8.4 Integration of Rational functions by Partial fraction.

Def A rational fn. $R(x)$ is defined as $R(x) = \frac{f(x)}{g(x)}$

where $f(x)$, $g(x)$ are Polynomial with no common factor

Case 1: If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as product of Distinct first degree factors

Case 2: If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as a repeated first degree factors

Case 3: If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as Product of Distinct Second degree factors

Case 4 If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as repeated Second degree factors

Case 1 If $\deg f(x) < \deg g(x)$, and $g(x)$ is written as a product of Distinct first degree factors.

Example: Evaluate the following integral

Page 461 Q16

① $\int \frac{x+3}{2x^3-8x}\,dx$

$= \int \frac{x+3}{2x(x^2-4)}\,dx = \int \frac{x+3}{2x(x-2)(x+2)}\,dx$

$\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2}$

Mulitply both side by "$\overbrace{2x(x-2)(x+2)}^{L.CD}$"

$x+3 = A(x-2)(x+2) + B(2x)(x+2) + C(2x)(x-2)$

* If $x=0$

$3 = -4A \longrightarrow A = \frac{-3}{4}$

* If $x=2$

$5 = 16B \longrightarrow B = \frac{5}{16}$

* If $x=-2$

$1 = 16C \longrightarrow C = \frac{1}{16}$

Solution

$\int \frac{x+3}{2x(x-2)(x+2)}\,dx = \int \frac{\frac{-3}{4}}{2x}\,dx + \int \frac{\frac{5}{16}}{x-2}\,dx + \int \frac{\frac{1}{16}}{x+2}\,dx$

$= \frac{-3}{8}\ln|x| + \frac{5}{16}\ln|x-2| + \frac{1}{16}\ln|x+2| + C$

② $\int \frac{x^2+3}{x^2-3x+2}$

If $\deg f(x) \geq \deg g(x)$
$\rightarrow$ Use long Division

$$\begin{array}{r|l} & 1 \\ x^2-3x+2 & x^2+3 \\ & x^2-3x+2 \\ \hline & 3x+1 \end{array}$$

$$\int 1 + \frac{3x+1}{x^2-3x+2}\,dx$$

$$\frac{3x+1}{x^2-3x+2} = \frac{3x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Mulitply by $L.C.D = (x-1)(x-2)$ by both side

$$3x+1 = A(x-2) + B(x-1)$$

$$3x+1 = Ax - 2A + Bx - B$$

$$A+B = 3 \quad \text{---- eq ①}$$

$$+ \quad -2A - B = 1 \quad \text{---- eq ②}$$

$$-A = 4 \rightarrow \boxed{A=-4}$$

$$\boxed{B=7}$$

Solution

$$\int \frac{x^2+3}{x^2-3x+2} = \int 1 - \frac{4}{x-1} + \frac{7}{x-2}\,dx$$

$$= x - 4\ln|x-1| + 7\ln|x-2| + C$$

Case 2. If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as a repeated first degree factors

③ $\int \frac{2x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$

$$\frac{2x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Multiply by L.C.D $= (x+1)^3$ both sides

$$2x^2+3x+3 = A(x+1)^2 + B(x+1) + C$$

* If $\boxed{x=-1}$

$$2-3+3 = C \longrightarrow \boxed{C=2}$$

* If $\boxed{x=0}$

$$3 = A + B + C \longrightarrow \boxed{A+B=1} \text{ ---- eq ①}$$

* If $\boxed{x=-2}$

$$8-6+3 = A - B + C \longrightarrow \boxed{A-B=3} \text{ --- eq ②}$$

$$\boxed{\begin{array}{l} A=2 \\ B=-1 \end{array}}$$

$$\text{Sol.} = 2\ln|x+1| + \frac{1}{x+1} - \frac{1}{(x+1)^2} + C$$

Sol $\int \frac{2x^2+3x+3}{(x+1)^3} = \int \frac{2}{x+1}\,dx + \int \frac{-1}{(x+1)^2}\,dx + \int \frac{2}{(x+1)^3}\,dx$

$= 2\ln|x+1| + \frac{1}{x+1} + \frac{2.}{-2}(x+1)^{-2} + C$

$= 2\ln|x+1| + \frac{1}{x+1} - \frac{1}{(x+1)^2} + C$

(4) $\int \frac{16x^3}{4x^2-4x+1}\,dx$

$$\begin{array}{r|l} & 4x+4 \\ \hline 4x^2-4x+1 & 16x^3 \\ & 16x^3-16x^2+4x \\ \hline & 16x^2-4x \\ & 16x^2-16x+4 \\ \hline & 12x-4 \end{array}$$

