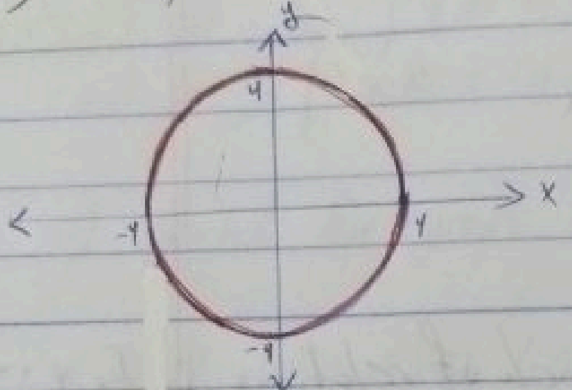
Exercises 4.1 :-)

Q.4 Sketch the graph of each of the following relations:-

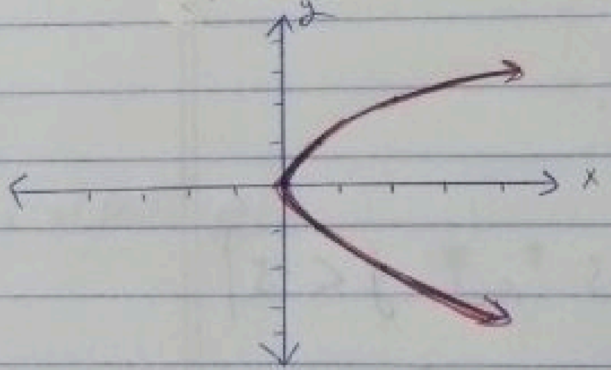
a) $S = \{(x,y) \in R \times R : x^2 + y^2 = 16\}$



⇒ Domain: $-4 \leq x \leq 4$

⇒ Range: $-4 \leq y \leq 4$

b) $S = \{(x,y) \in R \times R : y^2 = 2x\}$, $y^2 = 2x \Rightarrow x = \frac{1}{2} y^2$

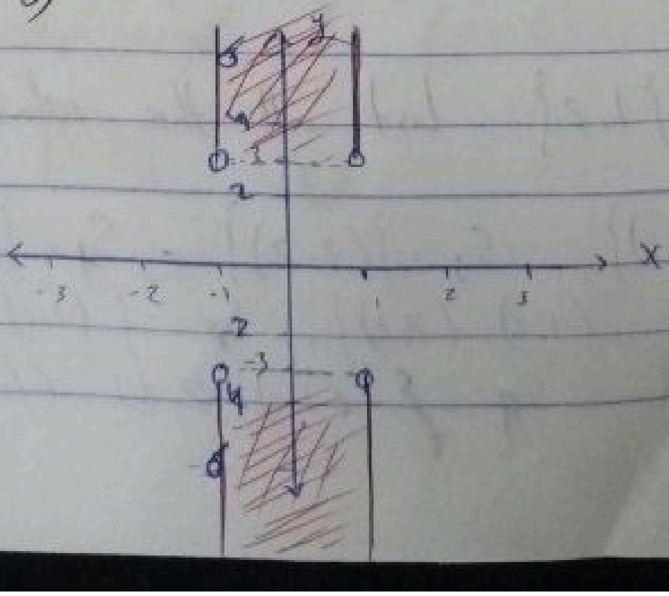


⇒ Domain: $x \geq 0$

⇒ Range: $\mathbb{R}$

c) $S = \{(x,y) \in R \times R : |x| \leq 1 \text{ and } |y| > 3\}$

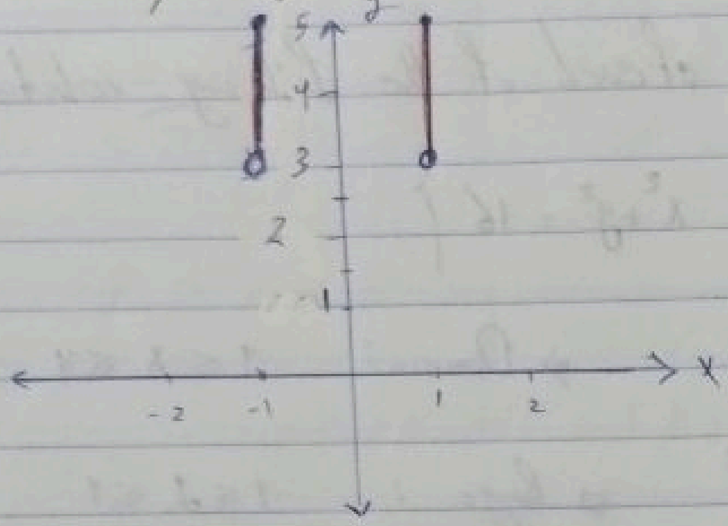
$$\left[\begin{array}{c} -1 \leq x \leq 1 \\ \text{and} \\ y < -3 \text{ or } y > 3 \end{array}\right]$$



⇒ Domain: $-1 \leq x \leq 1$

⇒ Range: $y < -3$ and $y > 3$

f) $S = \{(x,y) \in R \times R : |x| = 1 \text{ and } 3 < y \leq 5\}$



$\Rightarrow$ Domain : $x = 1$ and $x = -1$

$\Rightarrow$ Range : $3 < y \leq 5$

$\boxed{Q.5}$ evaluate $S[1]$, for each of relations in Ex.4 :-

a) $S[1] = \{-\sqrt{15}, \sqrt{15}\}$

b) $S[1] = \{-\sqrt{2}, \sqrt{2}\}$

e) $S[1] = \{y \in R : y > 3 \text{ and } y < -3\}$

f) $S[1] = \{y \in R : 3 < y \leq 5\}$.

