≡ **Computer Lab Assignment**

21. While the integral (12) can be graphed in the same manner discussed on page 758 to obtain Figure 15.3.5, it can also be expressed in terms of a special function that is built into a CAS.

(a) Use a trigonometric identity to show that an alternative form of the Fourier integral representation (12) of the function f in Example 2 (with $a = 1$) is

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha(x+1) - \sin\alpha(x-1)}{\alpha}\,d\alpha.$$

(b) As a consequence of part (a), $f(x) = f(x) = \lim_{b\to\infty} F_b(x)$, where

$$F_b(x) = \frac{1}{\pi}\int_0^b \frac{\sin\alpha(x+1) - \sin\alpha(x-1)}{\alpha}\,d\alpha.$$

Show that the last integral can be written as

$$F_b(x) = \frac{1}{\pi}[\mathrm{Si}(b(x+1)) - \mathrm{Si}(b(x-1))],$$

where Si(x) is the **sine integral function**. See Problem 51 in Exercises 2.3.

(c) Use a CAS and the sine integral form obtained in part (b) to graph $F_b(x)$ on the interval [–3, 3] for $b = 4$, 6, and 15. Then graph $F_b(x)$ for larger values of $b > 0$.

15.4 Fourier Transforms

≡ **Introduction** Up to now we have studied and used only one integral transform: the Laplace transform. But in Section 15.3 we saw that the Fourier integral had three alternative forms: the cosine integral, the sine integral, and the complex or exponential form. In the present section we shall take these three forms of the Fourier integral and develop them into three new integral transforms naturally called **Fourier transforms**. In addition, we shall expand on the concept of a transform pair; that is, an integral transform and its inverse. We shall also see that the inverse of an integral transform is itself another integral transform.

Transform Pairs The Laplace transform $F(s)$ of a function $f(t)$ is defined by an integral, but up to now we have been using the symbolic representation $f(t) = \mathscr{L}^{-1}\{F(s)\}$ to denote the inverse Laplace transform of $F(s)$. Actually, the inverse Laplace transform is also an *integral* transform. If

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s), \tag{1}$$

then the **inverse Laplace transform** is

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds = f(t). \tag{2}$$

The last integral is called a **contour integral;** its evaluation requires the use of complex variables and is beyond the scope of this discussion. The point here is this: Integral transforms appear in

transform pairs. If $f(x)$ is transformed into $F(\alpha)$ by an **integral transform**

$$F(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx, \qquad (3)$$

then the function f can be recovered by another integral transform

$$f(x) = \int_c^b F(\alpha)\, H(\alpha, x)\, d\alpha, \qquad (4)$$

called the **inverse transform**. The functions K and H in the integrands of (3) and (4) are called the **kernels** of their respective transforms. We identify $K(s, t) = e^{-st}$ as the kernel of the Laplace transform and $H(s, t) = e^{st}/2\pi i$ as the kernel of the inverse Laplace transform.

 Fourier Transform Pairs The Fourier integral is the source of three new integral transforms. From (8) and (9), (10) and (11), and (18) and (19) of the preceding section, we are prompted to define the following **Fourier transform pairs**.

Definition 15.4.1 Fourier Transform Pairs

(*i*) Fourier transform:

$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx = F(\alpha) \qquad (5)$$

Inverse Fourier transform:

$$\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\, d\alpha = f(x) \qquad (6)$$

(*ii*) Fourier sine transform:

$$\mathscr{F}_s\{f(x)\} = \int_0^{\infty} f(x)\sin \alpha x\, dx = F(\alpha) \qquad (7)$$

Inverse Fourier sine transform:

$$\mathscr{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\sin \alpha x\, d\alpha = f(x) \qquad (8)$$

(*iii*) Fourier cosine transform:

$$\mathscr{F}_c\{f(x)\} = \int_0^{\infty} f(x)\cos \alpha x\, dx = F(\alpha) \qquad (9)$$

Inverse Fourier cosine transform:

$$\mathscr{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\cos \alpha x\, d\alpha = f(x) \qquad (10)$$

□ **Existence** The conditions under which (5), (7), and (9) exist are more stringent than those for the Laplace transform. For example, you should verify that $\mathscr{F}\{1\}$, $\mathscr{F}_s\{1\}$, and $\mathscr{F}_c\{1\}$ do not exist. Sufficient conditions for existence are that *f* be absolutely integrable on the appropriate interval and that *f* and *f'* be piecewise continuous on every finite interval.