$\int 4x+4 + \frac{12x-4}{4x^2-4x+1}\,dx$

$\frac{12x-4}{4x^2-4x+1} = \frac{12x-4}{(2x-1)^2} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2}$

$12x-4 = A(2x-1) + B$

$12x-4 = 2Ax - A + B$

* Comparing the Coefficient in Both Sides

$12 = 2A \longrightarrow A = 6$

$-4 = -A + B \longrightarrow B = 2$

Sol. $\int \frac{16x^3}{4x^2-4x+1} = \int 4x+4 + \frac{6}{(2x-1)} + \frac{2}{(2x-1)^2}\,dx$

$= \frac{4x^2}{2} + 4x + 3\int \frac{2}{2x-1}\,dx + 2\int \frac{1}{(2x-1)^2}\,dx$

$= 2x^2 + 4x + 3\ln|2x-1| - \frac{1}{(2x-1)} + C$

(5) $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Mulitply by L.C.D $= (x-1)(x+1)^2$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

* If $x=-1 \longrightarrow -2C = 1 \longrightarrow C = \frac{-1}{2}$

* If $x = 1 \longrightarrow 4A = 1 \longrightarrow A = \frac{1}{4}$

* If $x = 0 \longrightarrow A - B - C = 0 \longrightarrow B = \frac{3}{4}$

$$\int \frac{x^2}{(x-1)(x^2+2x+1)} = \int \frac{\frac{1}{4}}{x-1}dx + \int \frac{\frac{3}{4}}{x+1}dx + \int \frac{\frac{-1}{2}}{(x+1)^2}\,dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$$

Case 3 If $\deg f(x) < \deg g(x)$ and $g(x)$ is written as product of Distinct Second degree factor

Page 461 Q30

6) $$\int \frac{x^2+x}{x^4-3x^2-4}$$

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{(x^2+1)(x^2-4)} = \frac{x^2+x}{(x^2+1)(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$$

If $\boxed{x=2}$

$$6 = 20A \longrightarrow A = \frac{3}{10}$$

If $\boxed{x=-2}$

$$2 = -20B \longrightarrow B = \frac{-1}{10}$$

If $\boxed{x=0}$

$$0 = 2A - 2B - 4D$$

$$0 = 2\left(\frac{3}{10}\right) - 2\left(\frac{-1}{10}\right) - 4D \longrightarrow D = \frac{2}{10} = \frac{1}{5}$$

If $\boxed{x=1}$

$$2 = 6A - 2B + -3(C+D) \longrightarrow C = \frac{-1}{5}$$

Sol $\int \frac{x^2+x}{x^4-3x^2-4} = \int \frac{\frac{3}{10}}{x-2}dx + \int \frac{\frac{-1}{10}}{x+2}dx + \int \frac{\frac{-1}{5}x+\frac{1}{5}}{x^2+1}dx$

$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{5.2}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$

$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}(x) + C$

Q_{38}

7) $\int \frac{2y^4}{y^3-y^2+y-1}dy$

$$\begin{array}{r|l} & 2y+2 \\ \hline y^3-y^2+y-1 & 2y^4 \\ & \underline{2y^4-2y^3+2y^2-2y} \\ & 2y^3-2y^2+2y \\ & \underline{2y^3-2y^2+2y-2} \\ & 2 \end{array}$$

$\int 2y+2 + \frac{2}{y^3-y^2+y-1}dy$

$\frac{2}{y^3-y^2+y-1} = \frac{2}{y^2(y-1)+(y-1)} = \frac{2}{(y-1)(y^2+1)}$

$\frac{2}{(y-1)(y^2+1)} = \frac{A}{(y-1)} + \frac{By+C}{(y^2+1)}$

$2 = A(y^2+1) + (By+C)(y-1)$

$2 = Ay^2 + A + By^2 - By + Cy - C$

$A+B=0$

$-B+C=0$

$A-C=2$

$\boxed{A=1}$ $\boxed{B=-1}$ $\boxed{C=-1}$

Case 4 If $\overset{deg}{f(x)} < \overset{deg}{g(x)}$ and $g(x)$ is written as repeated

second degree factors

(8) $\int \frac{x^3+4x^2-4x-4}{(x^2+1)^2}$

$$\frac{x^3+4x^2-4x-4}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^3+4x^2-4x-4 = (Ax+B)(x^2+1) + Cx+D$$

$$x^3+4x^2-4x-4 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$\boxed{A=1} \longrightarrow -4 = A + C \quad \boxed{C=-5}$

$\boxed{B=4} \longrightarrow -1 = B + D \quad \boxed{D=-5}$

Sol $\int \frac{x^3+4x^2-4x-4}{(x^2+1)^2} = \int \frac{x+4}{x^2+1}dx + \int \frac{-5x-5}{(x^2+1)^2}dx$

