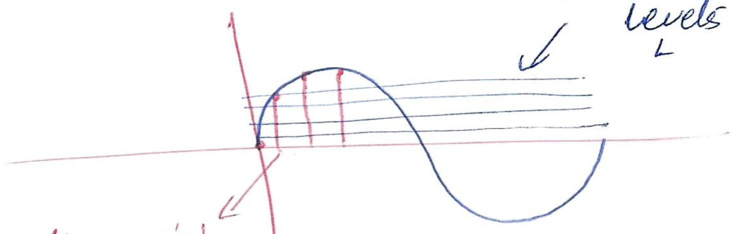
Chapter 4

Sampling of Continuous-Time Signal

* Periodic Sampling



T_s Sampling period

F_s Sampling rate (frequency) (no of samples/sec) Hz

$\Omega_s = \frac{2\pi}{T}$ (Sampling frequency) rad/s

- Analog
- Continous time
- discrete time
- digital

A/D Converter (analog / digital)

C/D Converter (continuous / discrete)

$F_s \uparrow \Rightarrow$ quality $\uparrow \Rightarrow$ memory $\uparrow$

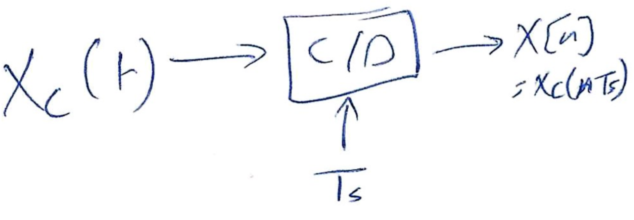
$F_s \downarrow \Rightarrow$ quality $\downarrow \Rightarrow$ memory $\downarrow$

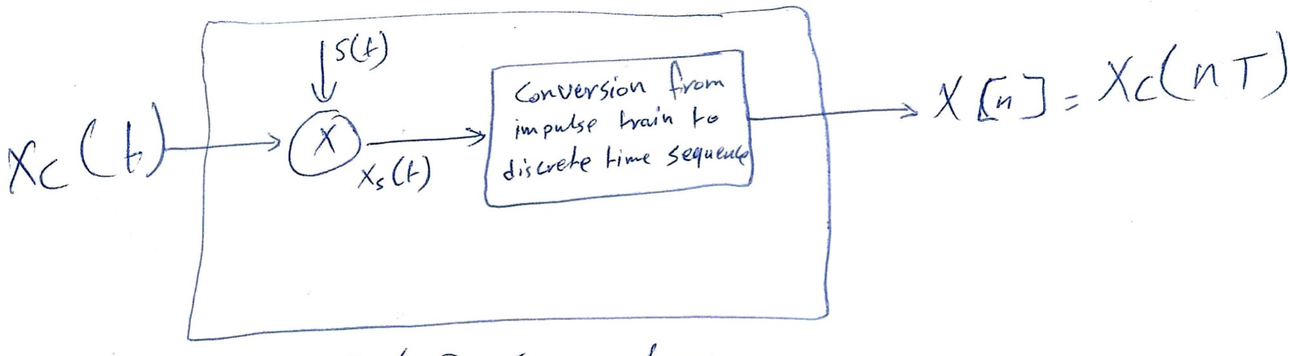
to determine the best F_s

Nyquist rate is presented

Nyquist rate (F_s) $\geq$ 2 (maximum frequency of the signal) = 2 Bw

If $F_s <$ Nyquist rat $\longrightarrow$ aliasing in sampling



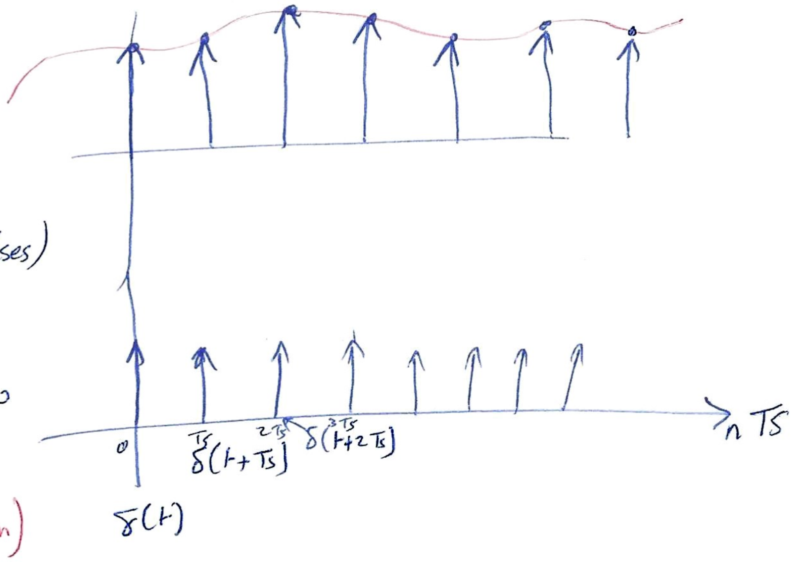


C/D Converter

Sampling process
↳ Two stages

① Impulse Train Modulator
(multiply the CT signal by train of impulses)

② Conversion of the impulse train to sequence (normalization)
Divide by Ts (in time domain)

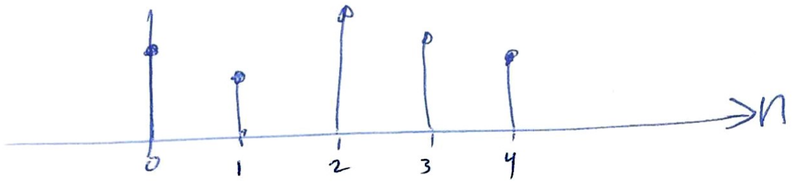


$$\sum_{K=-\infty}^{\infty} \delta(t + K\,T_s)$$

$$x(t) \cdot \delta(t) = x(0)\,\delta(t)$$

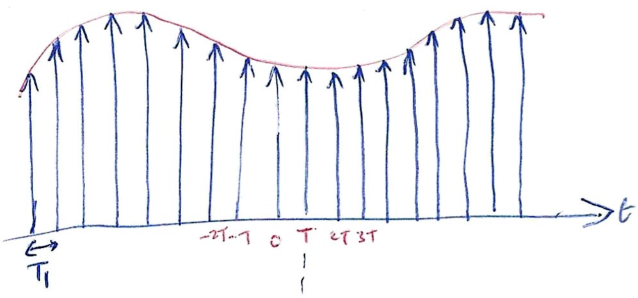
⇓

normalization $\frac{n\,T_s}{T_s}$

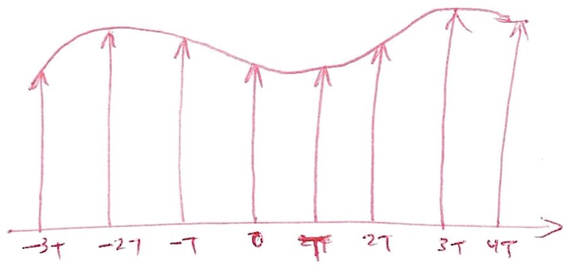


Sequenee

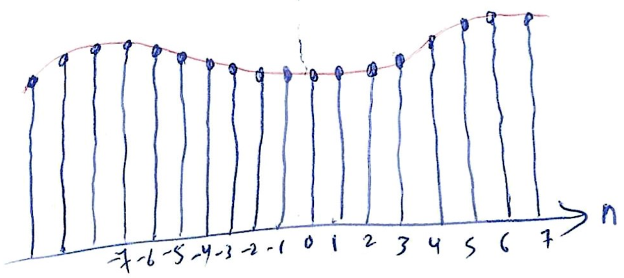
CT signal is the envelope of Impulse train



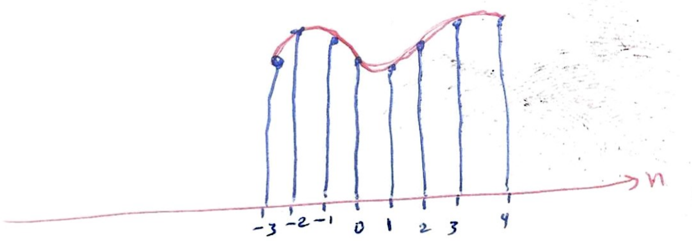
$T = T_1$



$T = 2T_1$



Compressed signal - distortion
aliasing



Frequency-Domain Representation of Sampling:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$\delta(t)$ unit impulse function

$$x_s(t) = x_c(t) \cdot s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

$$F\{x_s(t)\}$$

$F\{x_c(t) \cdot s(t)\}$ is a convolution of Fourier transform of $X_c(j\Omega)$ and $S(j\Omega)$

↓ multiplication to convolution

$\Omega = 2\pi f$ rad/s

$w = 2\pi \frac{f}{F_s}$ rad

* Fourier Transform of a periodic impulse train is a periodic impulse train

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

↓ non periodic

↳ many copies of the signal spectrum

↳ [illegible]

continuous convolution

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

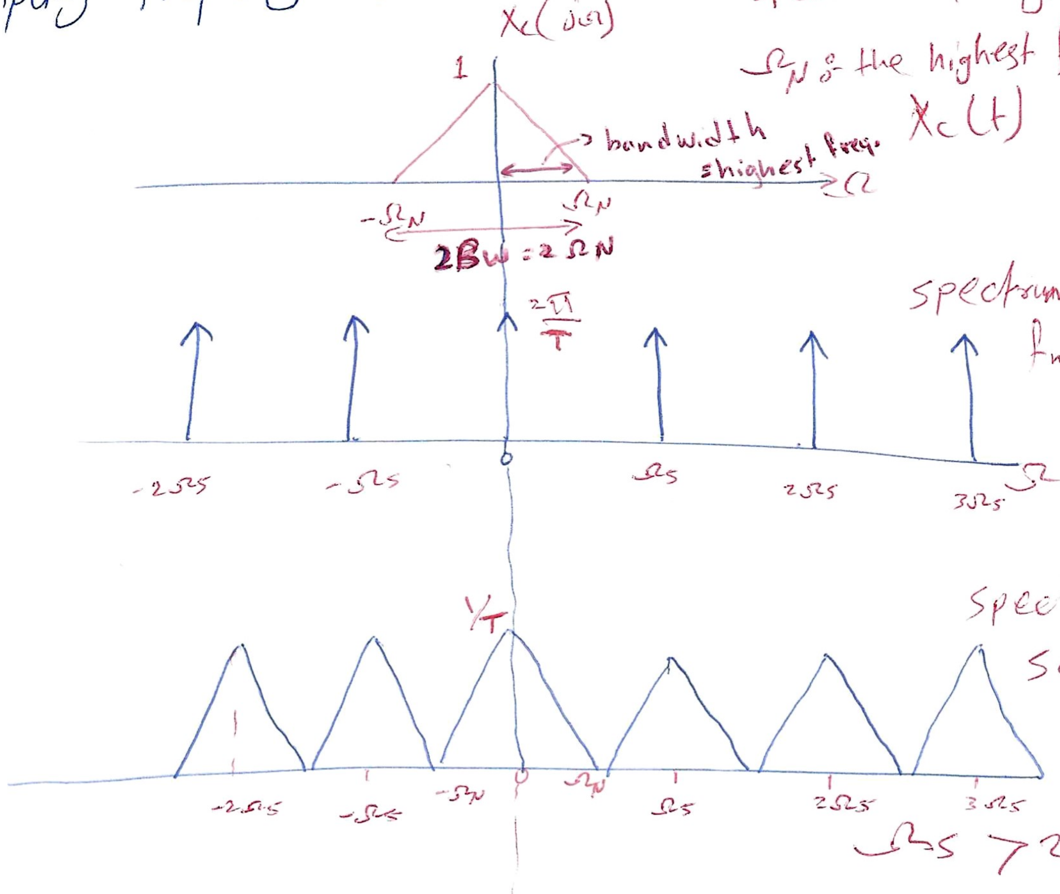
$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

periodically repeated copies of $X_c(j\omega)$

* copies of $X_c(j\Omega)$ are shifted by integer multiples of sampling frequency Ω_s

spectrum of orginal signal

Ω_N : the highest frequency in $X_c(t)$

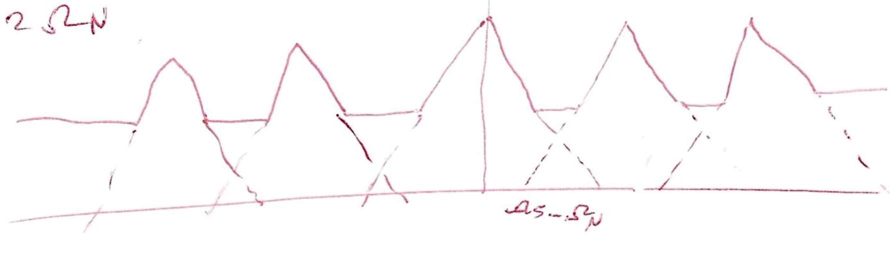


spectrum of sampling function

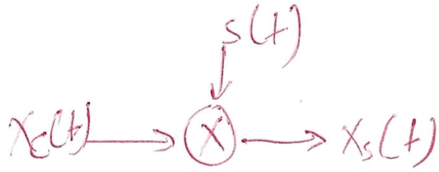
spectrum of sampled signal

$\Omega_s > 2\Omega_N$

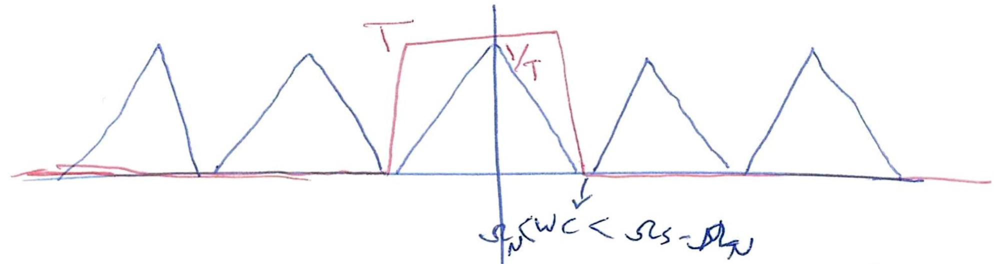
If $\Omega_s < 2\Omega_N$



when $\Omega_s - \Omega_N > \Omega_N \Rightarrow \Omega_s > 2\Omega_N$ no overlapping

