$\boxed{A.7}$ Complex Numbers

* Complex numbers have the form $a + ib = (a, b)$
  where • a and b are real numbers
  • a is called the real part
  • b is called the Imaginary part
  • $i = \sqrt{-1}$

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1 * Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ = positive integers

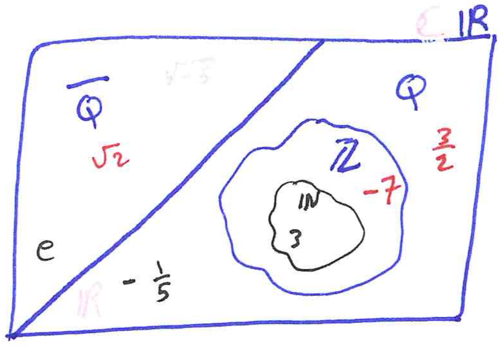
2 * Integer numbers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

3 * Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ with } n \neq 0\}$

4 * Irrational numbers $\overline{\mathbb{Q}}$

5 * Real numbers $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$

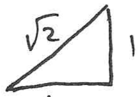
6 * Complex numbers $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$



$\mathbb{C}$ $\sqrt{-5}$

Note that :

- $\mathbb{N}$ is closed under $+$ and $\times$
- $\mathbb{Z}$ is closed under $+, -, \times$
- $\mathbb{Q}$ is closed under $+, -, \times, \div$ except division by zero
- There are some numbers that are not in $\mathbb{Q}$.
  $\overline{\mathbb{Q}}$ contains all such numbers like $\pm\sqrt{2}, \pm\sqrt{3}, \dots, e, \dots$



$\boxed{x = \sqrt{2} \text{ How to solve}}$

$\Rightarrow$ We can have a sequence of rational numbers

$$\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \dots$$

whose squares form a sequence

$$\frac{1}{1}, \frac{49}{25}, \frac{1681}{841}, \frac{57121}{28561}, \dots \quad \text{converges to } 2$$

$L^2 = 2$ and $L \notin \mathbb{Q}$

- Hence Real numbers $\mathbb{R}$ includes rational numbers and the limits of an increasing bounded of rational numbers.

- The complex numbers contain the solution of equations like $x^2+1=0$

* Properities of $\mathbb{C}$: $i=\sqrt{-1}$, $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i, \dots$

[1] Equality : $a+ib=c+id \iff a=c$ and $b=d$

[2] Addition : $(a+ib)+(c+id)=(a+c)+i(b+d)$

[3] Multiplication: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

- $c(a+ib)=ac+i(bc)$

[4] Division : If $a+ib\neq 0$, then

$$\frac{c+id}{a+ib}=\frac{c+id}{a+ib}\cdot\frac{a-ib}{\boxed{a-ib}}=\frac{(ac+bd)+i(ad-bc)}{a^2+b^2}$$

$$=\frac{ac+bd}{a^2+b^2}+i\,\frac{(ad-bc)}{(a^2+b^2)}$$

is called the complex conjugate of $a+ib$

Exp

[1] $(5+2i)+(3-4i)=(5+3)+i(2-4)=8-2i$

[2] $(5+2i)(3-4i)=15-20i+6i-8i^2$
$=15+8-14i=23-14i$

[3] $\frac{5+2i}{3-4i}\cdot\frac{3+4i}{3+4i}=\frac{15+20i+6i+8i^2}{9+16}=\frac{7+26i}{25}$
$=\frac{7}{25}+i\,\frac{26}{25}$

[4] $\overline{5+2i}=5-2i$

[5] $(\overline{5+2i})(5+2i)=(5-2i)(5+2i)=25-4i^2=29$

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* When the imaginary part in complex numbers is zero "$b=0$", then the complex numbers have all properities of real numbers.

* Recall that $a+ib=(a,b)$

⇒ The complex number $(0,0)$: $(0,0)\cdot(a,b)=(0,0)$

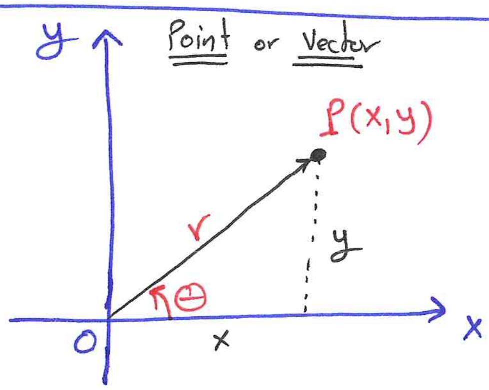
⇒ The complex number $(1,0)$: $(1,0)\cdot(a,b)=(a,b)$

⇒ The complex number $(0,1)=i$: $(0,1)\cdot(0,1)=(-1,0)=-1$

so $(0,1)^2+(1,0)=(0,0)$

Argand Diagrams:

Using this diagram



* we can represent the complex number $z=x+iy$ in the complex plane where:

→ x-axis is the real axis

→ y-axis is the imaginary axis.