$\boxed{Q.6}$ let $S = \{1, 2\}$. List all the relations on S :-

$S_1 = \emptyset$, $S_2 = \{(1,1)\}$, $S_3 = \{(2,2)\}$, $S_4 = \{(1,2)\}$,
$S_5 = \{(2,1)\}$, $S_6 = \{(1,1), (2,2)\}$, $S_7 = \{(1,1), (1,2)\}$
$S_8 = \{(1,1), (2,1)\}$, $S_9 = \{(1,1), (2,2), (1,2)\}$

$S_{10} = \{(1,1), (2,2), (2,1)\}$, $S_{11} = \{(2,2), (1,2)\}$, $S_{12} = \{(2,2), (2,1)\}$

$S_{13} = \{(2,2), (1,2), (2,1)\}$, $S_{14} = \{(1,1), (1,2), (2,1)\}$

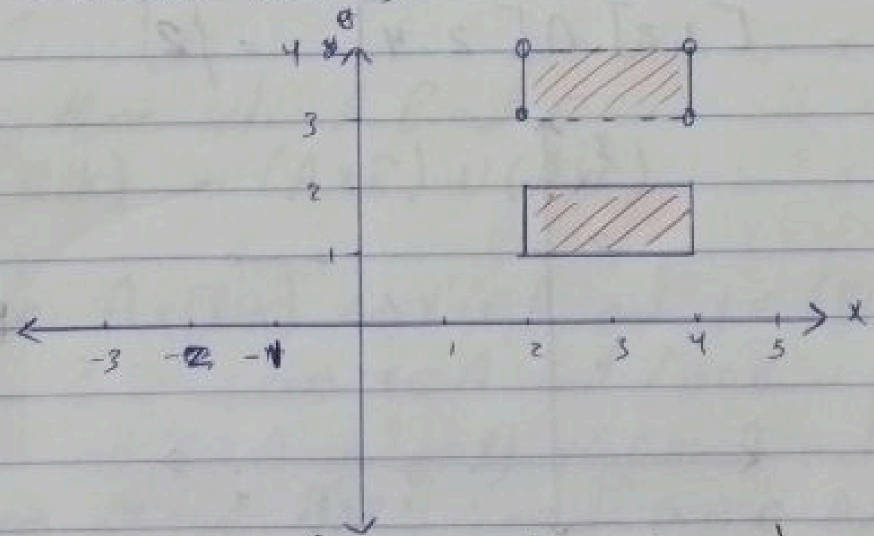
$S_{15} = \{(1,1), (1,2), (2,1), (2,2)\}$, $S_{16} = \{(1,2), (2,1)\}$

Q.7 let $A = \{x \in R : 1 \le x \le 2\}$, $B = \{x \in R : 3 < x < 4\}$ and $C = \{x \in R : 2 \le x \le 4\}$:-

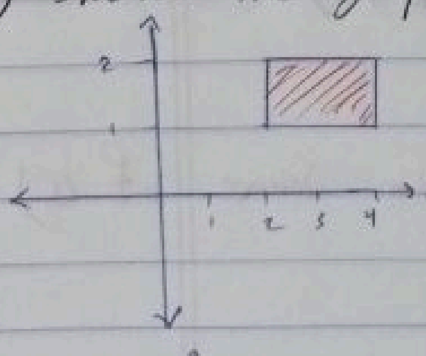
a) sketch the graph of $C \times (A \cup B)$:-