□ **Operational Properties** Since our immediate goal is to apply these new transforms to boundary-value problems, we need to examine the transforms of derivatives.

Fourier Transform

Suppose that *f* is continuous and absolutely integrable on the interval $(-\infty, \infty)$ and f' is piecewise continuous on every finite interval. If $f(x) \to 0$ as $x \to \pm\infty$, then integration by parts gives

$$\begin{aligned}\mathscr{F}\{f'(x)\} &= \int_{-\infty}^{\infty} f'(x)\, e^{i\alpha x}\, dx \\ &= f(x)e^{i\alpha x}\Big]_{-\infty}^{\infty} - i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx \\ &= -i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx;\end{aligned}$$

that is

$$\mathscr{F}\{f'(x)\} = -i\alpha F(\alpha). \tag{11}$$

Similarly, under the added assumptions that f' is continuous on $(-\infty, \infty)$, $f''(x)$ is piecewise continuous on every finite interval, and $f'(x) \to 0$ as $x \to \pm\infty$, we have

$$\mathscr{F}\{f''(x)\} = (-i\alpha)^2\, \mathscr{F}\{f(x)\} = -\alpha^2 F(\alpha). \tag{12}$$

In general, under conditions analogous to those leading to (12), we have

$$\mathscr{F}\{f^{(n)}(x)\} = (-i\alpha)^n\, \mathscr{F}\{f(x)\} = (-i\alpha)^n F(\alpha),$$

where $n = 1, 2, 3, \ldots$.

It is important to be aware that the sine and cosine transforms are not suitable for transforming the first derivative (or, for that matter, any derivative of *odd* order). It is readily shown that

$$\mathscr{F}_s\{f'(x)\} = -\alpha\mathscr{F}_c\{f(x)\} \qquad \text{and} \qquad \mathscr{F}_c\{f'(x)\} = \alpha\mathscr{F}_s\{f(x)\} - f(0).$$

The difficulty is apparent; the transform of $f'(x)$ is not expressed in terms of the original integral transform.

Fourier Sine Transform

Suppose that *f* and f' are continuous, *f* is absolutely integrable on the interval [0, 00), and f'' is piecewise continuous on every finite interval. If $f \to 0$ and $f' \to 0$ as $x \to \infty$, then

$$
\begin{aligned}
\mathcal{F}_s\{f''(x)\} &= \int_0^\infty f''(x)\sin\alpha x\,dx \\
&= f'(x)\sin\alpha x\Big]_0^\infty - \alpha\int_0^\infty f'(x)\cos\alpha x\,dx \\
&= -\alpha\left[f(x)\cos\alpha x\Big|_0^\infty + \alpha\int_0^\infty f(x)\sin\alpha x\,dx\right] \\
&= \alpha f(0) - \alpha^2\mathcal{F}_s\{f(x)\};
\end{aligned}
$$

that is,

$$\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0). \tag{13}$$

Fourier Cosine Transform

Under the same assumptions that lead to (9), we find the Fourier cosine transform of $f''(x)$ to be

$$\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0). \tag{14}$$

A natural question is "How do we know which transform to use on a given boundary-value problem?" Clearly, to use a Fourier transform, the domain of the variable to be eliminated must be the interval $(-\infty, \infty)$. To utilize a sine or cosine transform, the domain of at least one of the variables in the problem must be $[0, \infty)$. But the determining factor in choosing between the sine transform and the cosine transform is the type of boundary condition specified at zero.

◀ How do we know which transform to use?

In the examples that follow, we shall assume without further mention that both u and $\partial u/\partial x$ (or $\partial x/\partial y$) approach zero as $x \to \pm\infty$. This is not a major restriction since these conditions hold in most applications.

