Chapter 5: Transfere Analysis of LTI systems

$$x[n] \longrightarrow \underset{}{\overset{h[n]}{\boxed{LTI}}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$e^{jwn} \longrightarrow \overset{H(e^{jw})}{\boxed{LTI}} \longrightarrow H(e^{jw})\, e^{jwn}$$

$$Y(e^{jw}) = X(e^{jw}) \cdot \underbrace{H(e^{jw})}_{\text{frequency Response}}$$

$$Y(z) = X(z)\, \underbrace{H(z)}_{\text{transfer function}}$$

* frequency response of LTI systems:-

$$|Y(e^{jw})| = |X(e^{jw})| \cdot \underbrace{|H(e^{jw})|}_{\text{Magnitude response}}$$

(gain) → amplification
→ Attenuation

$$\angle Y(e^{jw}) = \angle X(e^{jw}) + \underbrace{\angle H(e^{jw})}_{\text{phase response}}$$

phase response
or phase shift

in real life systems always negative

*Ideal frequency selective filters:-

Ideal low pass filter:

$$H_{Lp}(e^{jw}) = \begin{cases} 1 & , |w| < w_c \\ 0 & , w_c < |w| \le \pi \end{cases}$$

$$h_{Lp}(n) = \frac{\sin w_c n}{\pi n} \quad , \quad -\infty < n < \infty$$

Ideal High pass filter

$$H_{hp}(e^{jw}) = \begin{cases} 0 & , |w| < w_c \\ 1 & , w_c < |w| \le \pi \end{cases}$$

$$h_{Hp}(n) = \delta(n) - \frac{\sin w_c n}{\pi n}$$

*phase distortion and delay:-

ideal delay System: $h_{id}(n) = \delta(n - n_d)$

$$H_{id}(e^{jw}) = 1 e^{-jwn_d}$$

$A e^{j\theta}$ phase

magnituded

$$|H_{id}(e^{jw})| = 1$$

$$\measuredangle H_{id}(e^{jw}) = \underbrace{-wn_d}_{\text{Linear equation} \rightarrow \text{Linear phase}}$$

$\angle H(e^{jw})$ is a function of $\underline{\underline{w}}$

each frequency has its (own) specific delay

so to calculate the average delay of all frequencies

the group delay definition is applied

$$\text{group delay}(\tau(w)) = -\frac{d}{dw}\left(\angle H(e^{jw})\right)$$

The negative slope of phase at a specific frequency

Examples- $h[n] = \delta(n-5)$ <u>ideal delay</u>

$$H(e^{jw}) = 1\, e^{-j5w}$$

$$|H(e^{jw})| = 1$$

$$\angle H(e^{jw}) = -5w$$

linear phase $\Rightarrow$ group delay = constant

$\tau(w) = 5$ samples

Example: $h[n] = \frac{1}{5}\{\underset{\underset{n=0}{\uparrow}}{1}, 1, 1, 1, 1\}$ 5-point Moving Average

$$h(n) = \frac{1}{5}\left(\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)\right)$$

$$H(z) = \frac{1}{5}\left(z^{0} + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)$$

$$H(e^{jw}) = \frac{1}{5}\left(e^{j0} + e^{-jw} + e^{-j2w} + e^{-j3w} + e^{-j4w}\right)$$

2 cos w

$$= \frac{1}{5}\left(e^{j2w} + e^{jw} + \underset{1}{e} + e^{-jw} + e^{-j2w}\right) e^{-j2w}$$