→ θ is called the polar angle

$\cos\theta=\frac{x}{r} \Rightarrow x=r\cos\theta$

$\sin\theta=\frac{y}{r} \Rightarrow y=r\sin\theta$

→ r is the length of the vector $\vec{OP}$ which is defined as the Absolute value of the complex number z:

$$r=|x+iy|=\sqrt{x^2+y^2}$$

→ θ is the argument of z and is written as $\theta=\arg z$

- $\theta+2\pi m$ is also an argument of the complex number z
- Note that $\boxed{z\cdot\bar{z}=|z|^2}$
- Exp Let $z=1+2i \Rightarrow \bar{z}=1-2i$

$\Rightarrow z\cdot\bar{z}=(1+2i)(1-2i)=1-4i^2=5$

$\Rightarrow |z|=\sqrt{1+4}=\sqrt{5} \Rightarrow |z|^2=5$ ✓

→ The complex number:

$$\left.\begin{aligned} z=x+iy&=r\cos\theta+i(r\sin\theta)\\ &=r(\cos\theta+i\sin\theta)\end{aligned}\right\}*$$

Euler's formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This comes from $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

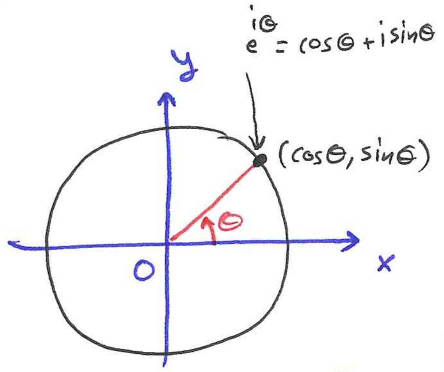
using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \cos\theta + i\sin\theta$$

- Now * becomes $z = x + iy = r e^{i\theta}$

→ Make $r = 1$ ⇒ means that the complex number z is a unit vector $e^{i\theta}$ that makes an angle θ with positive x-axis



Argand diagram for $e^{i\theta} = \cos\theta + i\sin\theta$ as a point.

Products of complex numbers : $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

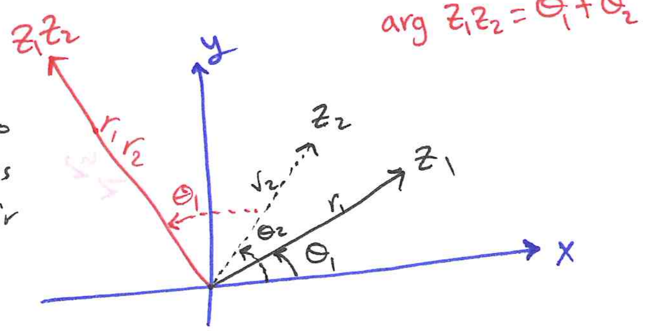
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$, where $r_1 = |z_1|$, $r_2 = |z_2|$

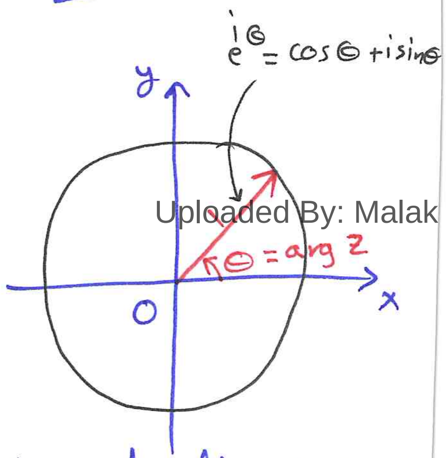
Hence,

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$$

$\arg z_1 z_2 = \theta_1 + \theta_2$

To multiply two omplex numbers e multiply their bsolute values nd add their rguments.





Argand diagram for $e^{i\theta} = \cos\theta + i\sin\theta$ as a vector

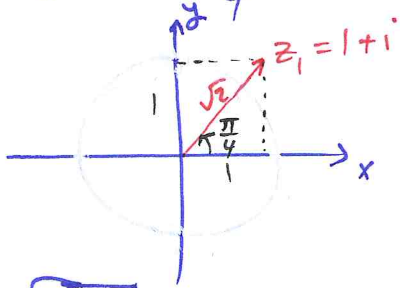
Exp Plot the following complex numbers in an Argand diagram:

[1] $z_1 = 1 + i \quad \Rightarrow r_1 = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$ The polar angle $\theta = \frac{\pi}{4}$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$