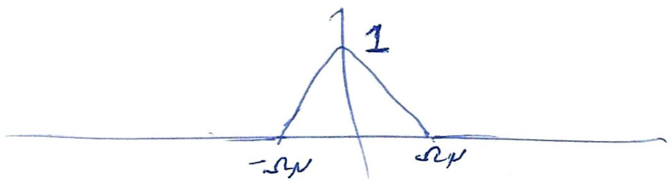


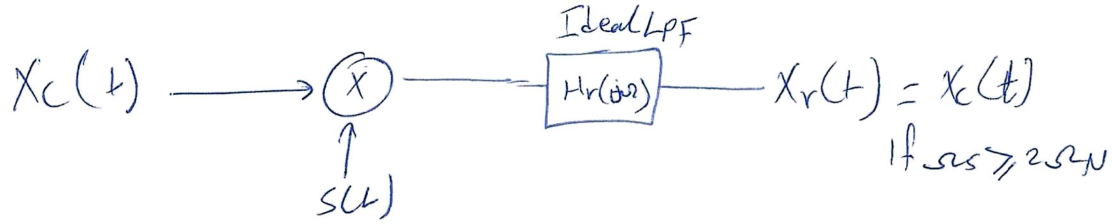
If we multiply Xs(t) with analog low pass filter (in frequency domain) then we can get the original signal Xc(t)



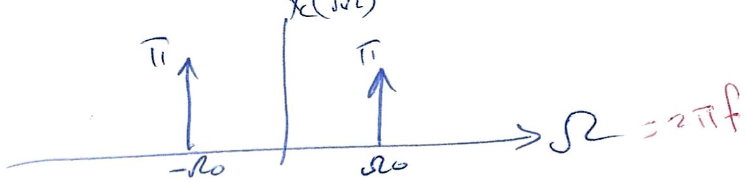
Analog not a digital filter because digital filter is periodic in frequency domain

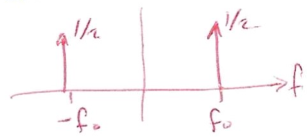
Analog low pass filter is non periodic

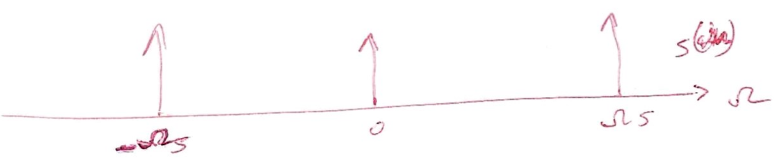




Example: assume $X_c(t) = \cos(\Omega_0 t)$ — periodic in time domain ⟹ not periodic in frequency domain



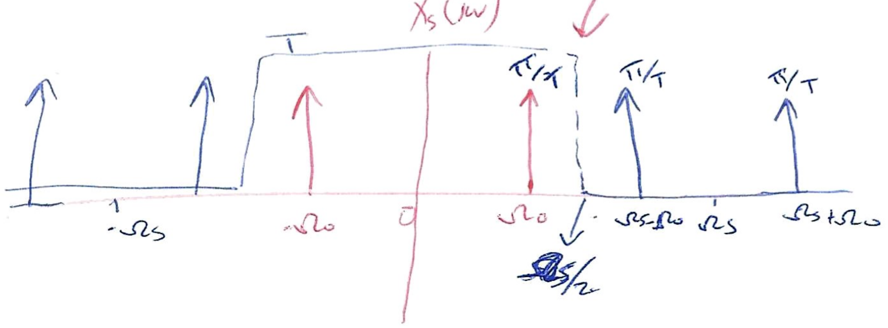




If $\Omega_s \geq 2\Omega_0$

Reconstruction filter $\Omega_c = \Omega_s/2$

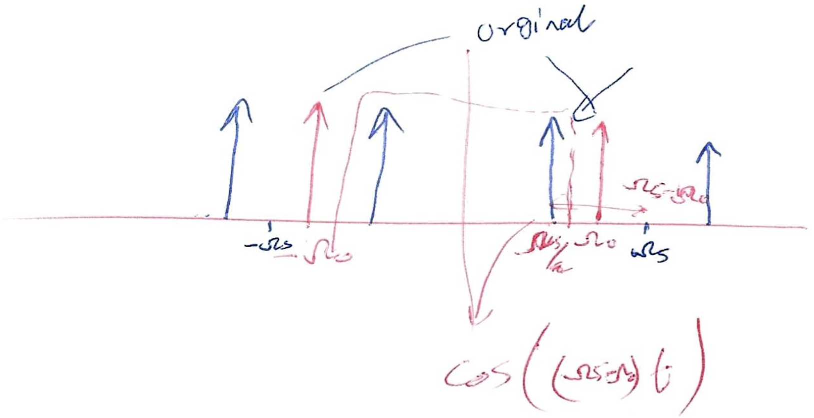
No aliasing



$\cos(\Omega_0 t)$

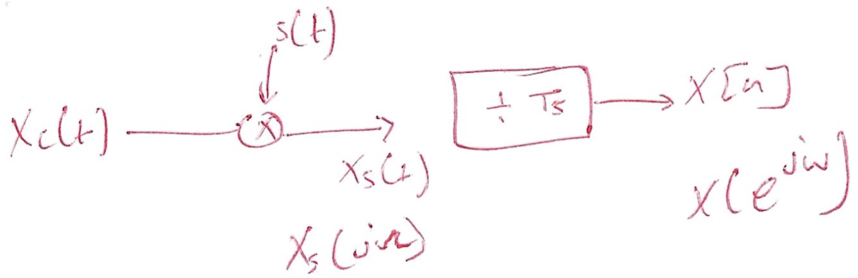
If $\Omega_s < 2\Omega_0$

aliasing



$\cos((\Omega_s-\Omega_0)t)$

- The second stage is the normalization



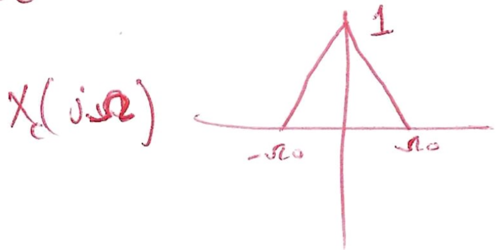
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)$$

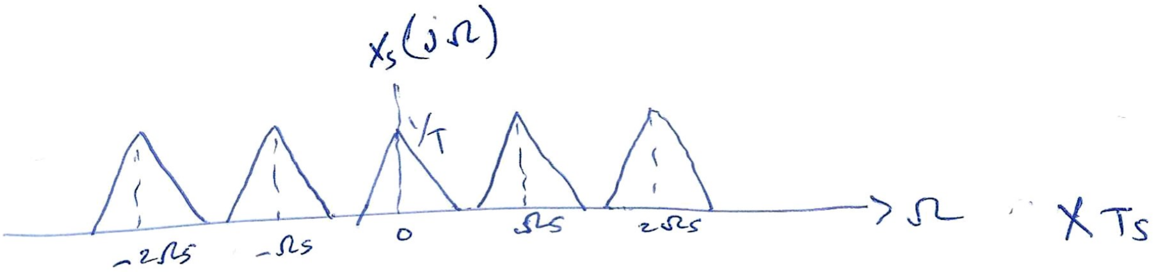
$$x[n] = x_c(nT)$$

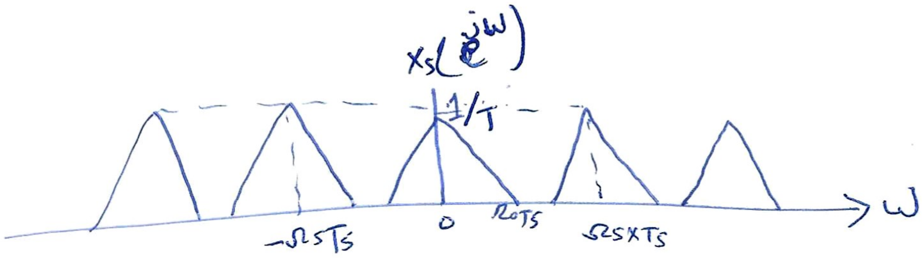
$$\Omega = \frac{\omega}{T}, \quad \omega = \Omega T$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Example:-







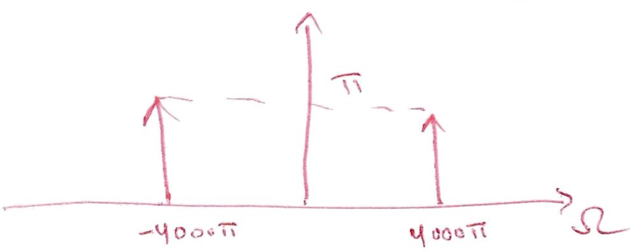
Example:- $x_c(t) = \cos(4000\pi t)$, $T = \frac{1}{6000}$ sec.

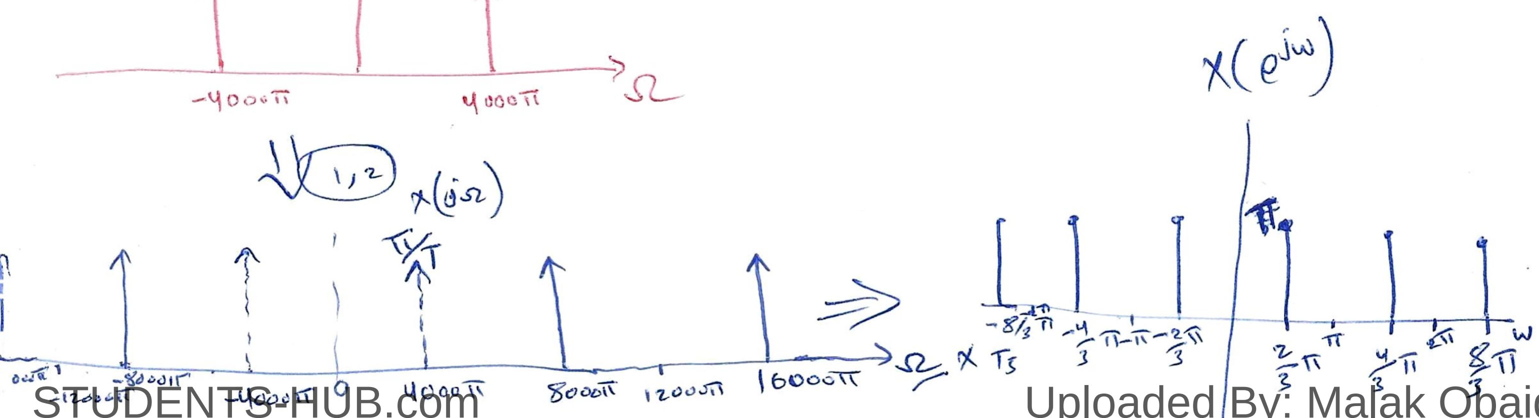
$f_0 = 2000\,KHz$ $\qquad f_s = 6\,KHz$

$\Omega = 4000\pi$ $\qquad \Omega_s = \frac{2\pi}{T_s} = 12000\pi$

$w = T\Omega$

$w_0 = \frac{2}{3}\pi$





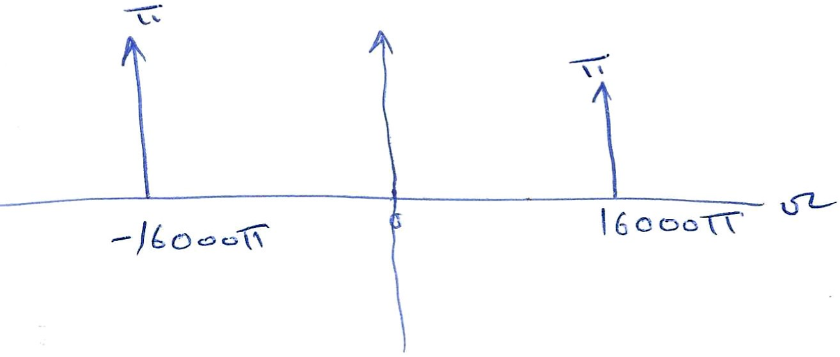
Example $x_c(t) = \cos(16000\pi t)$, sampling period $T = \frac{1}{6000}$