$* \; A \cup B = [1,2] \cup (3,4)$

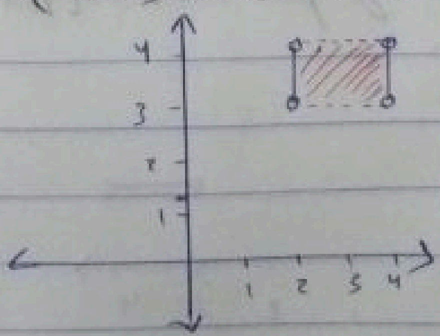
$C \times (A \cup B) \Rightarrow$



b) sketch the graph of $(C \times A) \cup (C \times B)$:-

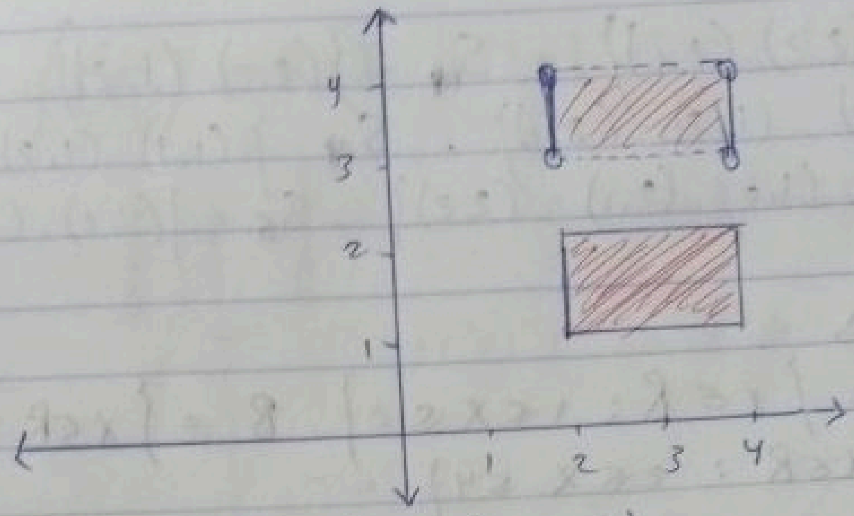


$\cup$



$\Rightarrow$

$C \times A$ $\qquad$ $C \times B$

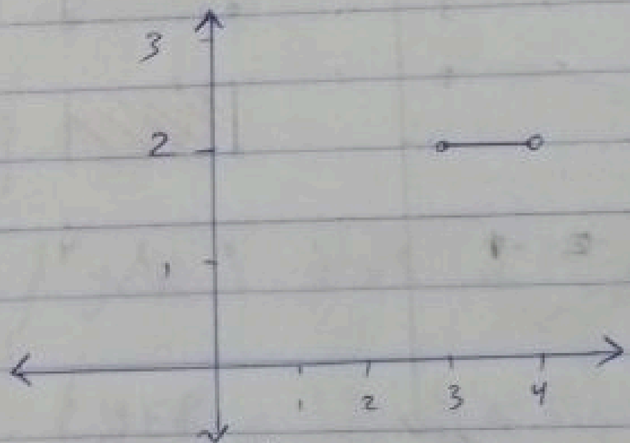


The Same of (a)

$(C \times A) \cup (C \times B)$

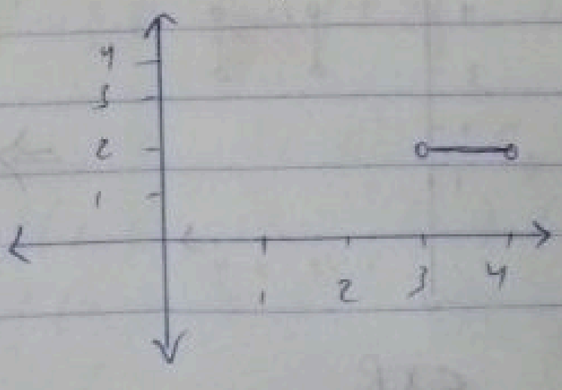
c) sketch the graph of $B \times (A \cap C)$:-

$\ast\ A \cap C = [1,2] \cap [2,4] = \{2\}$



$B \times (A \cap C)$

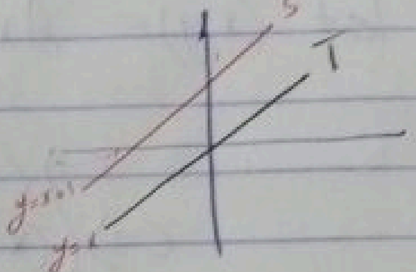
D) sketch the graph of $(B \times A) \cap (B \times C)$:-



The Same of (C)

Q.8 Give an example of two relations S and T such that $\mathbb{R}$ = the domain of S = the domain of T = the range of S = the range of T. but $S \cap T = \phi$:-

$S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x+1\}$



$T = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x\}$

Q.9 Give a proof of, or a counter example to, each of the following statment, :-

a) for any three sets A, B and C,
$A \times (B \cup C) = (A \times B) \cup (A \times C)$

let $(x,y) \in A \times (B \cup C) \Leftrightarrow x \in A$ and $y \in (B \cup C)$
$\Leftrightarrow x \in A$ and ($y \in B$ or $y \in C$)
$\Leftrightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$
$\Leftrightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$
$\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C)$ ✓

b) for any three sets A, B, and C

$A \times (B \cap C) = (A \times B) \cap (A \times C)$:-

let $(x,y) \in A \times (B \cap C) \Leftrightarrow x \in A$ and $y \in (B \cap C)$

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $((x \in A)$ and $y \in C)$

$\Leftrightarrow ((x,y) \in A \times B)$ and $((x,y) \in A \times C)$

$\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C)$ ✓

$\boxed{Q.11}$ a) let S and T be relations. Prove that if $S \subseteq T$ and x is a member of the domain of S, then $S[x] \subseteq T[x]$:-

