

10.9 Convergence of Taylor Series.

* Taylor's Formula:-

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}, \text{ for some } c \text{ between } a \text{ and } x$$

* If $R_n \rightarrow 0$ as $n \rightarrow \infty$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

* The Remainder Estimation Theorem:

If there is a positive constant M such that

$$|f^{(n+1)}(t)| \leq M \text{ for all } t \text{ between } x \text{ and } a$$

$$\text{then } |R_n(x)| \leq \frac{M |x-a|^{n+1}}{(n+1)!}$$

Questions: 10, 12, 18, 28, 35, 37, 41

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[10] Use substitution to find the Taylor series at $x=0$ of the function $\frac{1}{2-x}$

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \left(\frac{1}{1-\frac{x}{2}} \right)$$

$$\rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{geometric series } a=1, r=x)$$

$$\rightarrow \frac{1}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n$$

$$\begin{aligned} \rightarrow \frac{1}{2-x} &= \frac{1}{2} \left(\frac{1}{1-\frac{x}{2}} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \dots \end{aligned}$$

[12] Use power series operations to find the Taylor series at $x=0$ for the function $x^2 \sin x$

Taylor series generated by $\sin x$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\begin{aligned} \text{So } x^2 \sin x &= x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \dots \end{aligned}$$

18 Use power series ~~gen~~ operations to find the Taylor series at $x=0$ for the function $\sin^2 x$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$\begin{aligned} \sin^2(x) &= \frac{1}{2} - \frac{1}{2} \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right) \\ &= \frac{(2x)^2}{2 \cdot 2!} - \frac{(2x)^4}{2 \cdot 4!} + \frac{(2x)^6}{2 \cdot 6!} + \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x)^{2n}}{2 \cdot (2n)!} \end{aligned}$$

$$\sin^2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$$

28] use power series operations to find the Taylor series at $x=0$ for the function $\ln(x+1) - \ln(1-x)$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \right)$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$$

35 Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$|R_n(x)| \leq \frac{M |x|^{n+1}}{(n+1)!}$$

$$|R_3(x)| \leq \frac{M |x|^4}{4!}$$

find M , $f^{(4)}(x) = \sin(x) \leq 1$ for any $x \in [0, 0.1]$

take $M = 1$

$$|R_3(0.1)| \leq \frac{1 |0.1|^4}{4!} = 4.2 \times 10^{-6}$$

$$\text{Error} \leq 4.2 \times 10^{-6}$$

37] For approximately what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 5×10^{-4} ?

$$|E| < 5 \times 10^{-4}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$|E| < \frac{|x|^5}{5!}$$

by Alternating Series
Estimation

$$\frac{|x|^5}{5!} < 5 \times 10^{-4}$$

$$|x|^5 < 5 \times 10^{-4} \times 5!$$

$$|x|^5 < 600 \times 10^{-4}$$

$$|x| < \sqrt[5]{600 \times 10^{-4}} \approx 0.5697$$

$$-0.5697 < x < 0.5697$$

[41] The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation theorem to estimate the error when $|x| < 0.1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$|R_2(x)| = \left| \frac{f^{(3)}(c) x^3}{3!} \right| = \left| \frac{e^c x^3}{3!} \right|, \quad c \text{ between } 0 \text{ and } x$$

$$\rightarrow |x| < 0.1 \rightarrow |x|^3 < (0.1)^3$$

$$\rightarrow \text{if } -0.1 < x < 0, \text{ then } e^c < 1$$

$$\rightarrow |R_2(x)| < \frac{(0.1)^3}{3!} = 1.67 \times 10^{-4}$$

$$\text{if } 0 < x < 0.1 \rightarrow e^c < e^{0.1}$$

$$|R_2(0.1)| < \left| \frac{e^{0.1} (0.1)^3}{3!} \right| = 1.84 \times 10^{-4}$$

$$\text{So the error } E < 1.84 \times 10^{-4}$$