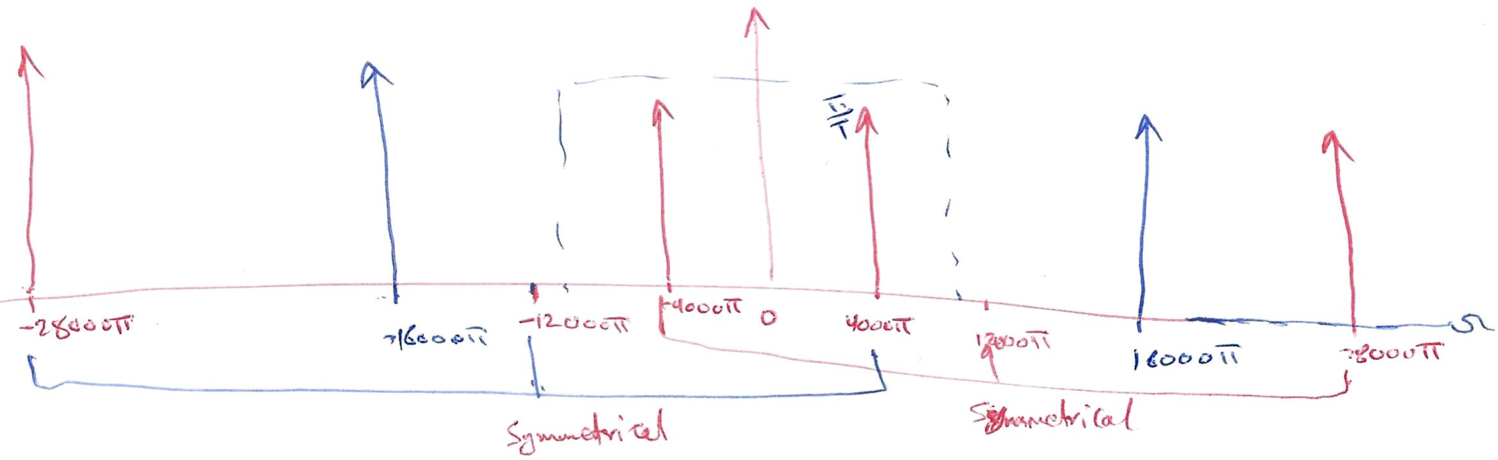
$\Omega = 16000\pi$

$f_0 = 8000\,Hz$ $\qquad$ $f_s = 6000\,Hz$

so f_s should be $2(8000) = 16000\,Hz$

but here f_s only $6000\,Hz$, so we expect to see aliasing





$12000\pi + 16000\pi = 28000\pi$

$12000\pi - 16000\pi = -4000\pi$

$-12000\pi + 16000\pi = 4000\pi$

$-12000\pi - 16000\pi = -28000\pi$

So if we apply a filter we will get $x_r(t) = \cos(4000\pi t)$ and it is not equal to $x_c(t)$

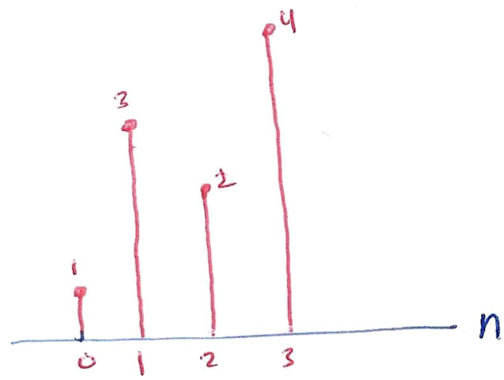
* Reconstruction of Bandlimited Signal from Its Samples

D/C converter

$x[n] \longrightarrow x_r(t)$

① $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)$

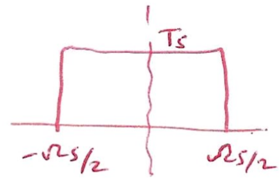
$x[n] = \{\underset{\uparrow}{1}, 3, 2, 4\}$

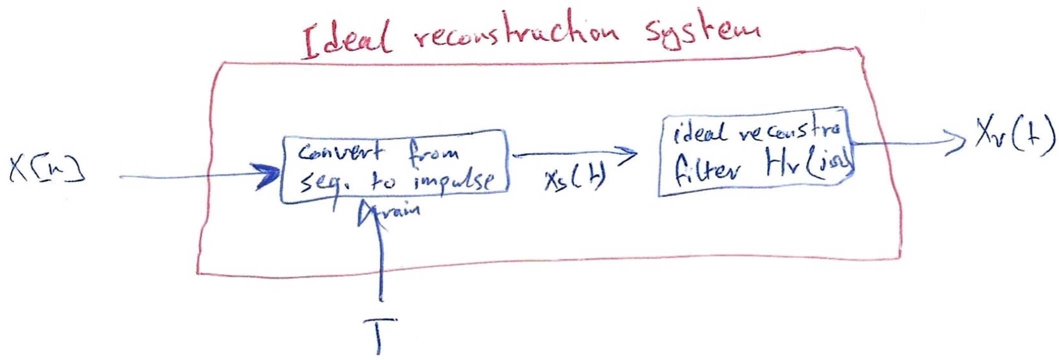


② $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT)$

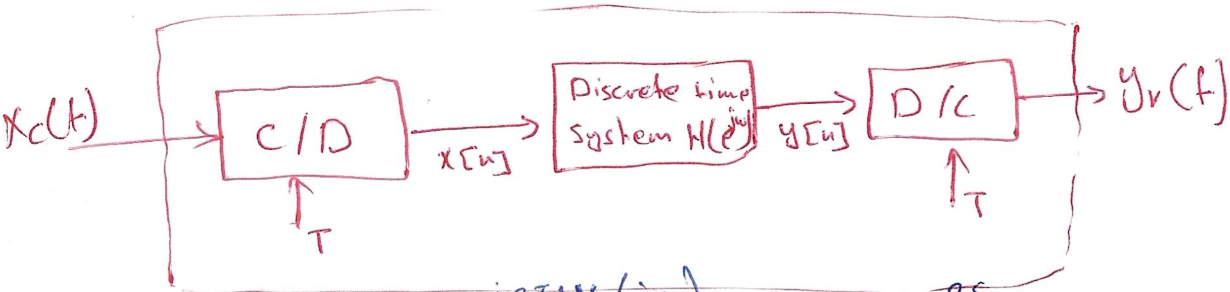
low pass filter

$x_s(t) \longrightarrow \boxed{h_r(t)} \longrightarrow x_r(t)$



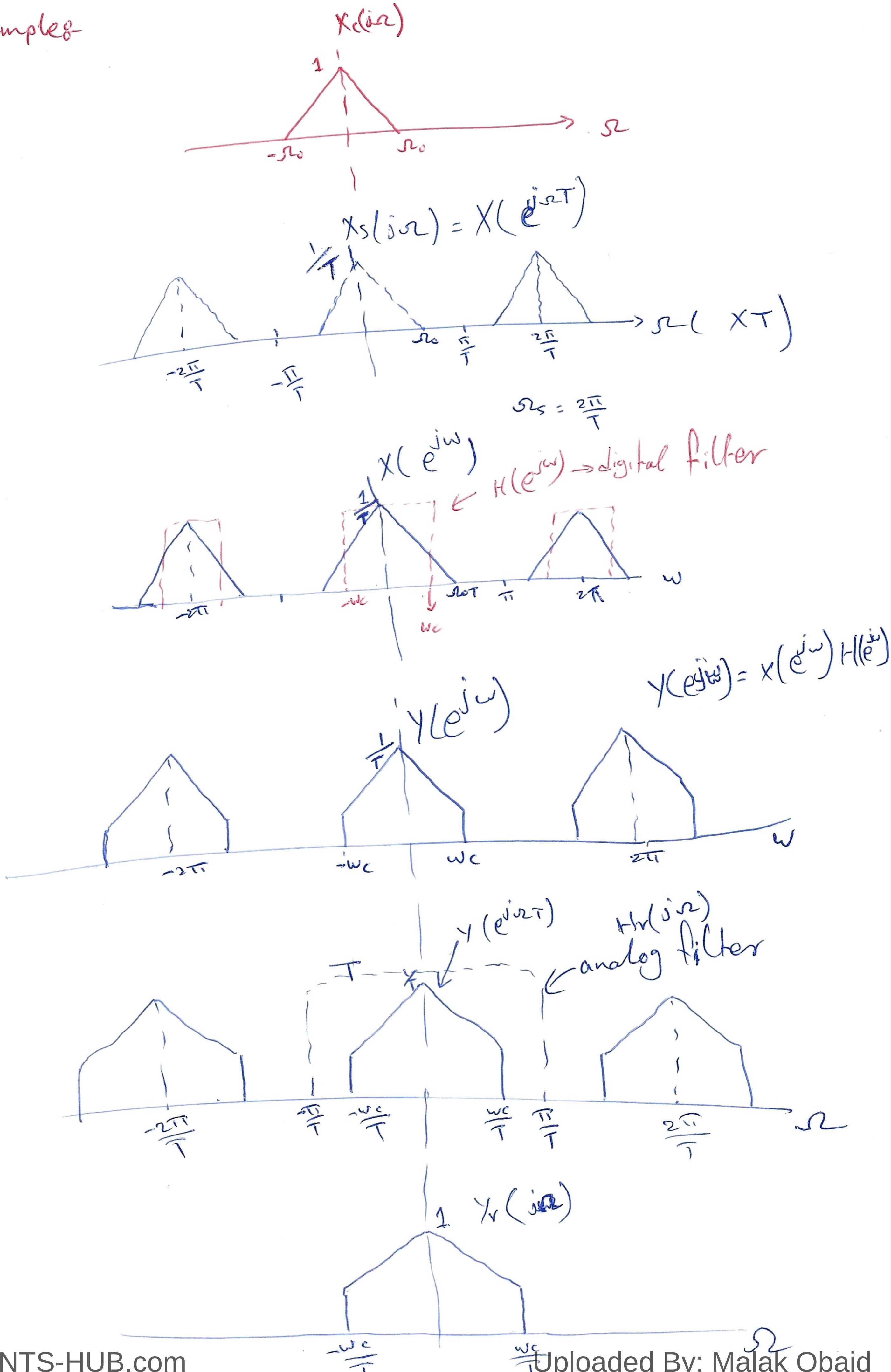


* Discrete-time processing of Continuous-time Signals



$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})\,X_c(j\Omega), & -\frac{\Omega_s}{2} \le \Omega < \frac{\Omega_s}{2} \\ 0, & \text{o.w} \end{cases}$$

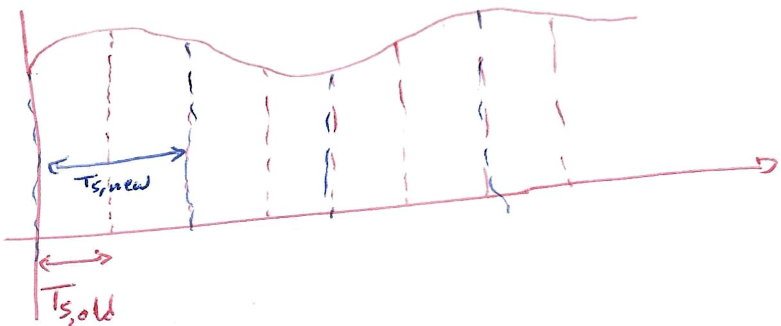
Examples


$X_c(j\Omega)$
1
$-\Omega_0$
Ω_0
Ω
$X_s(j\Omega) = X(e^{j\Omega T})$
1/T
$\Omega(\times T)$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
Ω_0
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
$\Omega_s = \frac{2\pi}{T}$
$X(e^{j\omega})$
$H(e^{j\omega})$ → digital filter
-2π
$-\omega_c$
ω_c
$\Omega_0 T$
π
2π
ω
$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
$Y(e^{j\omega})$
$Y(e^{j\Omega T})$
$H_r(j\Omega)$
analog filter
T
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
$Y_r(j\Omega)$

Decimation: is the two-step process of Lowpass filtering followed by an operation known as downsampling.

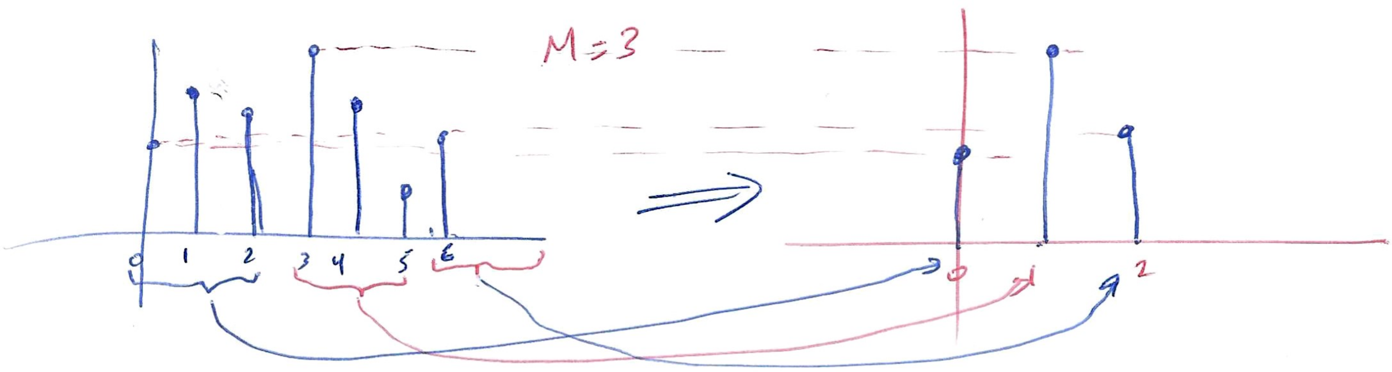
we can downsample the sequence of sampled signal values by a factor of M

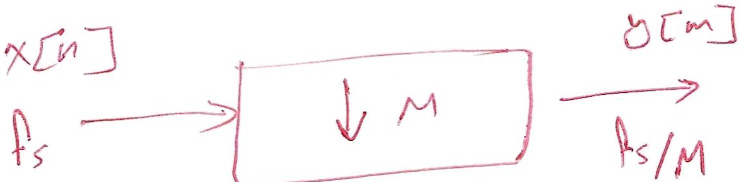
$$f_{s,new} = \frac{f_{s,old}}{M}$$



$f_{s,old} = 8\,kHz$ $\quad M = 2$ reducing the sampling frequency

$f_{s,new} = 4\,kHz$





$M = 3$ $\quad f_s$ = sampling frequency

The output signal $y[m]$ is obtained by taking every M_{th} sample of the input signal. If $M = 3$, we should just take every third sample of $x[n]$ to form the desired signal $y[m]$.