Suppose $S \subseteq T$ and $x \in$ Domain of S

$\Rightarrow x \in$ Domain of S $\Rightarrow y \in S[x]$

$\Leftrightarrow (x,y) \in S$

$\Leftrightarrow (x,y) \in T$ Since $S \subseteq T$

$\Leftrightarrow x \in$ Doman of T and $y \in T[x]$

$\Leftrightarrow S[x] \subseteq T[x]$

b) let A and B be nonempty sets. Prove that $(A \times B)[x] = B$, if $x \in A$ and $(A \times B)[x] = \phi$ if $x \notin A$:-

* if $x \notin A$ then $(A \times B)[x] = \phi$:-

Contradiction : Suppose $x \notin A$ and $(A \times B)[x] \neq \phi$

$\Rightarrow x \notin A$ and $\exists\, y \in (A \times B)[x]$

$\Rightarrow x \notin A$ and $(x, y) \in A \times B$

$\Rightarrow \underline{x \notin A}$ and $\underline{x \in A}$ and $y \in B$

a contradiction ✗

* if $x \in A$ then $(A \times B)[x] = B$

$\boxed{\subseteq}$ Suppose $x \in A$ and $y \in (A \times B)[x] \Rightarrow (x, y) \in (A \times B) \Rightarrow y \in B$

$\boxed{\supseteq}$ Suppose $x \in A$ and $y \in B \Rightarrow (x, y) \in (A \times B) \Rightarrow y \in (A \times B)[x]$

$\boxed{Q.16}$ let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$:-

① * if $A \times B = B \times A$ then $A \neq B$

let $x \in A$ and $y \in B$

$\Rightarrow (x, y) \in (A \times B)$

$\Rightarrow (x, y) \in (B \times A)$ since $A \times B = B \times A$

$\Rightarrow x \in B$ and $y \in A$

$\Rightarrow A \subseteq B$ since $x \in A$ and $x \in B$

$\Rightarrow B \subseteq A$ since $y \in B$ and $y \in A$

$\Rightarrow A = B$ ($A \subseteq B$ and $B \subseteq A$)

* if $A = B$ then $A \times B = B \times A$

let $x \in A$ and $y \in B$

$\Rightarrow x \in B$, $y \in A$ ($A = B$)

$\Rightarrow (x, y) \in A \times B$

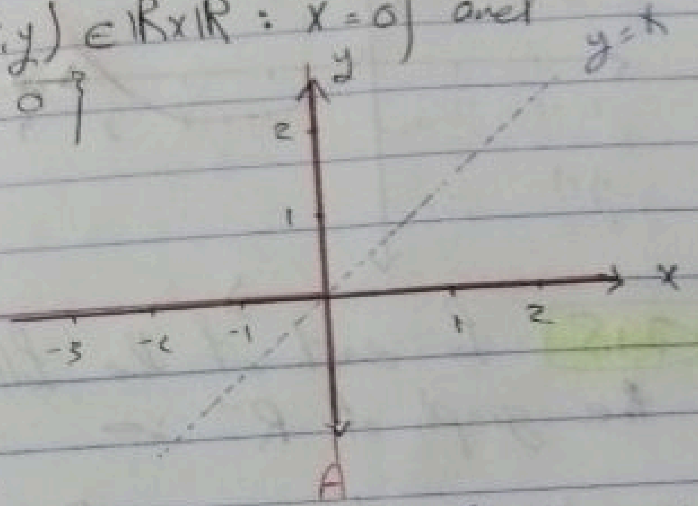
$\Rightarrow (x, y) \in B \times A$

$\Rightarrow A \times B = B \times A$ ✓

Exercises 4.2 :

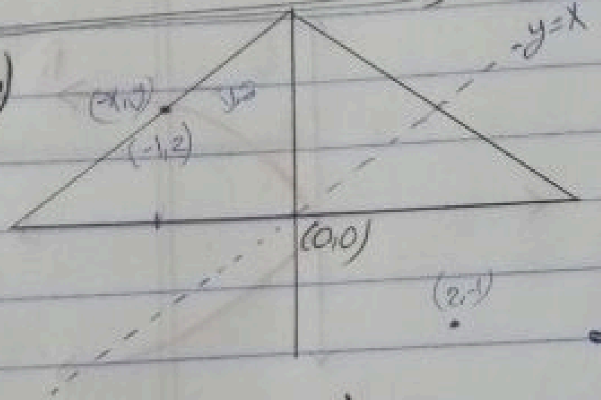
Q.17 which of the following relations are symmetric ??

a) $A \cup B$, where $A = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x = 0\}$ and $B = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = 0\}$