$\frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{4}{x^2+1}dx - \frac{5}{2}\int \frac{2x}{(x^2+1)^2}dx - 5\int \frac{1}{(x^2+1)^2}dx$

$u = x^2+1$
$du = 2x\,dx$

$x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

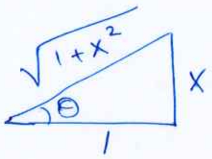
$\frac{1}{2}\ln|x^2+1| + 4\tan^{-1}(x) - \frac{5}{2}\int \frac{du}{u^2} - 5\int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$

$\frac{1}{2}\ln|x^2+1| + 4\tan^{-1}(x) + \frac{5}{2}\frac{1}{(x^2+1)} - 5\int \frac{\sec^2\theta}{\sec^4\theta}d\theta$

$$\frac{1}{2}\ln|x^2+1| + 4\tan^{-1}x + \frac{5}{2}\frac{1}{(x^2+1)} - 5\int \cos^2\theta \, d\theta$$

$x = \tan\theta$

$\tan^{-1}(x) = \theta$



$$\int \cos^2\theta \, d\theta = \int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\,\frac{\sin(2\theta)}{2}$$

$$= \frac{1}{2}\tan^{-1}(x) + \frac{1}{4}\left[2\sin(\theta)\cos(\theta)\right]$$

$$= \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\left[\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{2}\tan^{-1}(x) + \frac{x}{2(1+x^2)}$$

$$= \frac{1}{2}\ln|x^2+1| + 4\tan^{-1}(x) + \frac{5}{2}\frac{1}{(x^2+1)} - \frac{5}{2}\tan^{-1}(x) - \frac{5}{2}\frac{x}{(1+x^2)} + C$$

$$\frac{1}{2}\ln(x^2+1) + \frac{3}{2}\tan^{-1}(x) + \frac{5}{2(x^2+1)} - \frac{5x}{2(x^2+1)} + C$$

Page 462 Q42

9) $\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2} d\theta$

$U = \cos x$

$dU = -\sin x \, dx$

$$\int \frac{-dU}{U^2 + U - 2}$$

$$\frac{-1}{U^2+U-2} = \frac{-1}{(U+2)(U-1)} = \frac{A}{U+2} + \frac{B}{U-1}$$

$$-1 = A(U-1) + B(U+2)$$

$$-1 = AU - A + BU + 2B$$

$A + B = 0$

$-A + 2B = -1$

$B = \frac{-1}{3} \qquad A = \frac{1}{3}$

$$\int \frac{-dU}{U^2+U-2} = \int \frac{\frac{1}{3}}{U+2} dU + \int \frac{\frac{-1}{3}}{U-1} dU$$

$$= \frac{1}{3} \ln|U+2| - \frac{1}{3} \ln|U-1| + C$$

$$= \frac{1}{3} \ln|\cos x + 2| - \frac{1}{3} \ln|\cos x - 1| + C$$

10 $\int \frac{1}{x(x^{20}+1)}$

$u = x^{20}$

$du = 20x^{19}\,dx$

$\int \frac{1\,x^{19}}{x(x^{20}+1)x^{19}}\,dx$

$\frac{1}{20}\int \frac{du}{u(u+1)} \longrightarrow \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$

$1 = A(u+1) + B(u)$

$A + B = 0$ $\boxed{A = 1}$

$\boxed{B = -1}$

$\frac{1}{20}\int \frac{du}{u(u+1)} = \frac{1}{20}\int \frac{1}{u}\,du + \frac{1}{20}\int \frac{-1}{u+1}\,du$

$= \frac{1}{20}\ln|u| - \frac{1}{20}\ln|u+1| + C$

$= -\frac{1}{20}\ln\left|\frac{u+1}{u}\right| + C$

$= \frac{-1}{20}\ln\left|1 + \frac{1}{x^{20}}\right| + C$

$$\int \frac{x^4}{x^2-1}$$

$$\begin{array}{r} x^2+1 \\ x^2-1 \overline{\big)\, x^4 \qquad} \\ -x^4 \pm x^2 \\ \hline x^2 \\ -x^2 \pm 1 \\ \hline 1 \end{array}$$

$$\int x^2+1 + \frac{1}{x^2-1}\, dx$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1)$$

$A+B=0$ $\qquad A-B=1$

$A=\frac{1}{2}$ $\qquad B=\frac{-1}{2}$

$$\int \frac{x^4}{x^2-1}\, dx = \int x^2+1\, dx + \int \frac{\frac{1}{2}}{x-1} dx + \frac{-\frac{1}{2}}{x+1} dx$$

$$\frac{x^3}{3} + x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$$