0.5 Operations with Algebraic Expressions

$+, -, \times, \div, \cdots$

Def Polynomial is sum of finite number of terms with nonnegative integer powers. The general form of a Polynomial in x of degree n is

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \quad \text{where}$$

a_0 : constant term (coefficient)

a_1 : coefficient of x

a_2 : coefficient of x^2

$\vdots$

a_n : coefficient of x^n (leading coefficient), $a_n \neq 0$

n : degree of polynomial (n is nonnegative integer)

$a_0, a_1, \cdots, a_n$ are all real numbers

Exp Give the degree of the following polynomials, state the constant term, give the leading cofficient, decide whether it is polynomial of one or several variables or constant (no variables)

[1] $5x^4 - 2x^3 + 7$

- degree $n=4$ " poly. of degree 4 "
- constat term $7 = a_0$
- leading cofficient $5 = a_4$
- Poly of one variable x

[2] 8

- degree $n=0$ " Poly. of degree 0 "
- constant term $8 = a_0$
- leading cofficient $8 = a_0$
- Poly of no variables (constant)

(3) $7xy^2$

- degree $1+2=3$ "Poly of degree 3"
- constant term 0

- Poly of several variables x, y

[4] $9xy - 4x + 5y + 6$

- degree $1+1=2$ "poly of degree 2"
- constant term 6

- Poly of several variables

Exp Given the poly. $3x^4 - 7x^2 + 5$ Find degree and all coefficients

- degree $4 = n$ $\Rightarrow$ $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0$

$a_4 = 3$, $a_3 = 0$, $a_2 = -7$, $a_1 = 0$, $a_0 = 5$

Exp Evaluate the following algebraic expression at the indicated values of variables

(1) $7x - 2x^2 + 1$ at $x = 2$

$7(2) - 2(2)^2 + 1 = 14 - 2(4) + 1 = 14 - 8 + 1 = 6 + 1 = 7$

(2) $\frac{2x - y}{x^2 - 2y}$ at $x = -5$ and $y = -3$

$$\frac{2(-5) - (-3)}{(-5)^2 - 2(-3)} = \frac{-10 + 3}{25 + 6} = \frac{-7}{31}$$

Remark

- Poly with single term is called monomial

$x,\ 2y^3,\ 3x^5,\ 7,\ -8x^2, \dots$

- Poly with two terms is called binomial

$x^2-1,\ 2x+x^3,\ 5-x^4,\ 3x^7-x^5, \dots$

- Poly with three terms is called trinomial

$x^3-2x^2+1,\ x^2+2x+5,\ x^8-x+1, \dots$

Note: Terms with exactly the same variable factors are called like terms

Example of like terms:

$8xy$ and $-3xy$

$-5x^2$ and $8x^2$

$3x^3y$ and $-yx^3$

17 and -3

Exp Simplify by combining like terms (Remove parentheses)

① $(3x^3 + 4x^2y^2) + (3x^2y^2 - 7x^3) = -4x^3 + 7x^2y^2$

② $-[8 - 4(y+5) + y] = -[8 - 4y - 20 + y] = -[-12 - 3y]$

$= 12 + 3y$

③ $(3x^2 + 4xy + 5y^2 + 1) - (6x^2 - 2xy + 4)$

$= 3x^2 + 4xy + 5y^2 + 1 - 6x^2 + 2xy - 4$

$= -3x^2 + 6xy + 5y^2 - 3$

Exp (Products and Quotients)

Perform the indicated operations and simplify:

① $(-3x^2y)(2xy^3)(4x^2y^2)$

$(-3\cdot2\cdot4)(x^2\cdot x\cdot x^2)(y\cdot y^3\cdot y^2) = -24x^5y^6$

② $(-15m^3n)\div(5mn^4)$

$\frac{-15m^3n}{5mn^4} = -3\,m^{3-1}\,n^{1-4} = -3m^2n^{-3} = \frac{-3m^2}{n^3}$

Exp (Symbols of Grouping)