To insure that the nyquist condition is valid, a low pass filter is applied directly before the downsampling, the cutoff frequency of the low pass filter should be ½ of the new sampling rate

Example 8- $x[n] = \{1, 2, 3, 4, 5, -6, -8, 2, -3, 2\}$

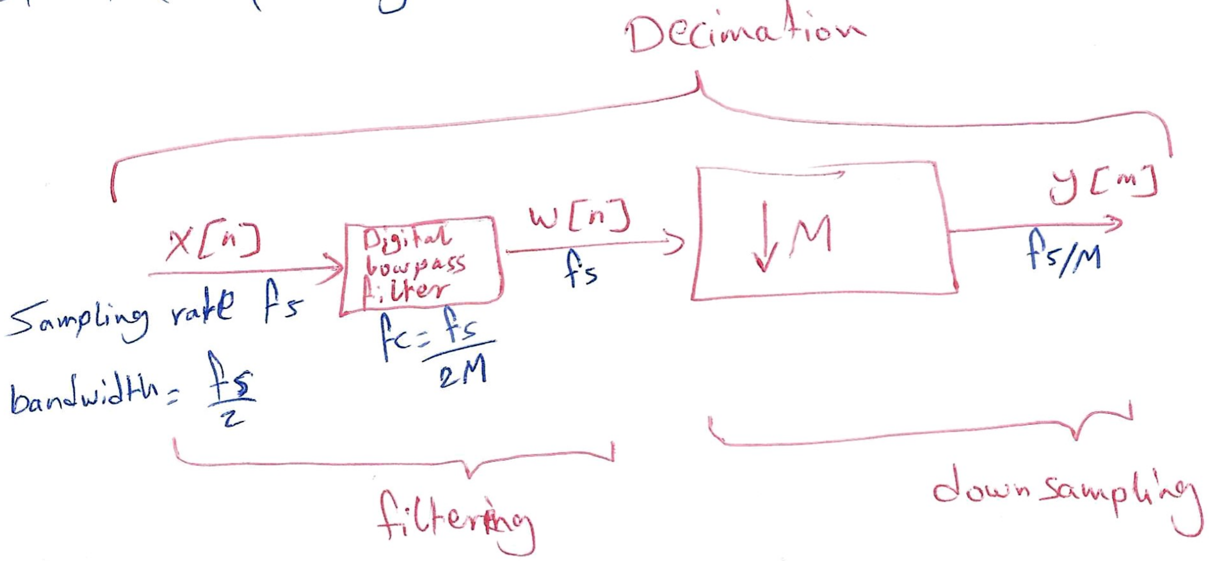
① Down sample by 2

$y[m] = \{1, 3, 5, -8, -3\}$ new bandwidth $= \frac{f_s}{4}$

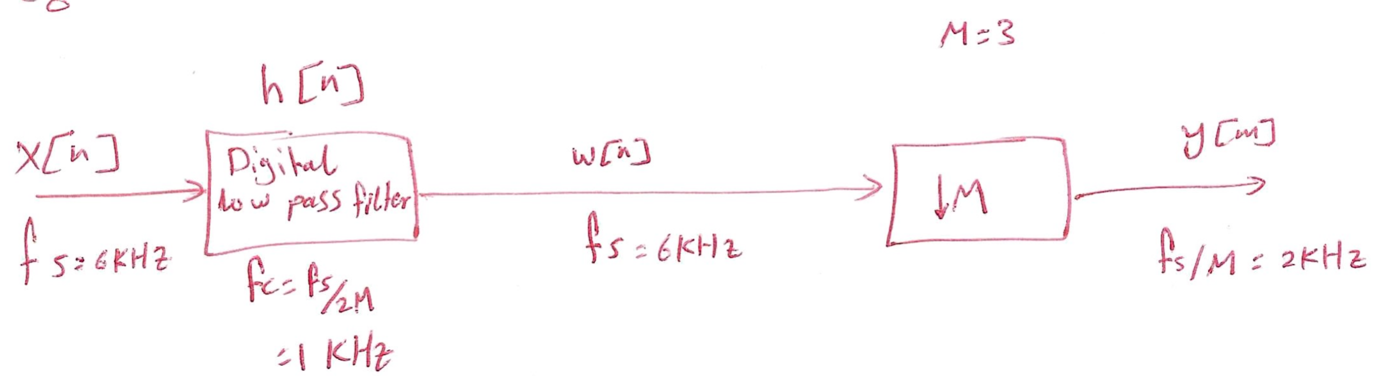
② Down sample by 3

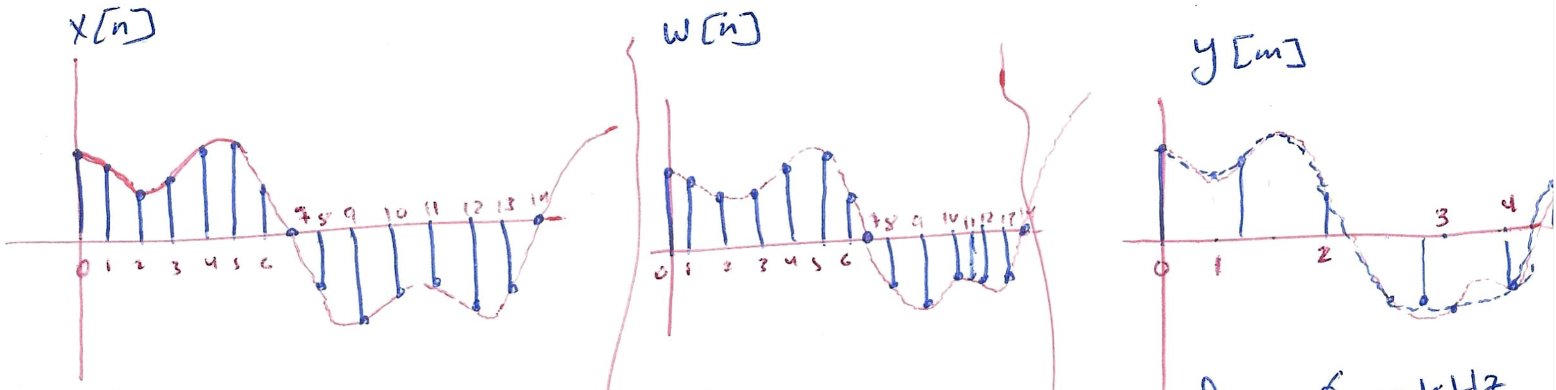
$y[m] = \{1, 4, -8, 2\}$ new bandwidth $= \frac{f_s}{6}$

The output signal $y[m]$ is obtained by taking every Mth sample of the input signal



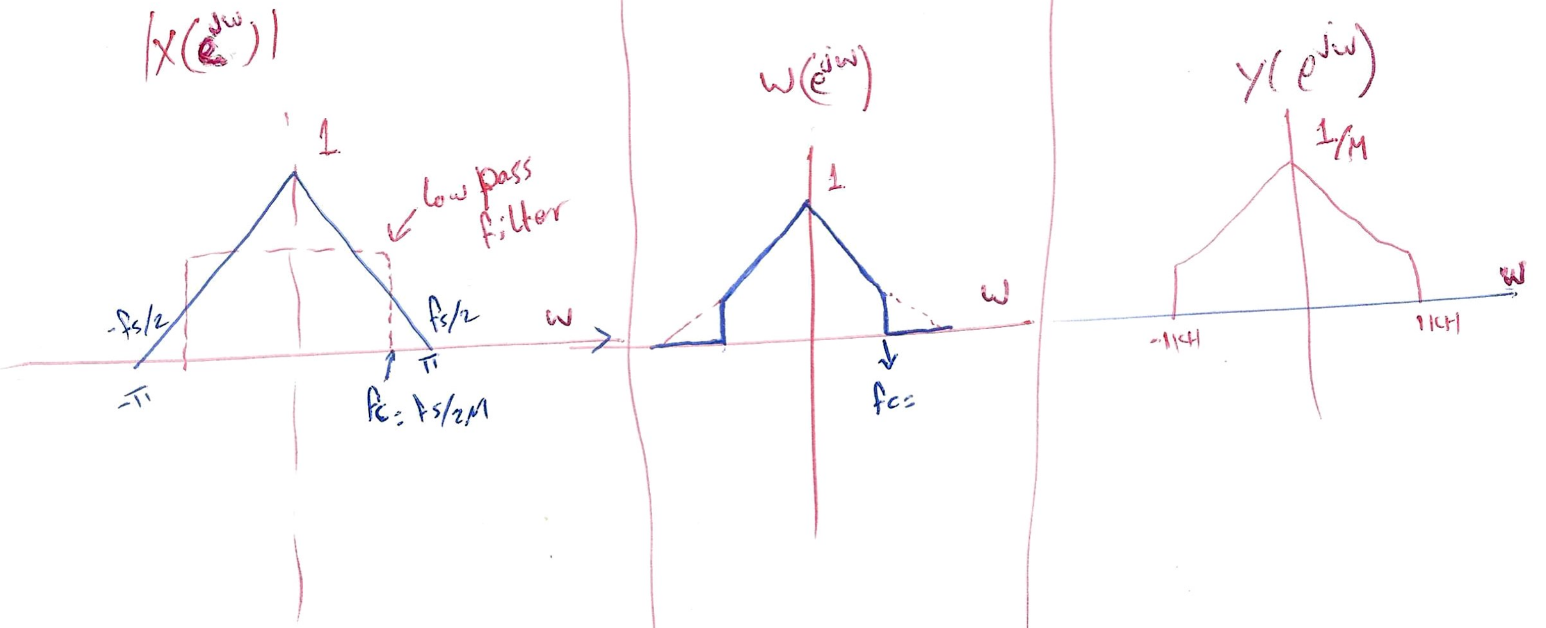
Example:-





$f_s = 6$ KHz $f_s = 6$ KHz $f_s = \frac{6}{3} = 2$ KHz

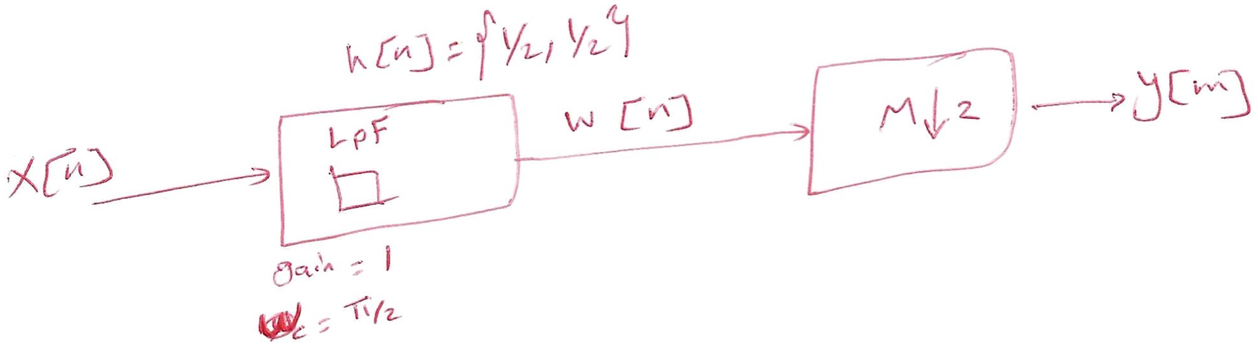
$B = 3$ KHz $B = f_s/2M = 1$ KH $B = 1$ KHz



FIR filter is used in this case, because we can do our computations at the rate of f_s/M. Thus, using an FIR filter in the decimation process will lead to a significantly lower computation rate.

Example: Decimation of $x[n] = \{2, 6, 4, 2, 6, 8, 4, 2, 4, 4\}$

$M = 2$, find $w[n]$ and $y[m]$?