EXAMPLE 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, -\infty < x < \infty, t > 0$, subject to

$$u(x, 0) = f(x), \qquad \text{where} \qquad f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

SOLUTION The problem can be interpreted as finding the temperature $u(x, t)$ in an infinite rod. Since the domain of x is the infinite interval $(-\infty, \infty)$, we use the Fourier transform (5) and define

$$\mathcal{F}\{u(x, t)\} = \int_{-\infty}^{\infty} u(x, t)e^{i\alpha x}\,dx = U(\alpha, t).$$

Transforming the partial differential equation and using (12),

$$\mathcal{F}\left\{k\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\}$$

yields

$$-k\alpha^2 U(\alpha, t) = \frac{dU}{dt} \qquad \text{or} \qquad \frac{dU}{dt} + k\alpha^2 U(\alpha, t) = 0.$$

Solving the last equation gives $U(\alpha, t) = ce^{-k\alpha^2 t}$. The initial temperature $u(x, 0) = f(x)$ in the rod is shown in **FIGURE 15.4.1** and its Fourier transform is

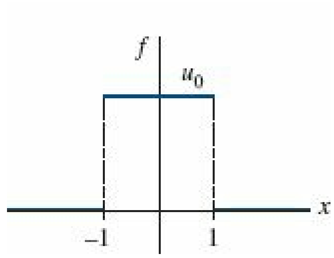

FIGURE 15.4.1 Initial temperature f in Example 1

$$\mathscr{F}\{u(x, 0)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \int_{-1}^{1} u_0 e^{i\alpha x}\,dx = u_0 \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha}.$$

This result is the same as $U(\alpha, 0) = 2u_0 \dfrac{\sin\alpha}{\alpha}$. Applying this condition to the solution $U(\alpha, t)$ gives $U(\alpha, 0) = c = (2u_0 \sin\alpha)/\alpha$, and so

$$U(\alpha, t) = 2u_0 \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}.$$

It then follows from the inversion integral (6) that

$$u(x, t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t} e^{-i\alpha x}\,d\alpha.$$

The last expression can be simplified somewhat by Euler's formula $e^{-i\alpha x} = \cos\alpha x - i\sin\alpha x$ and noting that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t} \sin\alpha x\,d\alpha = 0$ since the integrand is an odd function of α. Hence we finally have

$$u(x, t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha} e^{-k\alpha^2 t}\,d\alpha. \tag{15}$$

≡

It is left to the reader to show that the solution (15) can be expressed in terms of the error function. See Problem 23 in Exercises 15.4.

EXAMPLE 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \pi, \quad y > 0$$

$$u(0, y) = 0, \qquad u(\pi, y) = e^{-y}, \qquad y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \qquad 0 < x < \pi.$$

Solve for $u(x, y)$.

SOLUTION The domain of the variable y and the prescribed condition at $y = 0$ indicate that the Fourier cosine transform is suitable for the problem. We define

$$\mathscr{F}_c\{u(x, y)\} = \int_0^\infty u(x, y) \cos \alpha y \, dy = U(x, \alpha).$$

In view of (14),

$$\mathscr{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathscr{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathscr{F}_c\{0\}$$

becomes

$$\frac{d^2 U}{dx^2} - \alpha^2 U(x, \alpha) - u_y(x, 0) = 0 \qquad \text{or} \qquad \frac{d^2 U}{dx^2} - \alpha^2 U = 0.$$

Since the domain of x is a finite interval, we choose to write the solution of the ordinary differential equation as

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x. \tag{16}$$

Now $\mathscr{F}_c\{u(0, y)\} = \mathscr{F}_c\{0\}$ and $\mathscr{F}_c\{u(\pi, y)\} = \mathscr{F}_c\{e^{-y}\}$ are in turn equivalent to

$$U(0, \alpha) = 0 \qquad \text{and} \qquad U(\pi, \alpha) = \frac{1}{1 + \alpha^2}.$$

When we apply these latter conditions, the solution (16) gives $c_1 = 0$ and $c_2 = 1/[1 + \alpha^2) \sinh \alpha\pi]$. Therefore

$$U(x, \alpha) = \frac{\sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi},$$

and so from (10) we arrive at

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi} \cos \alpha y \, d\alpha. \tag{17}$$

≡

Had $u(x, 0)$ been given in Example 2 rather than $u_y(x, 0)$, then the sine transform would have been appropriate.