Perform the indicated operations and simplify:

① $3x^2 - [2x - (3x^2 - 2x)]$

$3x^2 - [2x - 3x^2 + 2x] = 3x^2 - 2x + 3x^2 - 2x = 6x^2 - 4x$

② $(3x-2) - 3x - 2(3x-2) + 5$

$-(3x-2) - 3x + 5 = -3x + 2 - 3x + 5 = -6x + 7$

Exp (Distributive Law)

Perform the indicated operations and simplify

① $ax^2(2x^2 + ax + ab) = 2ax^4 + a^2x^3 + a^2bx^2$

② $(4a + 5b + c)ac = 4a^2c + 5abc + ac^2$

③ $-5(4 - x^3) = -20 + 5x^3$

④ $(4x+1)(x-3) = (4x+1)(x) + (4x+1)(-3)$

$= 4x^2 + x - 12x - 3 = 4x^2 - 11x - 3$

⑤ $9(2x+1)(2x-1) = (18x+9)(2x-1) = 36x^2 - \cancel{18x} + \cancel{18x} - 9$

$= 36x^2 - 9$

Exp (Product of Two Polynomials)

Multiply the following

① $(3x+1)(5-x) = 15x - 3x^2 + 5 - x = 14x - 3x^2 + 5$

② $(x^3-1)(x^7-2x^4-5x^2+5)$

$x^{10} \underline{-2x^7} - 5x^5 + 5x^3 \underline{-x^7} + 2x^4 + 5x^2 - 5$

$x^{10} - 3x^7 - 5x^5 + 2x^4 + 5x^3 + 5x^2 - 5$

Special Products

- Binomial squared $(x+a)^2 = x^2 + 2ax + a^2$

  $(x-a)^2 = x^2 - 2ax + a^2$

- Difference of two squares $x^2 - a^2 = (x-a)(x+a)$

- Binomial cubed $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

  $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

Exp (Special Products)

Multiply the following

① $(x+3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9$

② $(3x-4y)^2 = (3x)^2 + 2(3x)(-4y) + (-4y)^2 = 9x^2 - 24xy + 16y^2$

③ $(x^2-y^3)^2 = (x^2)^2 + 2(x^2)(-y^3) + (-y^3)^2 = x^4 - 2x^2y^3 + y^6$

④ $(x+2)^3 = x^3 + 3(x)(2)^2 + 3(x)^2(2) + (2)^3 = x^3 + 12x + 6x^2 + 8$

⑤ $(x^2-\frac{1}{2})^2 = (x^2)^2 + 2(-\frac{1}{2})(x^2) + (-\frac{1}{2})^2 = x^4 - x^2 + \frac{1}{4}$

⑥ $(2x-3)^3 = (2x)^3 + 3(2x)(-3)^2 + 3(2x)^2(-3) + (-3)^3$

$= 8x^3 + (6x)(9) - 9(4x^2) + (-27) = 8x^3 + 54x - 36x^2 - 27$

Exp (Operations with Algebraic Expressions)

Perform the indicated operations and simplify

1) $3\sqrt{3} + 4x\sqrt{y} - 5\sqrt{3} - 11x\sqrt{y} - (\sqrt{3} - x\sqrt{y})$

$3\sqrt{3} - 5\sqrt{3} - \sqrt{3} + 4x\sqrt{y} - 11x\sqrt{y} + x\sqrt{y} = -3\sqrt{3} - 6x\sqrt{y}$

2) $x^{-\frac{2}{3}}\left(x^{\frac{5}{3}} - x^{-\frac{1}{3}}\right)$

$x^{-\frac{2}{3}} \cdot x^{\frac{5}{3}} - x^{-\frac{2}{3}} \cdot x^{-\frac{1}{3}} = x^{-\frac{2}{3}+\frac{5}{3}} - x^{\frac{-2}{3}+\frac{-1}{3}} = x^{\frac{3}{3}} - x^{\frac{-3}{3}} = x^{1} - x^{-1}$