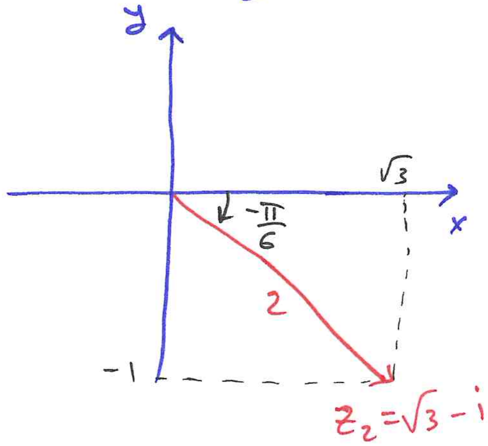


[2] $z_2 = \sqrt{3} - i \quad \Rightarrow r_2 = |\sqrt{3} - i| = \sqrt{3+1} = \sqrt{4} = 2$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$ The polar angle $\theta = -\frac{\pi}{6}$

$= 2\left(\cos\frac{-\pi}{6} + i \sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\, e^{-i\frac{\pi}{6}}$



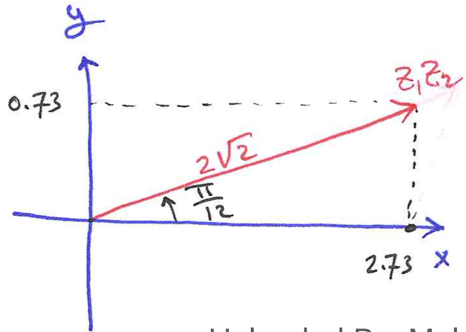
[3] $z_1 z_2 = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)\left(2\, e^{-i\frac{\pi}{6}}\right)$

$= 2\sqrt{2}\, e^{i\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}$

$= 2\sqrt{2}\, e^{i\frac{\pi}{12}}$

$= 2\sqrt{2}\left[\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right]$

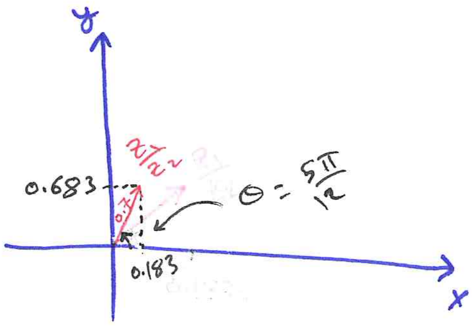
$\approx 2.73 + 0.73\, i$



[4] $\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2\, e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}}\, e^{i\frac{5\pi}{12}}$

$= 0.707\left[\cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}\right]$

$\approx 0.183 + 0.683\, i$



Powers of complex numbers

Recall the complex number $z = r e^{i\theta}$

$$z^n = \left(r e^{i\theta}\right)^n = r^n e^{in\theta}$$

Note that $e^{in\theta} = \left(e^{i\theta}\right)^n$

$$\boxed{\cos n\theta + i \sin n\theta = \left(\cos\theta + i \sin\theta\right)^n}$$

De Moivre's Theorem

Exp $z = 1 + \sqrt{3}\, i \Rightarrow r = |1+\sqrt{3}\, i| = \sqrt{1+3} = \sqrt{4} = 2$

$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \Rightarrow \theta = \frac{\pi}{3}$$

$$= 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$$

$$= 2 e^{i\frac{\pi}{3}}$$

Thus, $z^6 = \left(2 e^{i\frac{\pi}{3}}\right)^6$

$$= 2^6 e^{i 2\pi}$$

$$= 64\left(\cos 2\pi + i \sin 2\pi\right)$$

$$= 64\,(1+0)$$

$$= 64$$

Roots of complex numbers:

If $z = r e^{i\theta}$, then $\sqrt[n]{z} = \sqrt[n]{r e^{i\theta}} = \sqrt[n]{r}\, e^{i\frac{\theta}{n}}$

$$= \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right)$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

$k = 0, \pm 1, \pm 2, \dots$

Th "Fundemental Theorem of Algebra"

Every polynomial of degree n has exactly n roots.

Exp Find the four roots of -16

$-16+0i \Rightarrow r = |-16+0i| = \sqrt{(-16)^2+0^2} = 16$

$= 16(-1+0i) \Rightarrow$ The polar angle $\theta = \pi$

$= 16(\cos\pi + i\sin\pi)$ $\qquad z^4 = -16 \Leftrightarrow z = (-16)^{\frac{1}{4}}$

Thus, $z = (-16+0i)^{\frac{1}{4}}$

$$= (16)^{\frac{1}{4}}\left[\cos(\pi + 2\pi m) + i\sin(\pi + 2\pi m)\right]^{\frac{1}{4}}$$

$$= 2\left(\cos\left(\frac{\pi}{4} + \frac{\pi m}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)\right)$$

The four roots are when $m = 0, 1, 2, 3$

when $m=0 \Rightarrow w_0 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = 2\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = \sqrt{2} + \sqrt{2}i$

$m=1 \Rightarrow w_1 = 2\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right] = 2\left[\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = -\sqrt{2} + \sqrt{2}i$

$m=2 \Rightarrow w_2 = 2\left[\cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right)\right] = 2\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = -\sqrt{2} - \sqrt{2}i$

$m=3 \Rightarrow w_3 = 2\left[\cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)\right] = 2\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \sqrt{2} - \sqrt{2}i$