$w[n] = x[n] * h[n]$

$= \{2, 6, 4, 2, 6, 8, 4, 2, 4, 4\} * \{1/2, 1/2\}$

$= \{1, 4, 5, 3, 4, 7, 6, 3, 3, 4, 2\}$

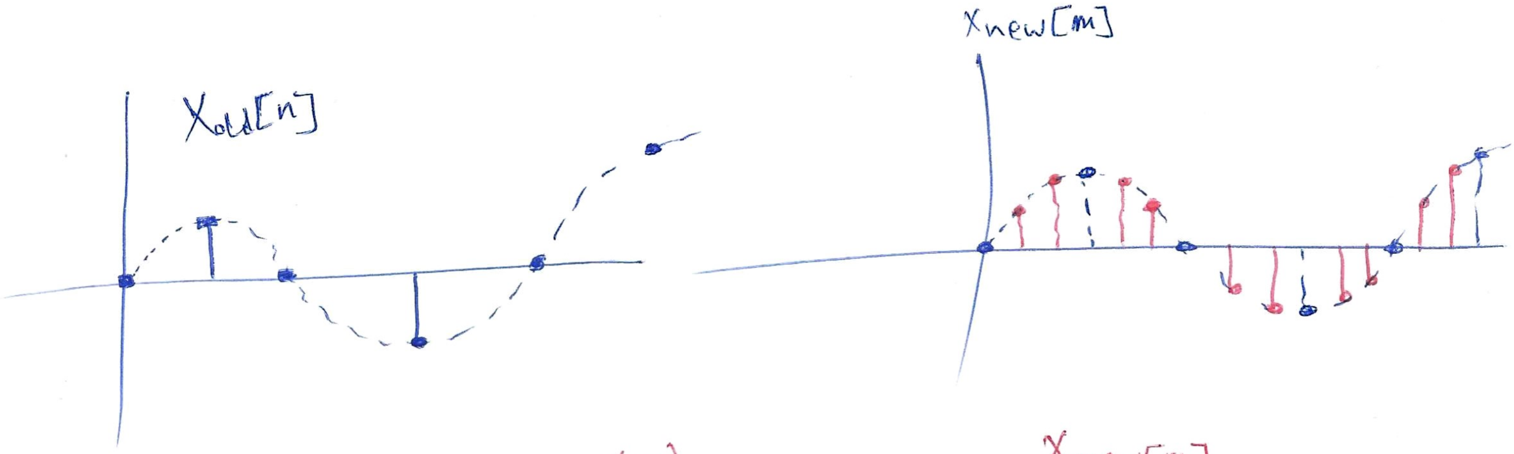
for $M = 2$

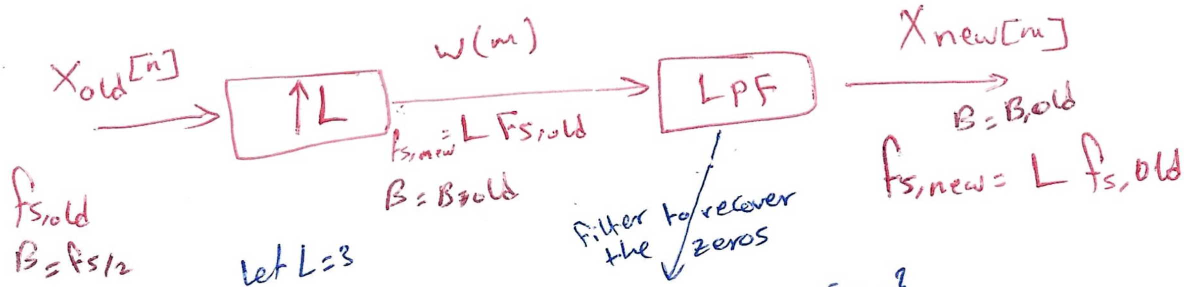
$y[m] = \{1, 5, 4, 6, 3, 2\}$

	2	6	4	2	6	8	4	2	4	4
1/2	1	3	2	1	3	4	2	1	2	2
1/2	1	3	2	1	3	4	2	1	2	2

Interpolation

Sample rate increase by interpolation. New sample values need to be calculated





let L=3

$x[n] = \{1, 0.9, -0.5\}$

$w[m] = \{1, 0, 0, 0.9, 0, 0, -0.5, 0, 0\}$

↑ insert L-1 zeros

Example:-

$x[n] = \{1, 2, 4, 3, -5, 6, -7, 2, 4, 3\}$

① for L = 2

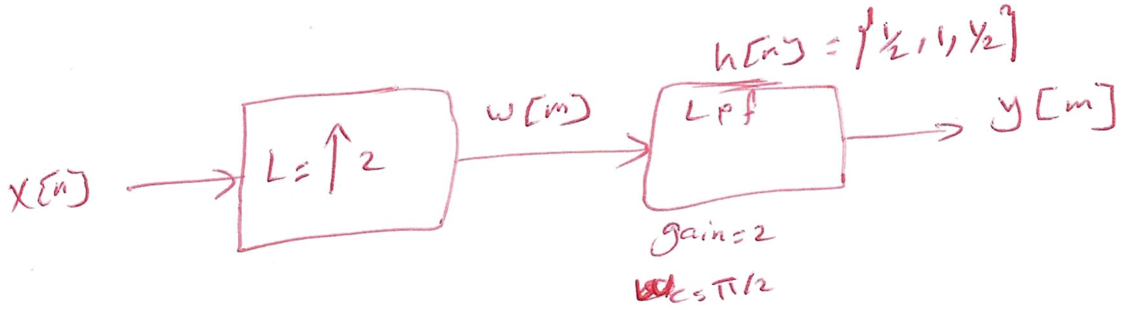
$w[m] = \{1, 0, 2, 0, 4, 0, 3, 0, -5, 0, 6, 0, -7, 0, 2, 0, 4, 0, 3, 0\}$

② for L = 3

$w[m] = \{1, 0, 0, 2, 0, 0, 4, 0, 0, 3, 0, 0, -5, 0, 0, 6, 0, 0, -7, 0, 0, 2, 0, 0, 4, 0, 0, 3, 0, 0\}$

The quality of the signal will not improved after the interpolation even the sampling rate increased

Example: interpolation of $x[n] = \{1, 3, 5, 3, 7\}$



find $w[m]$, $y[m]$?

$x[n] = \{1, 3, 5, 3, 7\}$

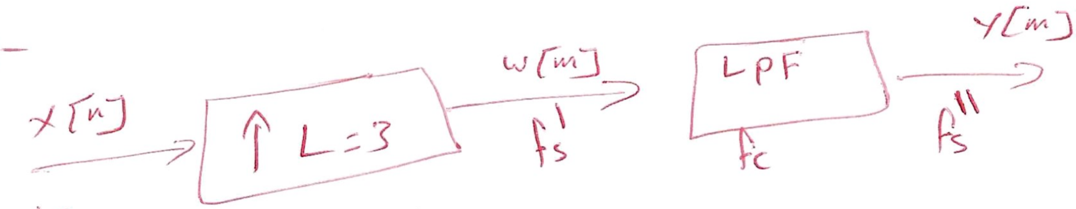
$w[m] = \{1, 0, 3, 0, 5, 0, 3, 0, 7, 0\}$ (insert zeros)

$y[m] = w[m] * h[n]$

$= \{1, 0, 3, 0, 5, 0, 3, 0, 7, 0\} * \{1/2, 1, 1/2\}$

$= \{1, 2, 3, 4, 5, 4, 3, 5, 7, 3.5\}$

Example:-



$f_s = 8\,KHZ$

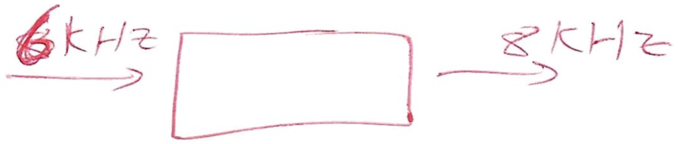
$Bw = 2\,KHZ$

what are the values of f_c, and f_s' ?

$f_s'' = f_s' = L f_s = 3(8\,KHZ) = 24\,KHZ$

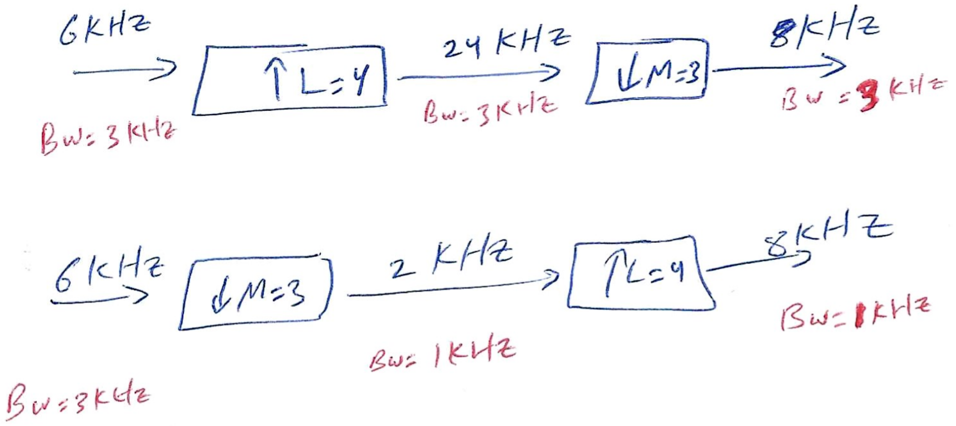
$f_c = Bw = 2\,KHZ$ or $\frac{f_s}{2}$ if Bw is not given

* Sampling Rate Conversion by Non-Integer Factor :-



$\frac{8}{6} = \frac{4}{3}$

in this case L is non-integer so we need to do the following steps

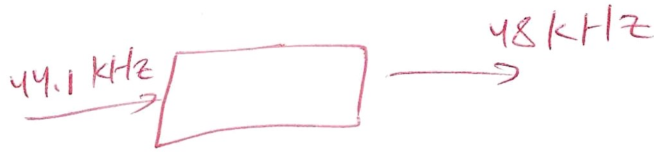


① This is better because we will not change the bandwidth of the signal

②

The overall process is interpolation since the ratio is $\frac{4}{3}$ more than 1

Example :-

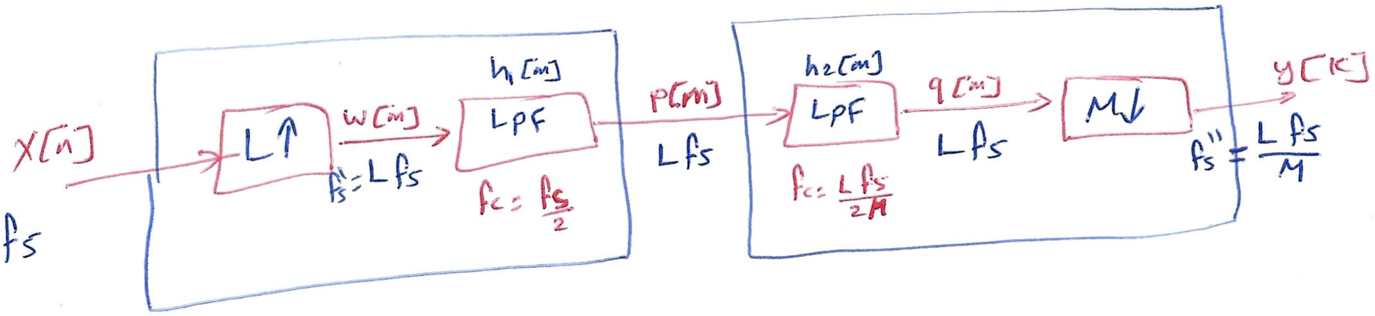


CD — Compact disk

DAT — Digital Audio Tape

$\frac{48000}{44100} = \frac{160}{147}$ ← L (160), ← M (147)

The overall process



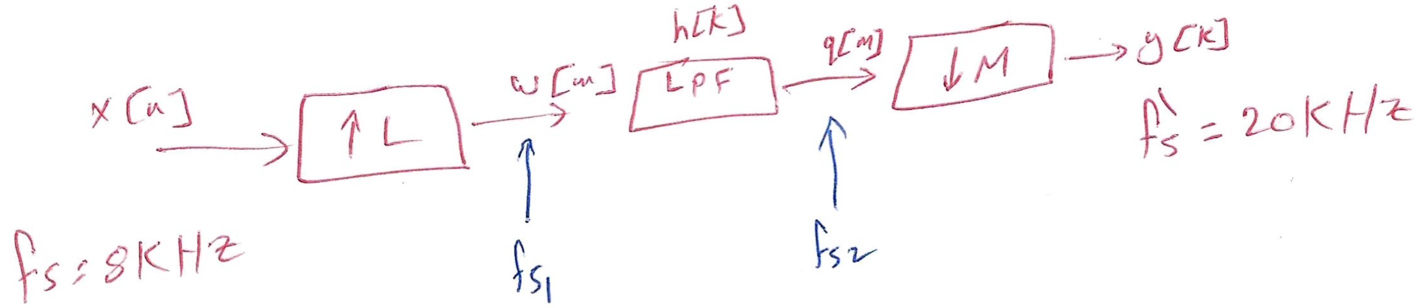
$h_1[m]$ and $h_2[m]$ can be combined into a single filter since they are in cascade and have a comon sampling frequency.

$f_c = \min\left(\frac{f_s}{2}, \frac{Lf_s}{2M}\right)$ or in rad $w_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$

if $M > L$ the resulting operation is a decimation

$M < L$ the resulting operation is an interpolation

Example:- Figure below shows sampling rate conversion by non-integer factor. Calculate the values of L and M ?