$= x - \frac{1}{x}$

3) $\left(\sqrt{x} - \sqrt[3]{x}\right)^2$

$\left(x^{\frac{1}{2}} - x^{\frac{1}{3}}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 + 2\left(x^{\frac{1}{2}}\right)\left(-x^{\frac{1}{3}}\right) + \left(-x^{\frac{1}{3}}\right)^2$

$= x^{\frac{1}{2}\cdot 2} - 2x^{\frac{1}{2}+\frac{1}{3}} + x^{\frac{1}{3}\cdot 2}$

$= x^{1} - 2x^{\frac{3}{6}+\frac{2}{6}} + x^{\frac{2}{3}}$

$= x - 2x^{\frac{5}{6}} + x^{\frac{2}{3}} = x - 2\sqrt[6]{x^5} + \sqrt[3]{x^2}$

4) $(\sqrt{x}+3)(\sqrt{x}-3)$

$(\sqrt{x})^2 - (3)^2 = x - 9$

or $(\sqrt{x}+3)(\sqrt{x}-3)$

$\sqrt{x}\,\sqrt{x} - \cancel{3\sqrt{x}} + \cancel{3\sqrt{x}} - (3)(3)$

$x - 9$

5) $\left(x^{\frac{1}{5}} + x^{\frac{1}{2}}\right)\left(x^{\frac{1}{5}} - x^{\frac{1}{2}}\right)$

$\left(x^{\frac{1}{5}}\right)^2 - \left(x^{\frac{1}{2}}\right)^2 = x^{\frac{2}{5}} - x$

or $\left(x^{\frac{1}{5}} + x^{\frac{1}{2}}\right)\left(x^{\frac{1}{5}} - x^{\frac{1}{2}}\right)$

$x^{\frac{1}{5}}x^{\frac{1}{5}} - \cancel{x^{\frac{1}{5}}x^{\frac{1}{2}}} + \cancel{x^{\frac{1}{2}}x^{\frac{1}{5}}} - x^{\frac{1}{2}}x^{\frac{1}{2}}$

$x^{\frac{1}{5}+\frac{1}{5}} - x^{\frac{1}{2}+\frac{1}{2}}$

$x^{\frac{2}{5}} - x^{1}$

Exp (Division of Polynomials - Long Division)

① Divide $4x^3 - 13x - 22$ by $x-3$, $x \neq 3$

$$\frac{4x^3-13x-22}{x-3} = \underbrace{4x^2+12x+23}_{\text{ناتج القسمة The quotient}} + \frac{47}{\underbrace{x-3}_{\text{المقسوم عليه}}}$$

The answer is $4x^2+12x+23+\frac{47}{x-3}$

$$\begin{array}{r|l} & 4x^2 + 12x + 23 \\ \hline x-3 & \cancel{4x^3} - 13x - 22 \\ & -\cancel{4x^3} + x^2 \\ \hline & \cancel{x^2} - 13x - 22 \\ & -\cancel{12x^2} + 36x \\ \hline & 23\cancel{x} - 22 \\ & -\cancel{23x} + 69 \\ \hline & 47 \end{array}$$

Remainder الباقي ← 47

② Use long division to find $(x^3+x-1) \div (x+2)$

$$\frac{x^3+x-1}{x+2} = x^2-2x+5 - \frac{11}{x+2}$$

The quotient is x^2-2x+5

Remainder is -11

Answer is $x^2-2x+5-\frac{11}{x+2}$

$$\begin{array}{r|l} & x^2 - 2x + 5 \\ \hline x+2 & \cancel{x^3} + x - 1 \\ & -\cancel{x^3} \mp 2x^2 \\ \hline & -2\cancel{x^2} + x - 1 \\ & +\cancel{2x^2} + 4x \\ \hline & 5\cancel{x} - 1 \\ & -\cancel{5x} \mp 10 \\ \hline & -11 \end{array}$$