There is a reflection about $y = x$ then the relation is symmetric

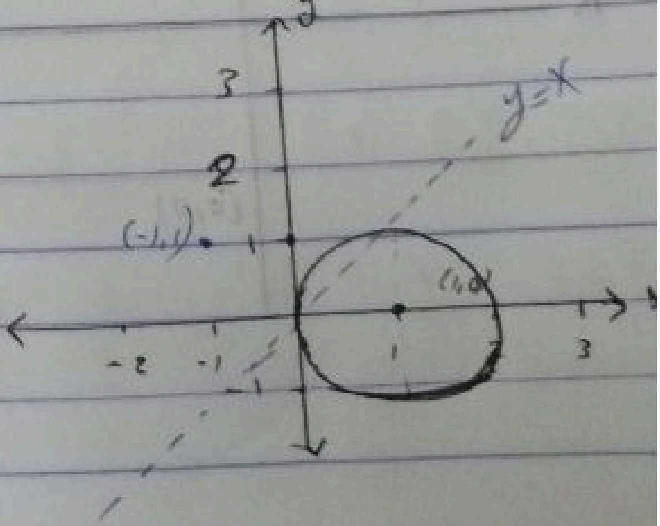
b)



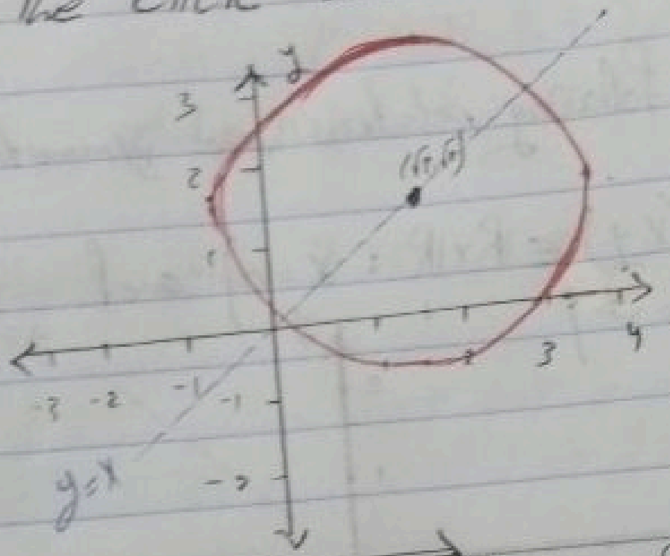
- There is ... reflection about $y=x$, then the relation is not symmetric.
- take $(-1,2) \in R$, but $(2,-1) \notin R$

c) The circle with center $(1,0)$ and radious 1 :

- There is no reflection about $y = x \Rightarrow$ not symmetric
- take $(1,-1) \in R$ but $(-1,1) \notin R$



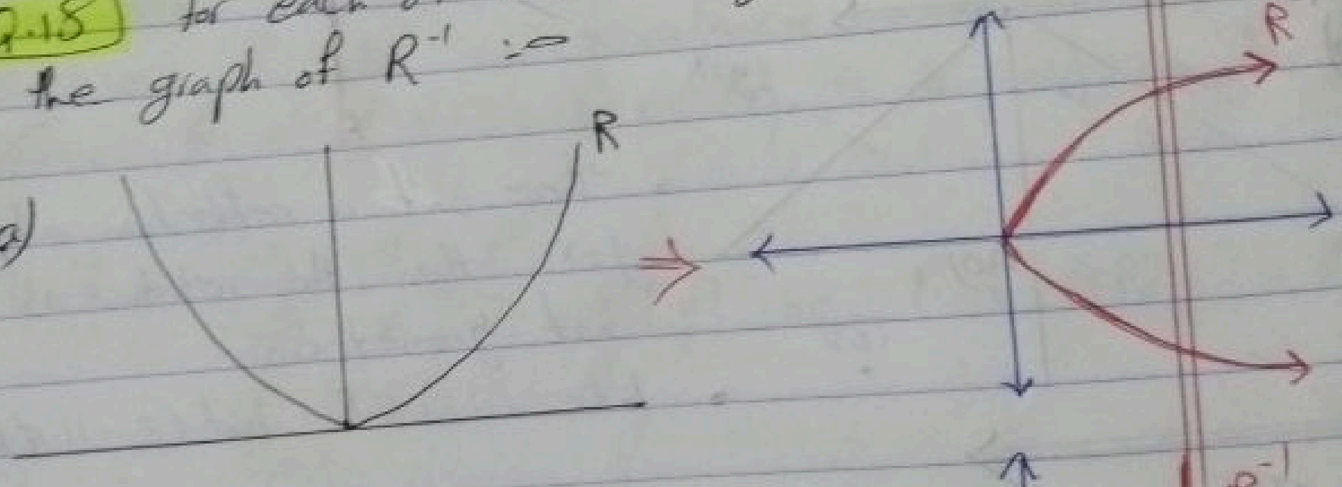
d) The circle with center $(\sqrt{2}, \sqrt{2})$ and radius 2 :-