$\frac{20}{8} = \frac{5}{2} = \frac{L}{M}$ $L = 5$

$M = 2$

$f_{s1} = f_{s2} = 8 \times 5 = 40KHz$

The process in an interpolation process

∴ Bw of the signal will not change = 4KHz

$f_c = 4KHz$

Decimation in frequency domain

$$X_s(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

$$X_s(e^{jw}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{w}{T} - \frac{2\pi n}{T}\right)$$

$$X_d(n) = X(nM) = X_c(nT') \quad \text{where } T' = MT$$

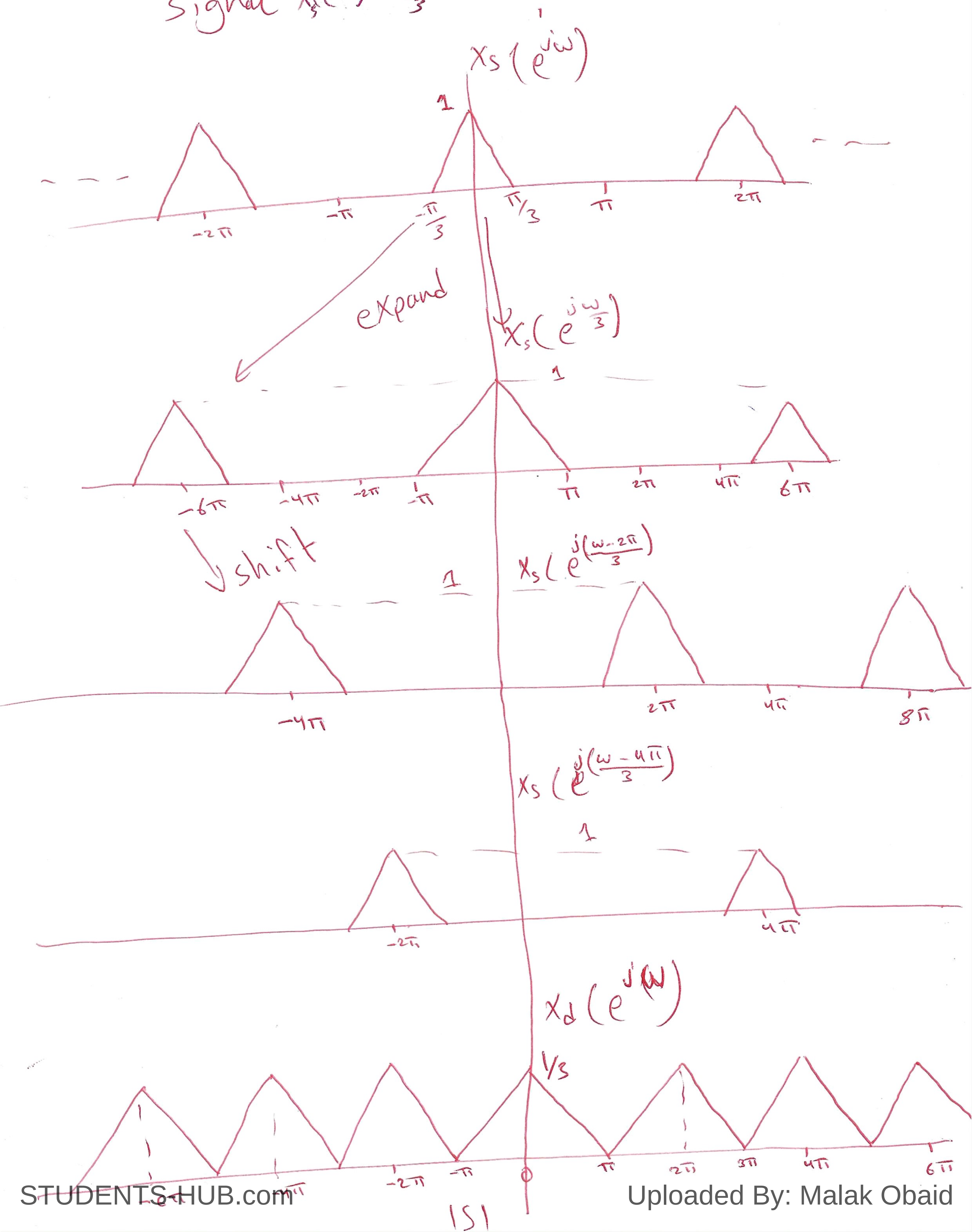
$$X_d(e^{jw}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{w}{MT} - \frac{2\pi r}{MT}\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X_s\left(e^{j\left(\frac{w}{M} - \frac{2\pi i}{M}\right)}\right)$$

So, to skitch $X_d(e^{jw})$ directly from $X_s(e^{jw})$, follow 3 steps

1. Expand $X_s(e^{jw})$ by a factor M to obtain $X_s(e^{j\frac{w}{M}})$. Note that the highest frequency of $X_s(e^{jw})$, w_H is repositioned to frequency $w = w_H \cdot M$
2. Create and put M Copies of $X_s(e^{j\frac{w}{M}})$ at freq $w = 2\pi i$ for $i = 0, 1, 2, 3, \ldots M-1$
3. add the M streched and shifted replicas and then divide by M to obtain the spectrum $X_d(e^{jw})$ of the down sampled sequence $x_d(n) = x(nM)$

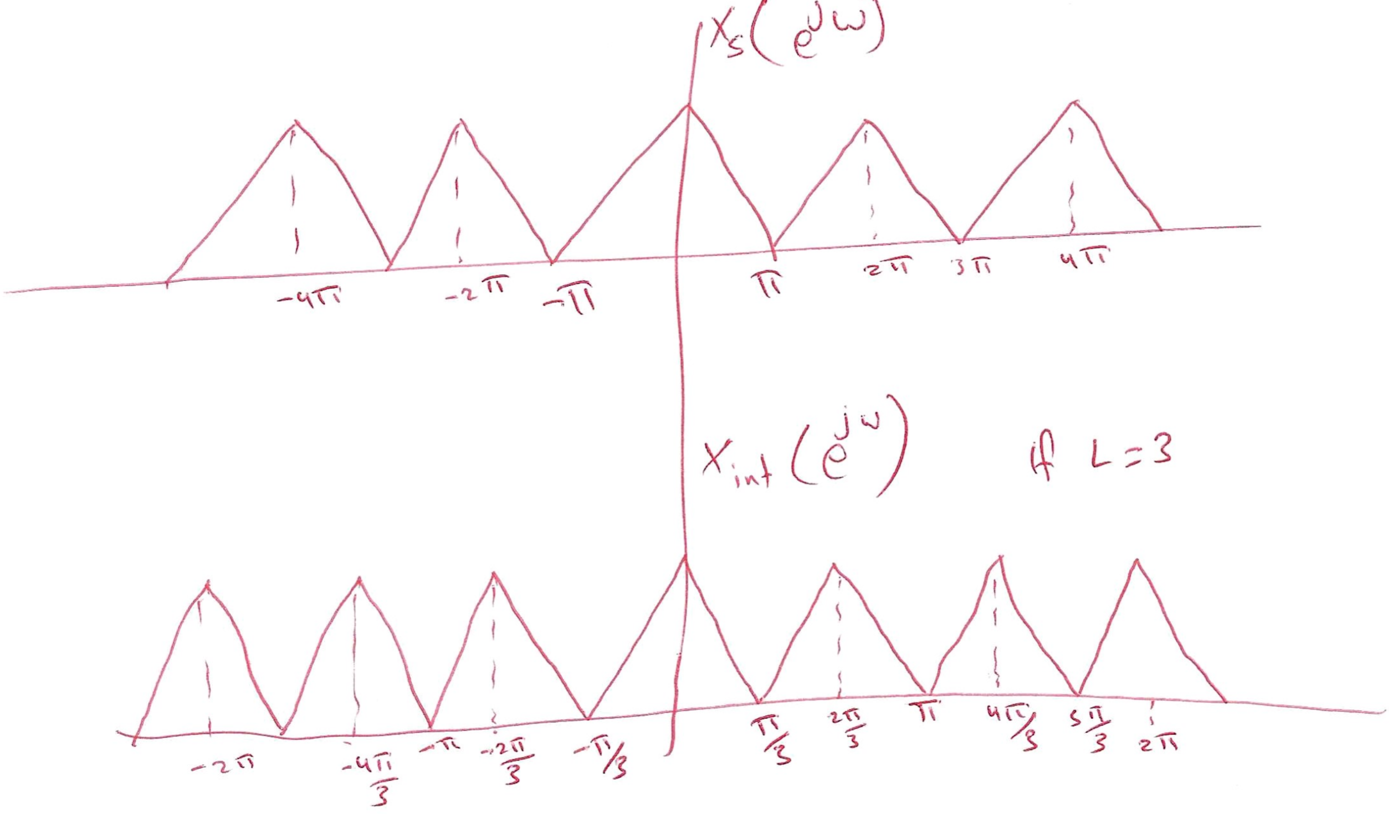
Example 8 - Set $M = 3$ and Bandwidth of Signal $X_s(e^{j\omega})$ is $\frac{\pi}{3}$



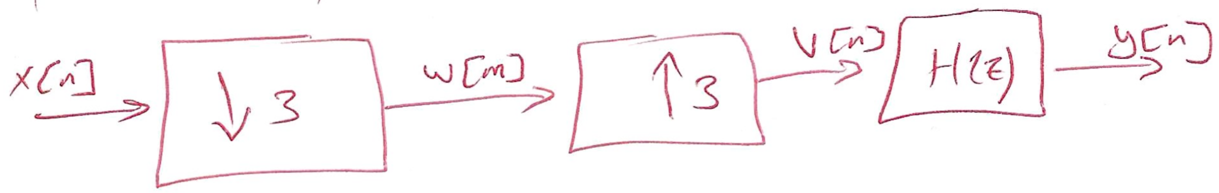
Interpolation in frequency domain:

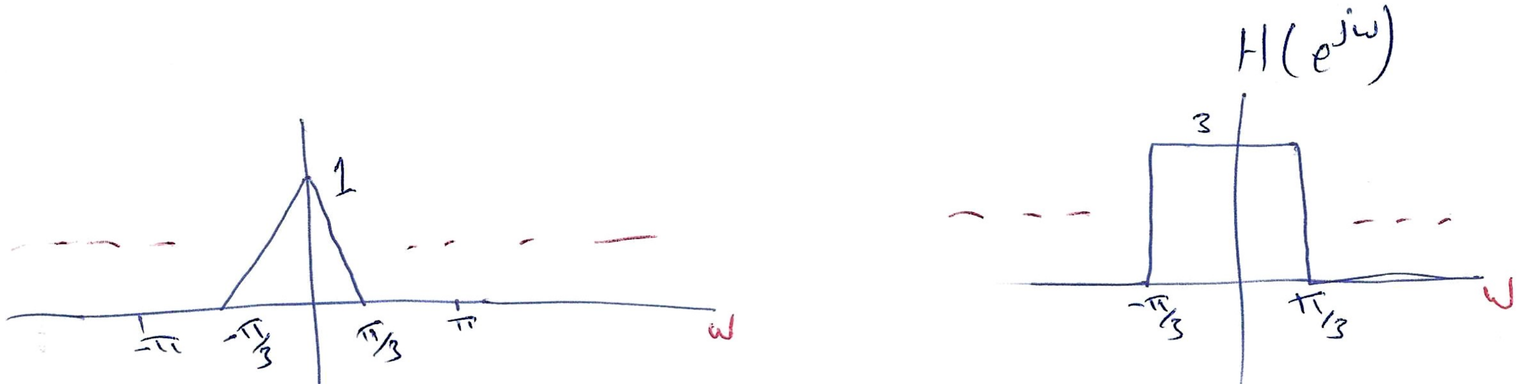
$x[n] \rightarrow \boxed{\uparrow L} \rightarrow X_{int}(m) = x(n/L)$

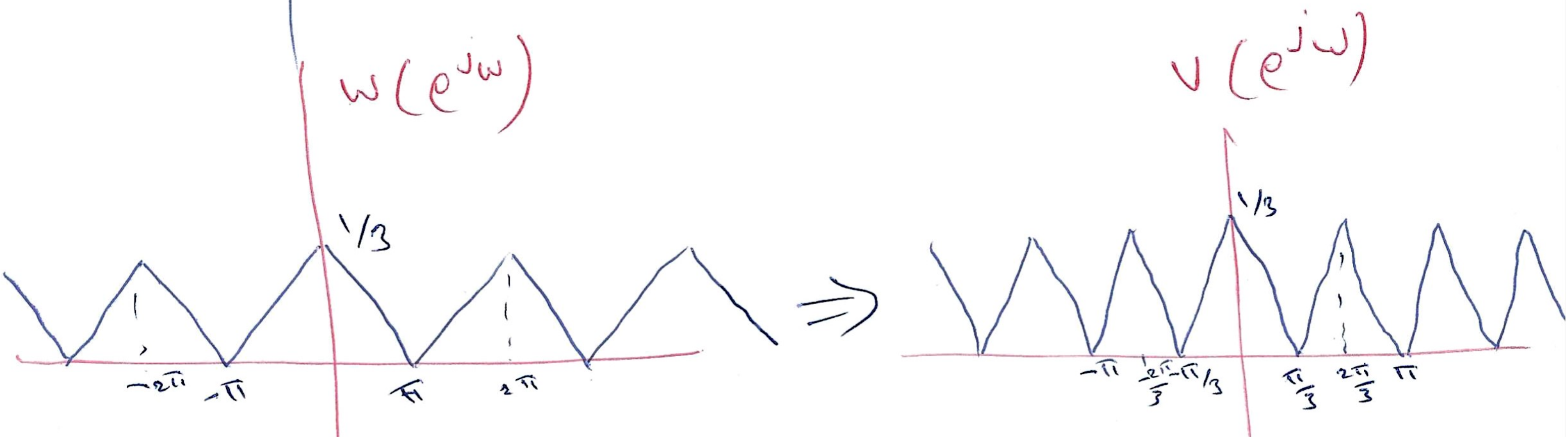
$$X_{int}(e^{j\omega}) = X_s(e^{j\omega L})$$

