \frac{2(1)}{3} - \frac{1}{5} - \left(3(-1) - \frac{2(-1)^3}{3} - \frac{(-1)^5}{5} \right) \right]$$

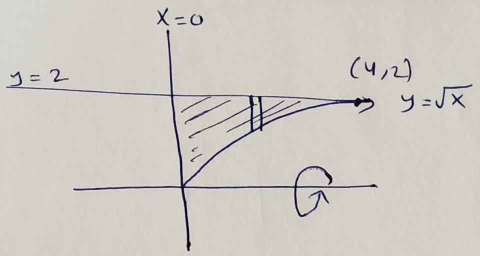
$$= \pi \left[3 - \frac{2}{3} - \frac{1}{5} + 3 - \frac{2}{3} - \frac{1}{5} \right]$$

$$= \pi \left[6 - \frac{4}{3} - \frac{2}{5} \right] = \pi \left[\frac{90 - 20 - 6}{15} \right]$$

$$= \frac{64\pi}{15}$$

51 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$, $x = 0$ about

a) x-axis

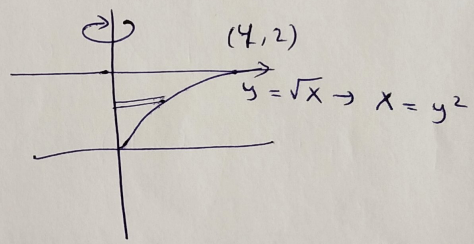


Washer method.

$$V = \pi \int_0^4 (2)^2 - (\sqrt{x})^2 \, dx$$

b) the y-axis

disk method.

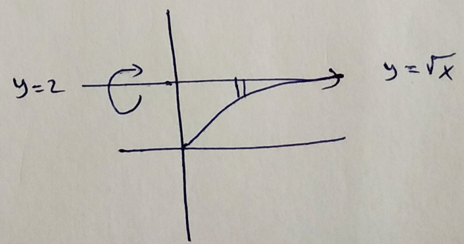


$$V = \pi \int_0^2 (y^2)^2 \, dy$$

$$= \pi \left. \frac{y^5}{5} \right|_0^2 = \pi \frac{(2)^5}{5} = \frac{32}{5}\pi$$

c) the line $y = 2$

disk



$$V = \pi \int_0^4 (2 - \sqrt{x})^2 \, dx$$

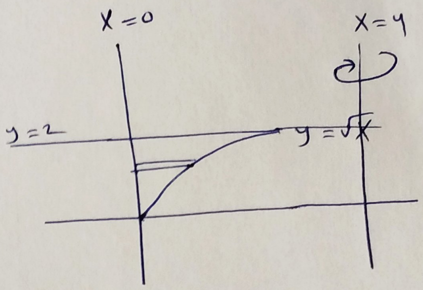
$$= \pi \int_0^4 4 - 4\sqrt{x} + x \, dx$$

$$= \pi \left(4x - 4\frac{x^{3/2}}{3/2} + \frac{x^2}{2} \right) \Big|_0^4$$

$$= \pi \left(4(4) - 4 \cdot \frac{2}{3} (4)^{3/2} + \frac{(4)^2}{2} - 0 \right) = \pi \left(16 - \frac{64}{3} + 8 \right)$$

d) The line $x = 4$

Washer method.



$$V = \pi \int_0^2 \left[R^2(y) - r^2(y)\right] dy$$

$$= \pi \int_0^2 \left[(4)^2 - (4 - y^2)^2\right] dy$$

$$= \pi \int_0^2 \cancel{16} - (\cancel{16} - 8y^2 + y^4)\, dy$$

$$= \pi \left[\frac{8y^3}{3} + \frac{y^5}{5} \right] \Big|_0^2$$

$$= \pi \left[\frac{8(2)^3}{3} + \frac{(2)^5}{5} - 0 \right]$$

$$= \pi \left[\frac{64}{3} + \frac{32}{5} \right]$$

$$= 32\pi \left[\frac{2}{3} + \frac{1}{5} \right] = 32\pi \left[\frac{10+3}{15} \right] = 32\pi \left(\frac{13}{15} \right)$$

$$= \frac{416}{15}\pi$$