3.4: The Normal distribution.

Def: A random variable X is called a Normal r.v with parameters a and b^2 if it has the following p.d.f :

$$f(x) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{(x-a)^2}{2b^2}}, \quad x \in \mathbb{R} \qquad \text{where } a \in \mathbb{R},\ b > 0.$$

Question: verify that $f(x)$ is a p.d.f.

$\rightarrow$ $f(x) > 0 \quad \forall x \in \mathbb{R}$,

$f(x) \geq 0, \ \forall x \in \mathbb{R}$.

$$\rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{(x-a)^2}{2b^2}}\,dx.$$

let $y = \frac{x-a}{b} \Rightarrow dy = \frac{dx}{b}$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} f(x)\,dx\right)^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\,dv.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(u^2+v^2)}{2}}\,du\,dv$$

By polar Coordinate

$$= \int_{0}^{2\pi}\int_{0}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta$$

$\rightarrow$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} r e^{-\frac{1}{2}r^2}\, dr\right) \rightarrow \quad \frac{1}{2}r^2 = w$$

$$= \frac{1}{2\pi}\, 2\pi \int_0^{\infty} e^{-w}\, dw \qquad r\,dr = dw$$

$$= \frac{e^{-w}}{-1}\Big|_0^{\infty} = 0-(-1) = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy > 0$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} f(x)\, dx\right)^2 = 1 \Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \mp 1 \quad \text{But it} > 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = 1 \rightsquigarrow \text{its p.d.f}$$

Question: Find the m.g.f of X.

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x)\, dx = \int_{-\infty}^{\infty} e^{tX} \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{(x-a)^2}{2b^2}}\, dx$$

$$= e^{at+\frac{1}{2}b^2t^2}, \quad t \in \mathbb{R}$$

$$\boxed{M(t) = e^{at+\frac{1}{2}b^2t^2}, \quad t \in \mathbb{R}}$$

Question : Find the mean , variance and standard deviation of X.

$$M(t) = e^{at + \frac{1}{2} b^2 t^2}$$

→ Mean :

$$\bar{M}(t) = M(t)\,(a + b^2 t)$$

$$\bar{M}(0) = M(0)\,(a + b^2(0)) = a.$$

$$\Rightarrow \boxed{E(x) = \mu = a}$$

→ Var(x) :

$$\bar{\bar{M}}(t) = M(t)\, b^2 + (a + b^2 t)\, \bar{M}(t)$$

$$\bar{\bar{M}}(0) = M(0)\, b^2 + (a + b^2(0))\, \bar{M}(0)$$

$$= b^2 + a\,(a)$$

$$= b^2 + a^2$$

$$\Rightarrow Var(x) = \bar{\bar{M}}(0) - (\bar{M}(0))^2 = b^2 + a^2 - a^2 = b^2.$$

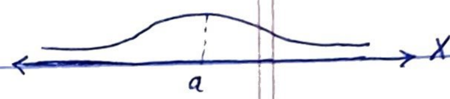
$$\Rightarrow \boxed{Var(x) = \sigma^2 = b^2}$$

→ standard deviation :

$$\boxed{\sigma = \sqrt{\sigma^2} = b}.$$

Summary: X Normal r.V with parameters a and b^2 :

1. $f(x) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{(x-a)^2}{2b^2}}$, $x \in \mathbb{R}$, $a \in \mathbb{R}$, $b > 0$.



2. $M(t) = e^{at + \frac{1}{2}b^2t^2}$, $t \in \mathbb{R}$.

3. $\bar{M}(0) = E(X) = \mu = a$.

4. $\bar{\bar{M}}(0) - (\bar{M}(0))^2 = Var(X) = \sigma^2 = b^2$.

Rename: X Normal r.V with parameters μ and σ^2 ($\mu \in \mathbb{R}$, $\sigma > 0$)

1. $f(x) = \frac{1}{\sqrt{2\pi}\, \sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

2. $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $t \in \mathbb{R}$

3. $E(X) = \bar{M}(0) = \mu$

4. $Var(X) = \bar{\bar{M}}(0) - (\bar{M}(0))^2 = \sigma^2$.

Remark: X Normal r.V with mean Zero and st. dev, is called standard normal random variable.

Remark: c.d.f of standard Normal r.V is denoted by

$$\Phi(x) = Pr(X \leq x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx .$$

check table III

Notation :

- $X \sim N(\mu, \sigma^2)$: X has Normal dist. with mean μ and Variance σ^2
- $X \sim N(0,1)$: X has standard Normal distribution.

Thm 1 :

$$X \sim N(\mu, \sigma^2)$$
$$Z = \frac{X-\mu}{\sigma} \quad \Rightarrow \quad Z \sim N(0,1).$$

proof:

X has p.d.f f and c.d.f F

Z has p.d.f g and c.d.f G.

$$G(z) = Pr(Z \le z) = Pr\left(\frac{X-\mu}{\sigma} \le z\right)$$

$$\overset{\sigma>0}{=} Pr(X \le \mu + z\sigma)$$
$$= F(\mu + z\sigma)$$

$$g(z) = \bar{G}(z) = \frac{d}{dz} F(\mu + z\sigma) = f(\mu + z\sigma) . \sigma$$

$$g(z) = \sigma \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\mu + z\sigma - \mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 \cancel{\sigma^2}}{2\cancel{\sigma^2}}}$$

$$\boxed{g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Longrightarrow Z \sim N(0,1)}$$

Thm 2 :

$$X \sim N(\mu, \sigma^2)$$
$$V = \left(\frac{X-\mu}{\sigma}\right)^2 \Rightarrow V \sim \chi^2(1).$$

Proof :

X has p.d.f f and c.d.f F.
$W = \frac{X-\mu}{\sigma}$ has p.d.f ϕ and c.d.f Φ
$V = W^2$ has p.d.f g and c.d.f G.

$$G(v) = Pr(V \leq v) = Pr(W^2 < v)$$
$$= \begin{cases} Pr(-\sqrt{v} \leq W \leq \sqrt{v}) & , v > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} \int_{-\sqrt{v}}^{\sqrt{v}} \phi(w)\, dw = \Phi(\sqrt{v}) - \Phi(-\sqrt{v}) & , v > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$g(v) = G'(v) = \frac{d}{dv} \begin{cases} \Phi(\sqrt{v}) - \Phi(-\sqrt{v}) & , v > 0 \\ 0 & , v < 0 \end{cases}$$

$$= \begin{cases} \phi(\sqrt{v}) \frac{1}{2\sqrt{v}} - \phi(-\sqrt{v}) \left(-\frac{1}{2\sqrt{v}}\right) & , v > 0 \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{v}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{v})^2}{2}} + \frac{1}{2\sqrt{v}} \frac{1}{2\sqrt{\pi}} e^{-\frac{\pi^2}{2}} & , v > 0 \\ 0 & , \text{else} \end{cases}$$

$\rightarrow$

$$g(v) = \begin{cases} \frac{1}{\sqrt{\pi}\, 2^{\frac{1}{2}}} \; v^{\frac{1}{2}-1} \; e^{-\frac{v}{2}} & , \; v > 0 \\ 0 & , \; \text{else} \end{cases}$$

$\Rightarrow \quad V \sim \text{Gamma}\left(\frac{1}{2}, 2\right)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{2}$.

$\Rightarrow \quad V \sim \chi^2(1)$.

check exp $1 \to 4$.