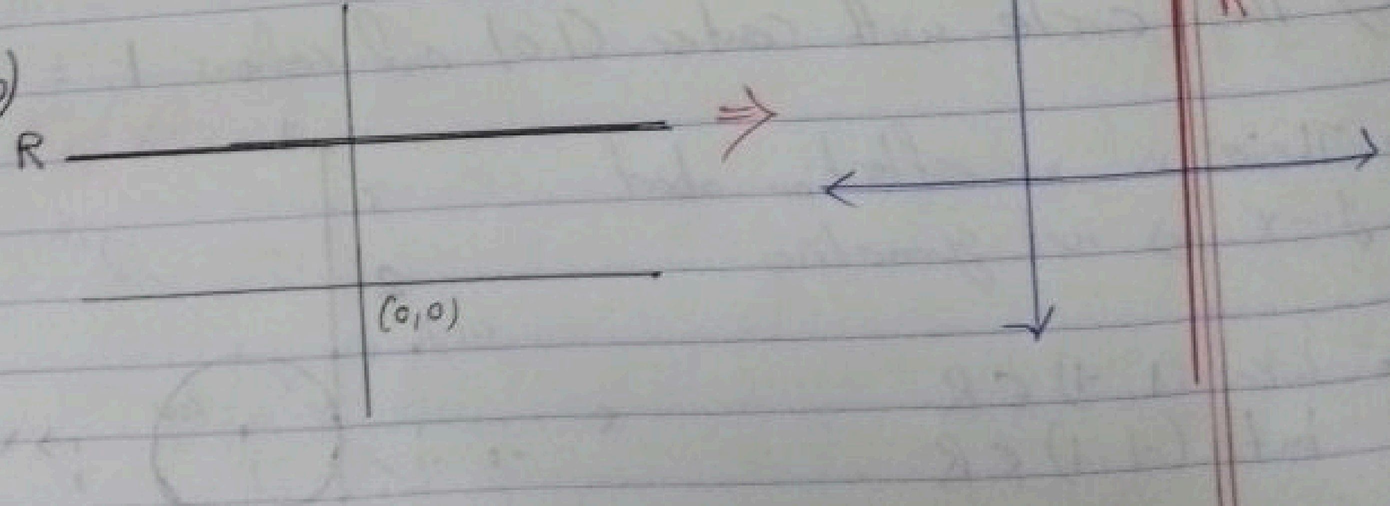
- There is a reflection about $y=x$, then the relation is symmetric

Q.18 For each of the following relations R, sketch the graph of R^{-1} :-

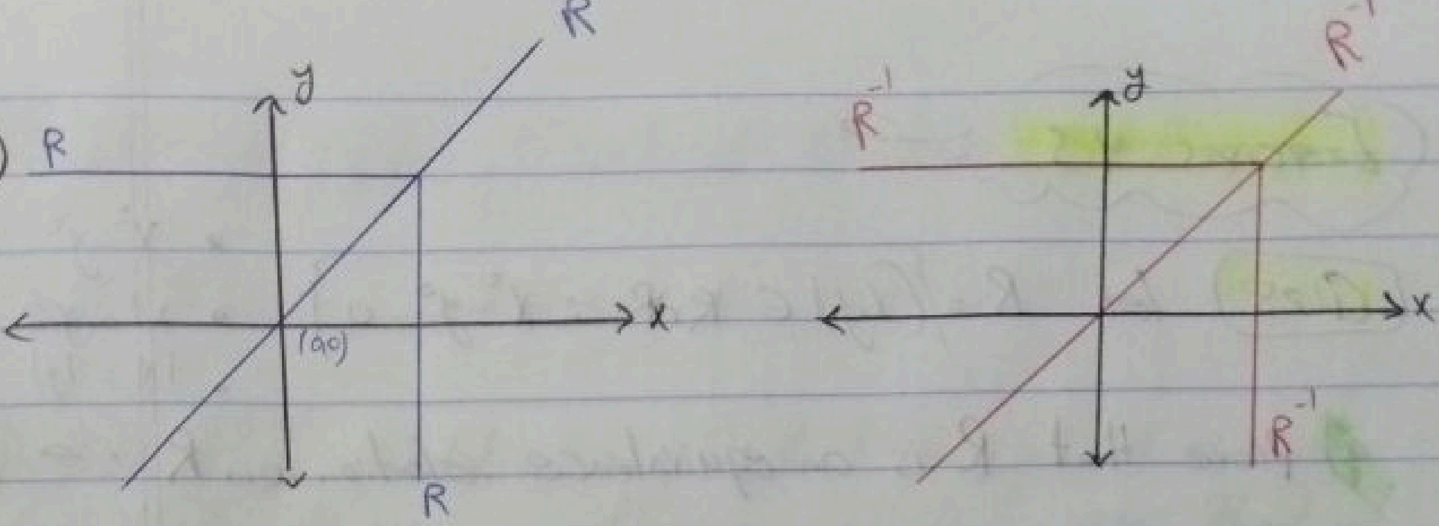
a)