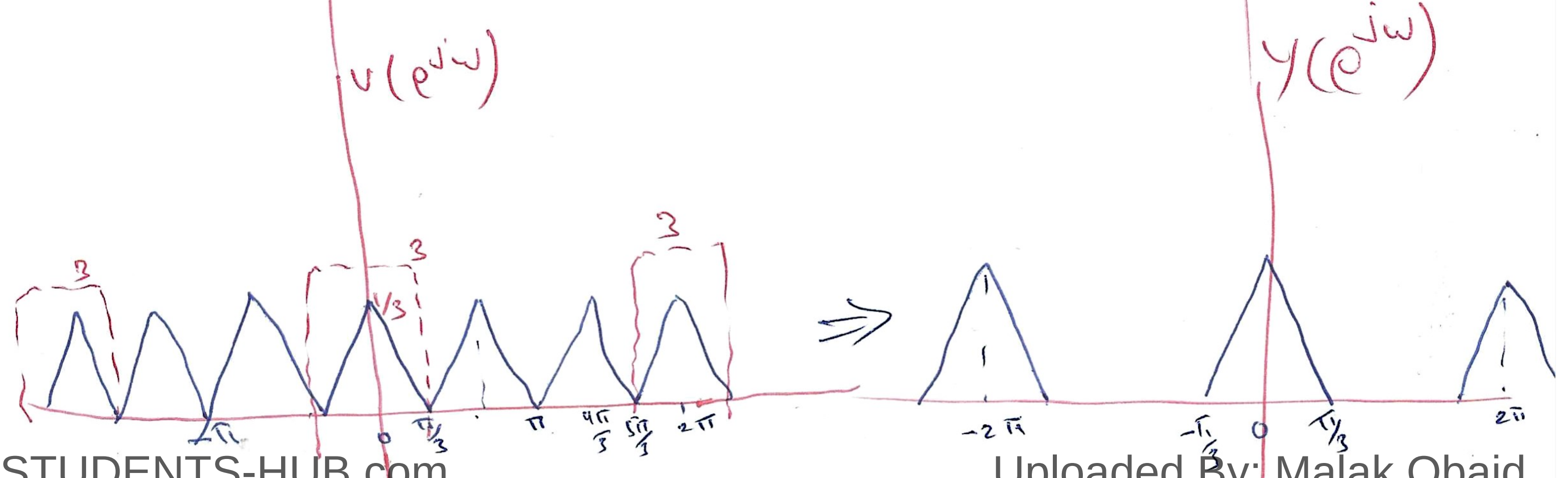


Example:- An input signal $x[n]$ with spectrum $X(e^{j\omega})$ shown below. The input signal is applied to the system shown below. sketch $|X(e^{j\omega})|$, $|W(e^{j\omega})|$, $|V(e^{j\omega})|$, and $|Y(e^{j\omega})|$ against ω.

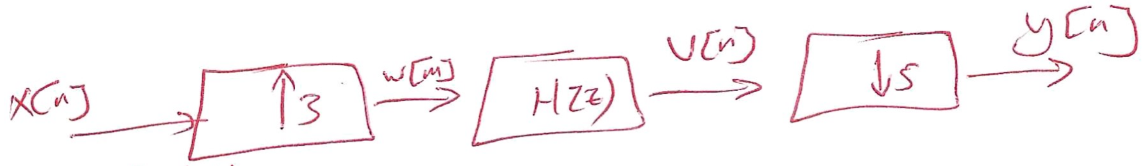


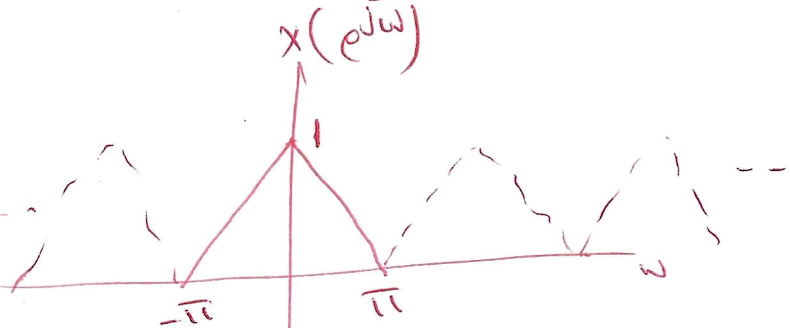


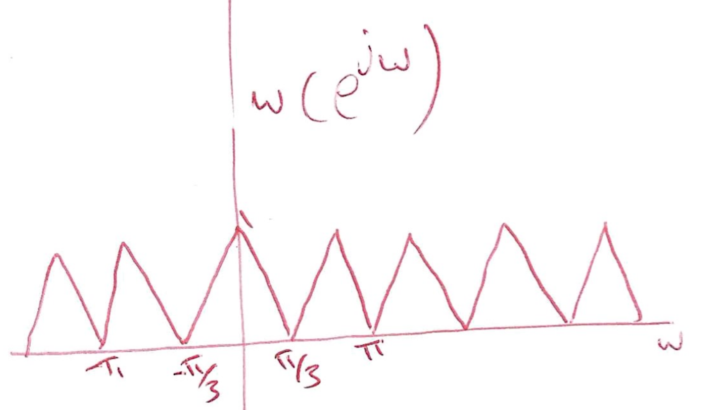


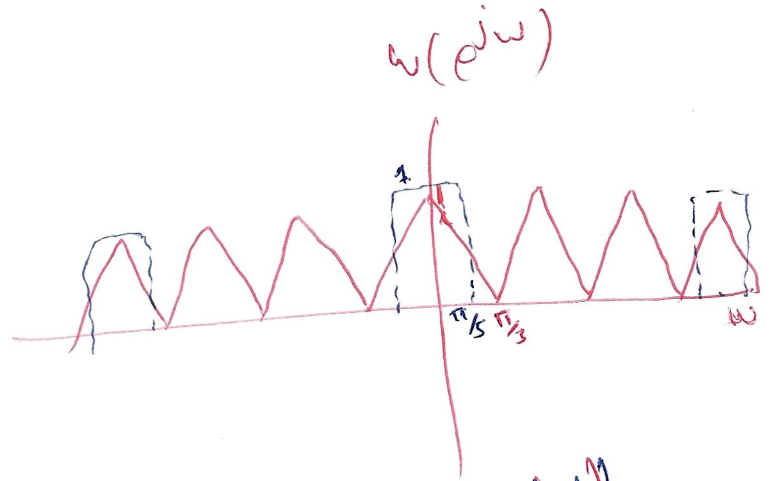


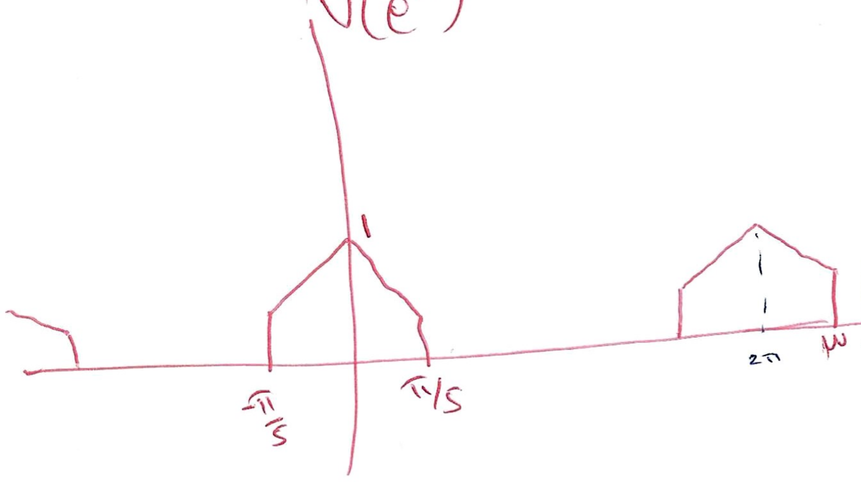
Example :- A signal $x[n]$ has a spectrum $X(e^{jw})$ as shown below. The signal is applied to the system shown below. The ideal low pass filter $H(z)$ has a gain factor of 1 in the passband and a cut-off frequency $w_c = \pi/5$. Sketch $X(e^{jw})$, $W(e^{jw})$, $V(e^{jw})$, $Y(e^{jw})$ against w.

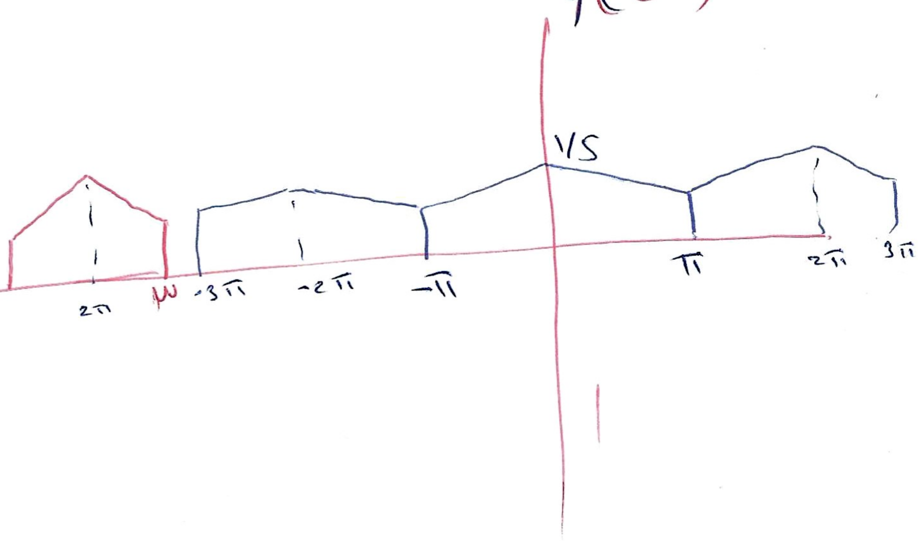




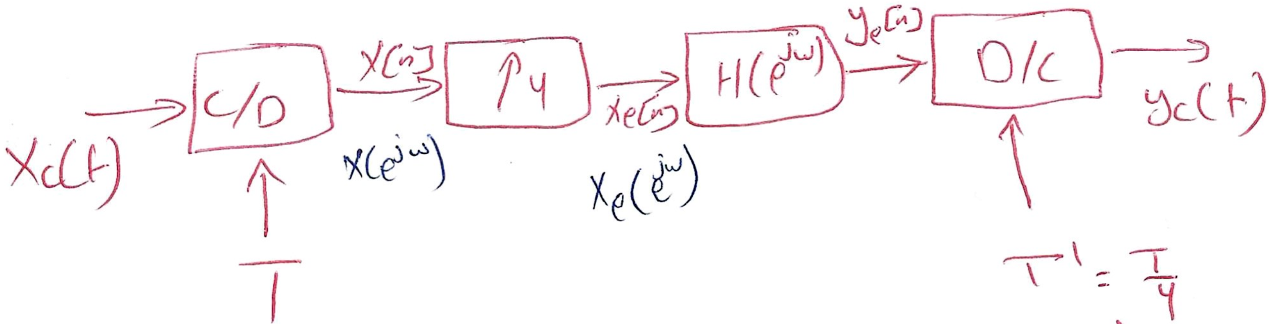


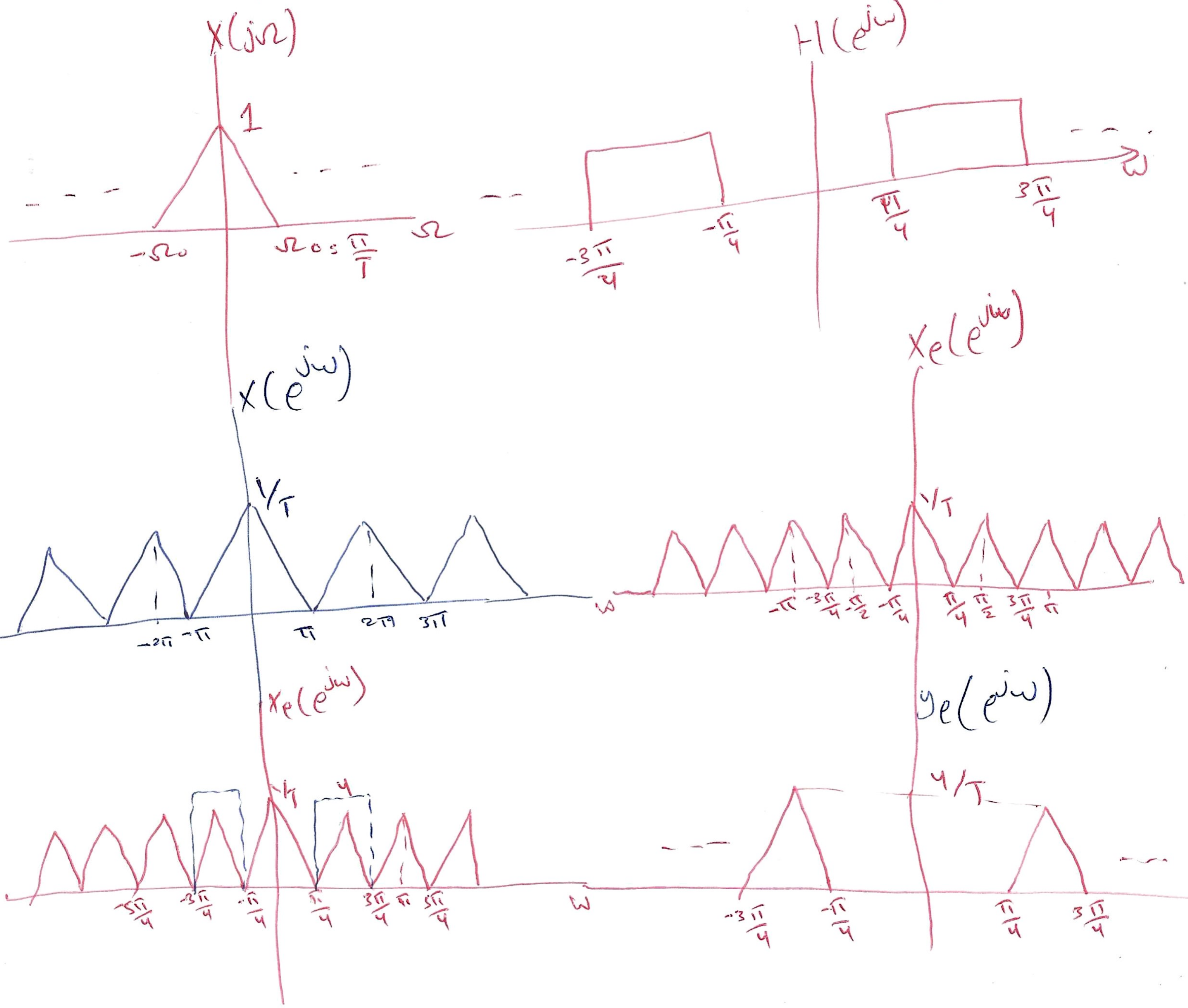






Example: Consider the system shown below. The input to this system is a bandlimited signal whose F.T. is shown below with $\Omega_0 = \frac{\pi}{T}$. The discrete-time LTI system has the frequency response shown below