15.4 Exercises Answers to selected odd-numbered problems begin on page ANS-35.

In Problems 1–18, use an appropriate Fourier integral transform to solve the given boundary-value problem. Make assumptions about boundedness where necessary.

1.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ -\infty < x < \infty, \ t > 0$$

$$u(x, 0) = e^{-|x|}, \ -\infty < x < \infty$$

2.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ -\infty < x < \infty, \ t > 0$$

$$u(x, 0) = \begin{cases} 0, & x < -1 \\ -100, & -1 < x < 0 \\ 100, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

3. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = u_0$, $t > 0$ and $u(x, 0) = 0$, $x > 0$.

4. Use the result $\int_0^\infty \frac{\sin \alpha x}{\alpha} d\alpha = \frac{\pi}{2}, x > 0$, to show that the solution in Problem 3 can be written as

$$u(x, t) = u_0 - \frac{2u_0}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t}\, d\alpha.$$

5. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = 0$, $t > 0$, and

$$u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

6. Solve Problem 3 if the condition at the left boundary is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A, \quad t > 0.$$

7. Solve Problem 5 if the end $x = 0$ is insulated.

8. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = 1$, $t > 0$, and $u(x, 0) = e^{-x}$, $x > 0$.

9. **(a)**

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ -\infty < x < \infty$$

(b) If $g(x) = 0$, show that the solution of part (a) can be written as $u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)]$.

10. Find the displacement $u(x, t)$ of a semi-infinite string if

$$u(0, t) = 0, \quad t > 0$$

$$u(x, 0) = xe^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ x > 0.$$

11. Solve the problem in Example 2 if the boundary conditions at $x = 0$ and $x = \pi$ are reversed:

$$u(0, y) = e^{-y}, \ u(\pi, y) = 0, \ y > 0.$$

12. Solve the problem in Example 2 if the boundary condition at $y = 0$ is $u(x, 0) = 1$, $0 < x < \pi$.

13. Find the steady-state temperature $u(x, y)$ in a plate defined by $x \geq 0$, $y \geq 0$ if the boundary $x = 0$ is insulated and, at $y = 0$,

$$u(x, 0) = \begin{cases} 50, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

14. Solve Problem 13 if the boundary condition at $x = 0$ is $u(0, y) = 0$, $y > 0$.

15.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ x > 0,\ 0 < y < 2$$
$$u(0, y) = 0, \quad 0 < y < 2$$
$$u(x, 0) = f(x), \quad u(x, 2) = 0, \quad x > 0$$

16.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < y < \pi, y > 0$$
$$u(0, y) = f(y), \quad \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 0, y > 0$$
$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ 0 < x < \pi$$

In Problems 17 and 18, find the steady-state temperature $u(x, y)$ in the plate given in the figure. [*Hint*: One way of proceeding is to express Problems 17 and 18 as two and three boundary-value problems, respectively. Use the superposition principle (see Section 13.5).]

17.

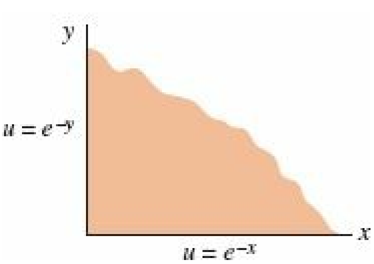

FIGURE 15.4.2 Infinite plate in Problem 17

18.

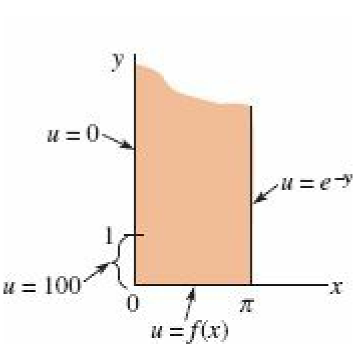

FIGURE 15.4.3 Semi-infinite plate in Problem 18

19. Use the result $\mathcal{F}\{e^{-x^2/4p^2}\} = 2\sqrt{\pi}pe^{-p^2\alpha^2}$ to solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = e^{-x^2}, \quad -\infty < x < \infty.$$