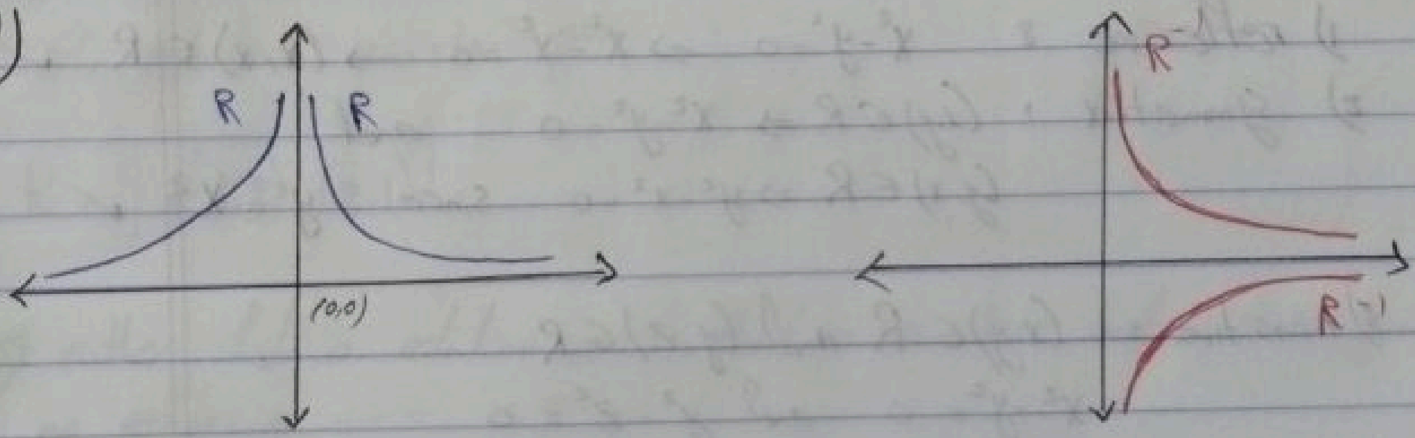
b)



c)



d)



Exercises 4.3 :-

Q.28 let $R = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x^2 - y^2 = 0\}$

- $x^2 - y^2 = 0$
- $x^2 = y^2$
- $|x| = |y|$

a) Prove that R is an equivalence relation on R :-

1) reflexive : $x^2 - y^2 = 0 \Rightarrow x^2 - x^2 = 0 \rightarrow (x,x) \in R$ ✓

2) symmetrix : $(x,y) \in R \Rightarrow x^2 - y^2 = 0$ and
$(y,x) \in R \Rightarrow y^2 - x^2 = 0$ since $y^2 = x^2$ ✓

3) transitive : $(x,y) \in R$ and $(y,z) \in R$
$x^2 - y^2 = 0$ and $y^2 \cdot z^2 = 0$
$\Rightarrow x^2 - z^2 = 0 \Rightarrow (x,z) \in R$ ✓

Hence, R is an equivalence relation

b) list all members of [3] :-

$[3] = \{3, -3\}$

Q.30 let $A = \{1, 2, 3, 4\}$:-

a) either find a relation that is reflexive on A but neither symmetric nor transitive or prove that no such relation exists :-

$R = \{(1,1), (1,2), (2,2), (3,3), (4,4), (2,3)\}$

is reflexive, neither symmetric nor transitive.

b) Either find a relation on A that is symmetric but neither transitive nor reflexive on A or prove that no such relation exists :-

$R = \{(1,3)\ (3,1)\}$

c) either find a relation on A that is transitive but neither symmetric no reflexive --- :-

$R = \{(1,3), (3,1), (1,1)\}$

d) either find a relation that is reflexive on A and symmetric but not transitive --- :-

$R = \{(1,1), (2,2), (3,3), (4,4)\}$

e) find a relation that is reflexive on A and transitive but not symmetric :-

No such relation exist since all reflexive is symmetric.

f) find a relation on A that is symmetric and transi
but not reflexive :-

$R = \{(1,1), (1,4), (4,1)\}$

Q.34 let $\mathbb{R}$ be the set of all real numbers and consider the following subsets of $R \times R$:-

$R_1 = \{(x,y) \in R \times R : xy = 0\}$

$R_2 = \{(x,y) \in R \times R : |x-y| < 5\}$

$R_3 = \{(x,y) \in R \times R : xy \neq 0\}$

$R_4 = \{(x,y) \in R \times R : x \geq y\}$

$R_5 = \{(x,y) \in R \times R : x^2 + y^2 = 1\}$

which of the above is ① reflexive on R, ② symm. ③ trans.

	reflexive	Symmetric	transitive	equivalence. relation
R_1	X	✓	X	X
R_2	✓	✓	X	X
R_3	X	✓	✓	X
R_4	✓	X	✓	X
R_5	X	✓	X	X

Q.35 for each real number x, let $f(x) = x^2$. For any two real numbers a and b, define $a \simeq b$ provided that $f(a) = f(b)$. Prove that $\simeq$ is an equivalence relation on $\mathbb{R}$ and list all the members of $[-7]$:-

$\Rightarrow f(x) = x^2, \quad a \simeq b, \quad f(a) = f(b)$

* reflexive : $(x,x) \Rightarrow x^2 = x^2, \quad f(x) = f(x)$ ✓

* Symmetric : $(x,y) \Rightarrow f(x) = f(y) \Rightarrow x^2 = y^2$ and

$(y,x) \rightarrow f(y) = f(x) \Rightarrow y^2 = x^2$ ✓

* transitive : (x,y) and (y,z) , $f(x)=f(y)$ and $f(y)=f(z)$
$\Rightarrow f(x) = f(z)$, $(x,z) \in \mathbb{R}$ ✓