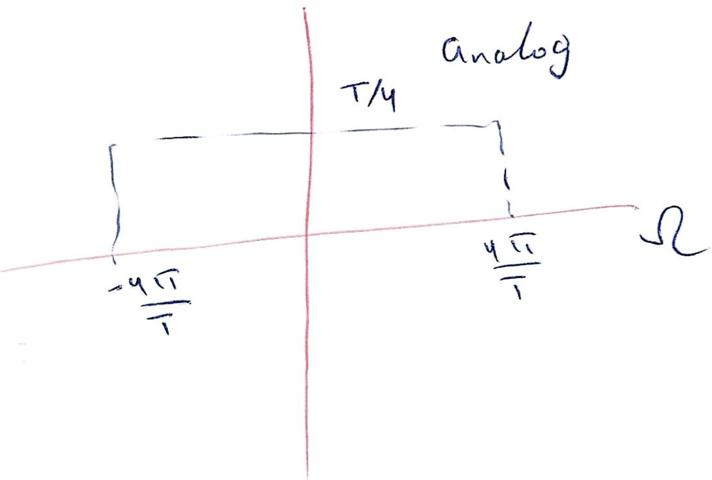


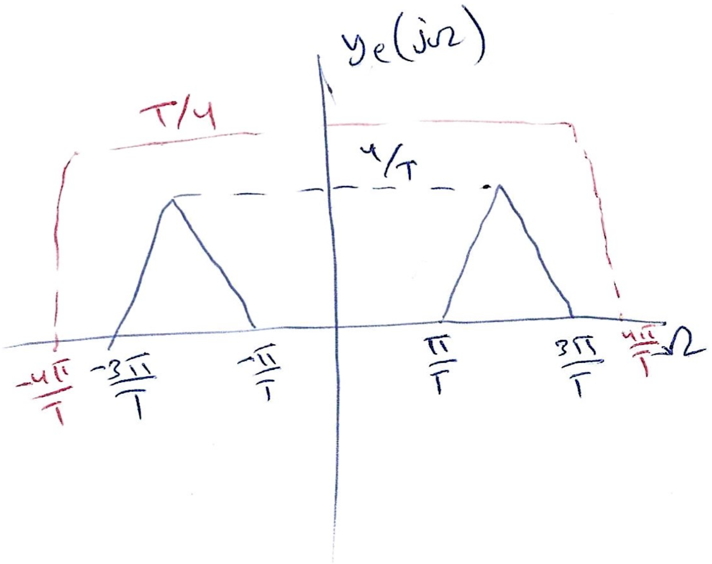


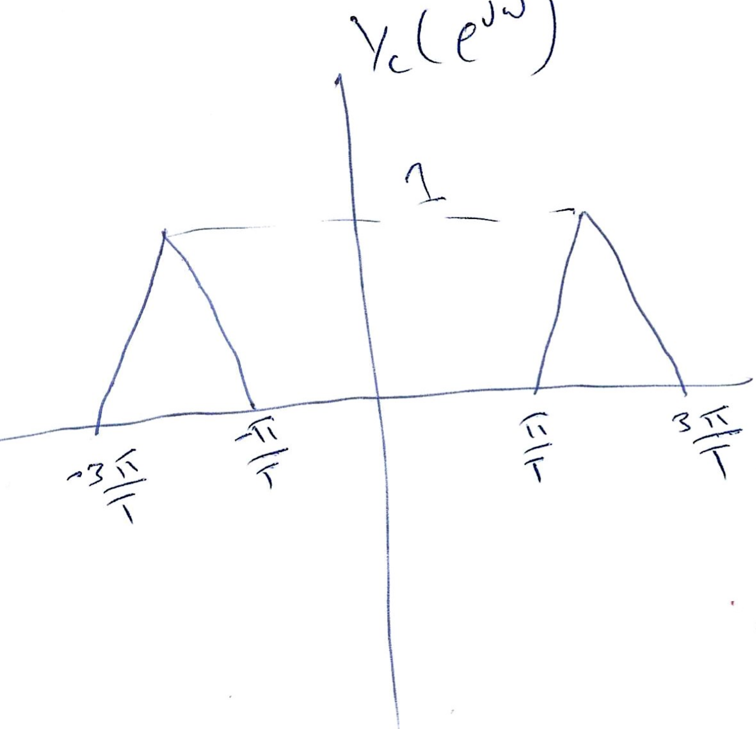
$$T' = \frac{T}{4}$$

$$\Omega_{s,new} = \frac{2\pi}{T'} = \frac{2\pi}{\frac{T}{4}} = \frac{8\pi}{T}$$

$$\Omega_c = \frac{\Omega_s}{2} = \frac{4\pi}{T}$$





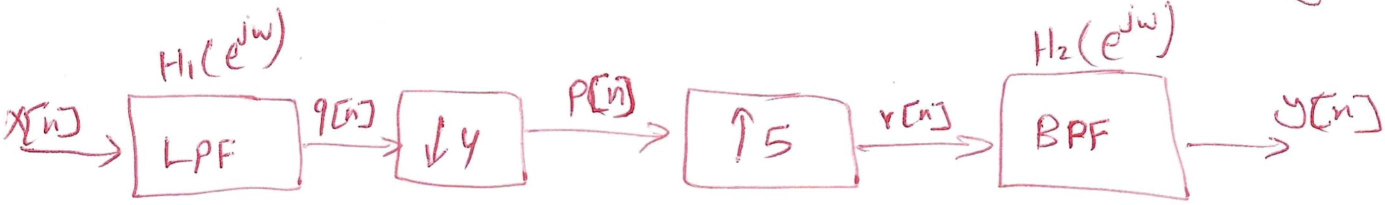


$$Y_c(j\Omega) = X_c\left(j\left(\Omega - \frac{2\pi}{T}\right)\right) + X_c\left(j\left(\Omega + \frac{2\pi}{T}\right)\right)$$

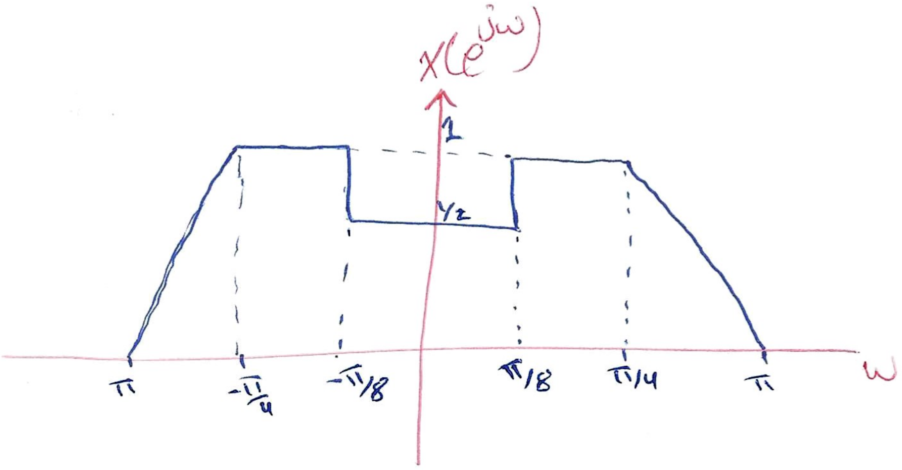
Therefor

$$y_c(t) = 2\,X_c(t)\cos\left(\frac{2\pi}{T}t\right)$$

Example:- Consider the multi-rate system shown in the figure below



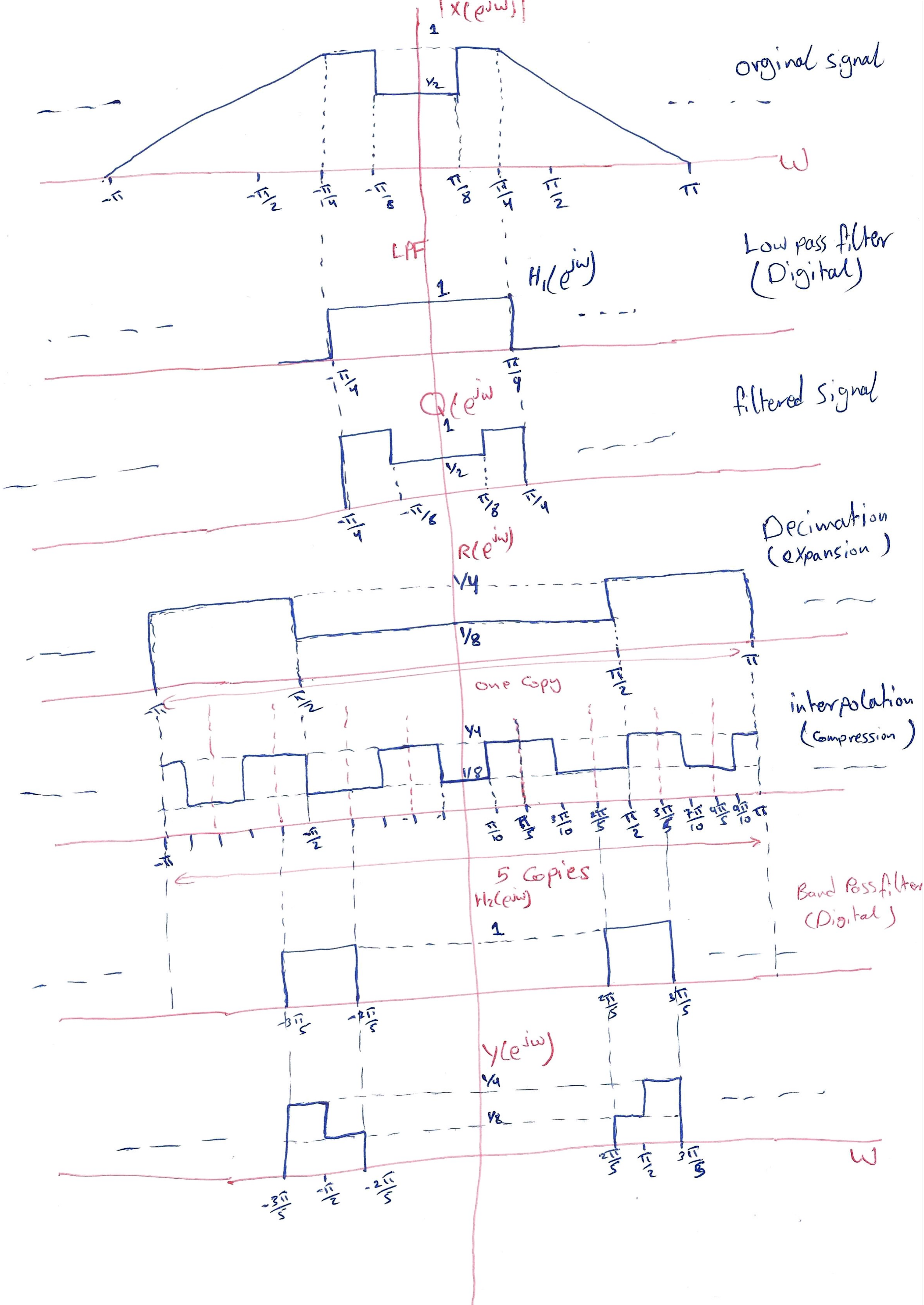
If the magnitude frequency spectrum of x[n], $|X(e^{jw})|$, is given as follows:-



and frequency response of low pass filter, $H_1(e^{jw})$ and Band Pass filter $H_2(e^{jw})$ are as follows:-

$$H_1(e^{jw}) = \begin{cases} 1, & |w| \le \pi/4 \\ 0, & \text{otherwise} \end{cases} \qquad H_2(e^{jw}) = \begin{cases} 1, & \frac{\pi}{3} \le |w| \le \frac{3\pi}{5} \\ 0, & \text{otherwise} \end{cases}$$

sketch $|Q(e^{jw})|$, $|P(e^{jw})|$, $|R(e^{jw})|$, and $|Y(e^{jw})|$?


|X(e^jw)|
orginal signal
Low pass filter (Digital)
LPF
H1(e^jw)
filtered signal
Q(e^jw)
Decimation (expansion)
R(e^jw)
one copy
interpolation (compression)
5 copies
Band Pass filter (Digital)
H2(e^jw)
Y(e^jw)
W