20. If $\mathscr{F}\{f(x)\} = F(\alpha)$ and $\mathscr{F}\{g(x)\} = G(\alpha)$, then the **convolution theorem** for the Fourier transform is given by

$$\int_{-\infty}^{\infty} f(\tau)g(x-\tau)\,d\tau = \mathscr{F}^{-1}\{F(\alpha)G(\alpha)\}.$$

Use this result and the transform $\mathscr{F}\{e^{-x^2/4p^2}\}$ given in Problem 19 to show that a solution of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

is

$$u(x, t) = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(\tau)e^{-(x-\tau)^2/4kt}\,d\tau.$$

21. Use the transform $\mathscr{F}\{e^{-x^2/4p^2}\}$ given in Problem 19 to find the steady-state temperature $u(x, y)$ in the infinite strip shown in **FIGURE 15.4.4**.

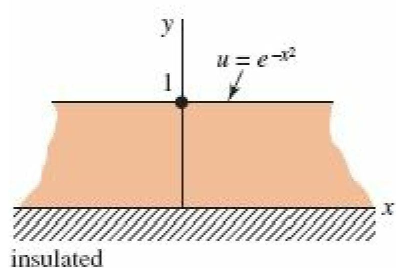

FIGURE 15.4.4 Infinite plate in Problem 21

22. The solution of Problem 14 can be integrated. Use entries 42 and 43 of the table in Appendix III to show that

$$u(x, y) = \frac{100}{\pi}\left[\arctan\frac{x}{y} - \frac{1}{2}\arctan\frac{x+1}{y} - \frac{1}{2}\arctan\frac{x-1}{y}\right].$$

23. Use Problem 20, the change of variables $v = (x-\tau)/2\sqrt{kt}$, and Problem 11 in Exercises 15.1, to show that the solution of Example 1 can be expressed as

$$u(x, t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{kt}}\right)\right].$$

24. The steady-state temperatures in a semi-infinite cylinder are described by the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, \quad z > 0$$
$$u(1, z) = 0, \quad z > 0$$
$$u(r, 0) = u_0, \quad 0 < r < 1.$$

Use an appropriate Fourier transform to find $u(r, z)$. [*Hint*: See Example 3 in Section 14.2.]

25. Find the steady-state temperatures $u(r, z)$ in the semi-infinite cylinder in Problem 24 if the base of the cylinder is insulated and

$$u(1, z) = \begin{cases} u_0, & 0 < z < 1 \\ 0, & z > 1. \end{cases}$$

≡ Computer Lab Assignment

26. Assume $u_0 = 100$ and $k = 1$ in the solution of Problem 23. Use a CAS to graph $u(x, t)$ over the rectangular region $-4 \le x \le 4$, $0 \le t \le 6$. Use a 2D plot to superimpose the graphs of $u(x, t)$ for $t = 0.05, 0.125, 0.5, 1, 2, 4, 6$, and 15 for $-4 \le x \le 4$. Use the graphs to conjecture the values of $\lim_{t\to\infty} u(x, t)$ and $\lim_{x\to\infty} u(x, t)$. Then prove these results analytically using the properties of $\operatorname{erf}(x)$.

≡ Discussion Problem

27. **(a)** Suppose

$$\int_0^\infty f(x)\cos \alpha x \, dx = F(\alpha),$$

where

$$F(\alpha) = \begin{cases} 1 - \alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1. \end{cases}$$

Find $f(x)$.

(b) Use part (a) to show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

15.5 Fast Fourier Transform

≡ **Introduction** Consider a function f that is defined and continuous on the interval $[0, 2p]$. If $x_0, x_1, x_2, \ldots, x_n, \ldots$ are equally spaced points in the interval, then the corresponding function values $f_0, f_1, f_2, \ldots, f_n, \ldots$ shown in **FIGURE 15.5.1** are said to represent a discrete *sampling* of the function f. The notion of discrete samplings of a function is important in the analysis of continuous signals.

In this section, the complex or exponential form of a Fourier series plays an important role in the discussion. A review of Section 12.4 is recommended.