4.2 Homogeneous Equations with Constant Coefficients

Let's consider the homogeneous linear nth-order differential equation

(1) $$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0 ,$$

where $a_n (\neq 0), a_{n-1}, \ldots, a_0$ are real constants.†

If we let L be the differential operator defined by the left-hand side of (1), that is,

(3) $$L[y] := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y ,$$

then we can write (1) in the operator form

(4) $$L[y](x) = 0 .$$

For $y = e^{rx}$, we find

(5) $$\begin{aligned} L[e^{rx}](x) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0) = e^{rx} P(r) , \end{aligned}$$

where $P(r)$ is the polynomial $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$. Thus, e^{rx} is a solution to equation (4), provided r is a root of the **auxiliary** (or **characteristic**) **equation**

(6) $$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 .$$

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 . \quad (6)$$

According to the fundamental theorem of algebra, the auxiliary equation has n roots (counting multiplicities), which may be either real or complex. However, there are no formulas for determining the zeros of an arbitrary polynomial of degree greater than four, although if we can determine one zero r_1, then we can divide out the factor $r - r_1$ and be left with a polynomial of lower degree.

We proceed to discuss the various possibilities.

Distinct Real Roots

If the roots $r_1, \ldots, r_n$ of the auxiliary equation (6) are real and distinct, then n solutions to equation (1) are

$$y_1(x) = e^{r_1 x},\ y_2(x) = e^{r_2 x},\ \ldots,\ y_n(x) = e^{r_n x} \quad . \tag{7}$$

Example 1 Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0 \,. \tag{15}$$

Solution The auxiliary equation is

$$r^3 - 2r^2 - 5r + 6 = 0\,. \tag{16}$$

By inspection we find that $r = 1$ is a root. Then, using polynomial division, we get

$$r^3 - 2r^2 - 5r + 6 = (r-1)(r^2 - r - 6)\,,$$

which further factors into $(r-1)(r+2)(r-3)$. Hence the roots of equation (16) are $r_1 = 1$, $r_2 = -2$, $r_3 = 3$. Since these roots are real and distinct, a general solution to (15) is

$$y(x) = C_1 e^x + C_2 e^{-2x} + C_3 e^{3x}\,. \quad \blacklozenge$$

Complex Roots

If $\alpha + i\beta$ (α, β real) is a complex root of the auxiliary equation (6), then so is its complex conjugate $\alpha - i\beta$.

To find two real-valued solutions corresponding to the roots $\alpha \pm i\beta$, we can just take the real and imaginary parts of $e^{(\alpha+i\beta)x}$. That is, since

(17) $$e^{(\alpha+i\beta)x} = e^{\alpha x} \cos \beta x + i e^{\alpha x} \sin \beta x ,$$

then two linearly independent solutions to (1) are

(18) $$e^{\alpha x} \cos \beta x , \qquad e^{\alpha x} \sin \beta x .$$

Example 2 Find a general solution to

(19) $$y''' + y'' + 3y' - 5y = 0 .$$

Solution The auxiliary equation is

$$r^3 + r^2 + 3r - 5 = (r - 1)(r^2 + 2r + 5) = 0 , \quad (20)$$

which has distinct roots $r_1 = 1, r_2 = -1 + 2i, r_3 = -1 - 2i$. Thus, a general solution is

$$y(x) = C_1 e^x + C_2 e^{-x} \cos 2x + C_3 e^{-x} \sin 2x . \quad (21)$$

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Repeated Roots

If r_1 is a root of multiplicity m, then the n solutions given in (7) are not even distinct, let alone linearly independent. Recall that for a second-order equation, when we had a repeated root r_1 to the auxiliary equation, we obtained two linearly independent solutions by taking e^{r_1x} and xe^{r_1x}. So if r_1 is a root of (6) of multiplicity m, we might expect that m linearly independent solutions are

$$(22) \qquad e^{r_1x}, \quad xe^{r_1x}, \quad x^2e^{r_1x}, \quad \ldots, \quad x^{m-1}e^{r_1x}.$$

If $\alpha + i\beta$ is a repeated complex root of multiplicity m, then we can replace the $2m$ complex-valued functions

$$e^{(\alpha+i\beta)x}, \quad xe^{(\alpha+i\beta)x}, \quad \ldots, \quad x^{m-1}e^{(\alpha+i\beta)x},$$
$$e^{(\alpha-i\beta)x}, \quad xe^{(\alpha-i\beta)x}, \quad \ldots, \quad x^{m-1}e^{(\alpha-i\beta)x}$$

by the $2m$ linearly independent real-valued functions

$$(28) \qquad \begin{array}{llll} e^{\alpha x}\cos\beta x, & xe^{\alpha x}\cos\beta x, & \ldots, & x^{m-1}e^{\alpha x}\cos\beta x, \\ e^{\alpha x}\sin\beta x, & xe^{\alpha x}\sin\beta x, & \ldots, & x^{m-1}e^{\alpha x}\sin\beta x. \end{array}$$

Example 3 Find a general solution to

$$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0\,. \tag{29}$$

Solution The auxiliary equation is

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r-1)^3(r+2) = 0\,,$$

which has roots $r_1 = 1$, $r_2 = 1$, $r_3 = 1$, $r_4 = -2$. Because the root at 1 has multiplicity 3, a general solution is

(30) $$y(x) = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-2x}\,. \quad ◆$$

Example 4 Find a general solution to

$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0 , \tag{31}$$

whose auxiliary equation can be factored as

$$r^4 - 8r^3 + 26r^2 - 40r + 25 = (r^2 - 4r + 5)^2 = 0 . \tag{32}$$

Solution The auxiliary equation (32) has repeated complex roots: $r_1 = 2+i$, $r_2 = 2+i$, $r_3 = 2-i$, and $r_4 = 2-i$. Hence, a general solution is

$$y(x) = C_1e^{2x}\cos x + C_2xe^{2x}\cos x + C_3e^{2x}\sin x + C_4xe^{2x}\sin x\,. \quad \blacklozenge$$

PROBLEMS

In each of Problems 11 through 28, find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{(4)} - 4y''' + 4y'' = 0$

37. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?