* $[-7] = \{49\}$

(Q. 36) a) Prove that $\simeq$ is an equivalence relation on $\mathbb{R}\times\mathbb{R}$

* $[(a,b),(a,b)] \in \simeq \Rightarrow a^2+b^2 = a^2+b^2 \Rightarrow$ reflexive
* $[(a,b),(c,d)] \in \simeq \Rightarrow a^2+b^2 = c^2+d^2$
$\Rightarrow c^2+d^2 = a^2+b^2$
$\Rightarrow [(c,d),(a,b)] \in \simeq \Rightarrow$ symmetric
* $[(a,b),(c,d)] \in \simeq$ and $[(c,d),(e,f)] \in \simeq$
$\Rightarrow a^2+b^2 = c^2+d^2$ and $c^2+d^2 = e^2+f^2$
$\Rightarrow a^2+b^2 = e^2+f^2 \Rightarrow [(a,b),(e,f)] \in \simeq \Rightarrow$ transitive

Hence, $\simeq$ is equivalence relation.

b) list all the members of $[(0,0)]$:-

$R[(0,0)] = \{(0,0)\}$

c) Give a geometric description of $[(5,11)]$

$x^2+y^2 = 146$

Q. 39 ① reflexive : $((x,y),(x,y)) \in S$ since $xy = yx$ $\forall\, x,y \in \mathbb{N}$

② symmetric : suppose $(x,y),(z,r) \in S \Rightarrow xr = yz$
$\Rightarrow zy = rx \Rightarrow ((z,r),(x,y)) \in S$

③ transitive : suppose $((x,y),(z,r)) \in S$ and $((z,r),(e,f)) \in S$
$\Rightarrow xr = yz$ and $zf = re \Rightarrow r = \frac{zf}{e}$
$\Rightarrow x\left(\frac{zf}{e}\right) = yz \Rightarrow xf = ye \Rightarrow ((x,y),(e,f)) \in S$ ✓

$*$ $R[(6,8)] = \{(3,4),(6,8),(9,12),(12,16)\}$

Q. 43 :- let R and S be equivalence relations on a set X :-

a) Is $R \cup S$ reflexive on X ??

- let $a \in X$, $\Rightarrow (a,a) \in R$ and $(a,a) \in S$
since R, S reflexive on X.
- $\Rightarrow (a,a) \in R\cup S \Rightarrow$ reflexive

b) Is $R\cup S$ symmetric ??

let $(x,y) \in R\cup S \Rightarrow (x,y) \in R$ or $(x,y) \in S$
$\Rightarrow (y,x) \in R$ or $(y,x) \in S$
$\Rightarrow (y,x) \in R\cup S \Rightarrow$ symmetric

c) Is RUS transitive ??

let $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$S = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

$R \cup S = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

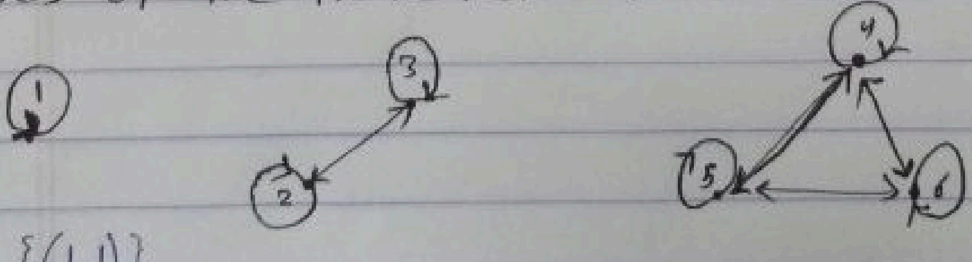
$(1,2) \in R \cup S$ and $(2,3) \in R \cup S$ but $(1,3) \notin R \cup S$

$\Rightarrow$ not transitive

$\boxed{Q.50}$ let $S = \{1,2,3,4,5\}$ and let $\mathcal{A} = \{\{1,2\},\{3\},\{4,5\}\}$ list all ~~members~~ ordered pairs of the equivalence relation promised by Theorem 4.5 :-

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (4,5), (5,4)\}$

$\boxed{Q.51}$ The accompanying digraph represents an equivalance relation on $S = \{1,2,3,4,5,6\}$. list the equivalence classes of the indicated relation :



$A_1 = \{(1,1)\}$

$A_2 = \{(2,2), (2,3), (3,2), (3,3)$

$A_3 = \{ (4,4), (5,5), (6,6), (4,5), (5,4), (4,6), (6,4)$
$(5,6), (6,5) \}$

* $[1] = \{1\}$
* $[2] = \{2,3\} = [3]$
* $[4] = [5] = [6] = \{4,5,6\}$

$R = \{ \{1\}, \{2,3\}, \{4,5,6\} \}$