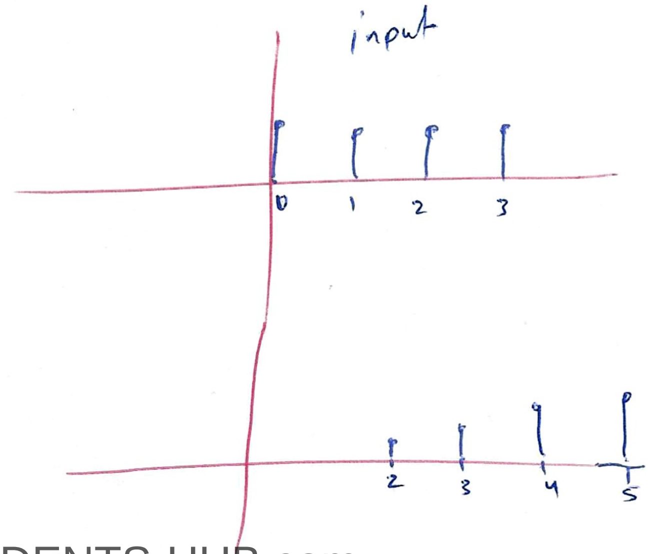
2 cos 2w

$$= \underbrace{\frac{1}{5}\left(1 + 2\cos w + 2\cos(2w)\right)}_{\text{magnitude}} e^{-j2w} \leftarrow \text{phase}$$

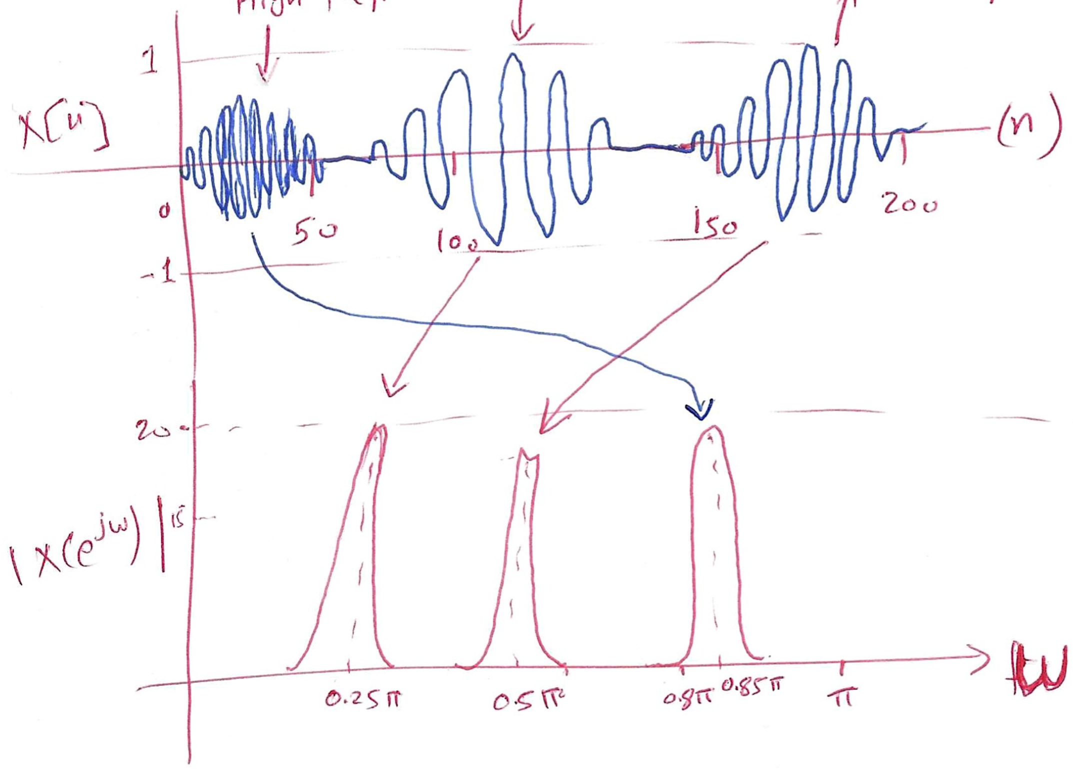
$$A e^{j\theta}$$

$$\measuredangle H(e^{jw}) = -2w$$

$$\tau(w) = 2 \text{ Samples}$$

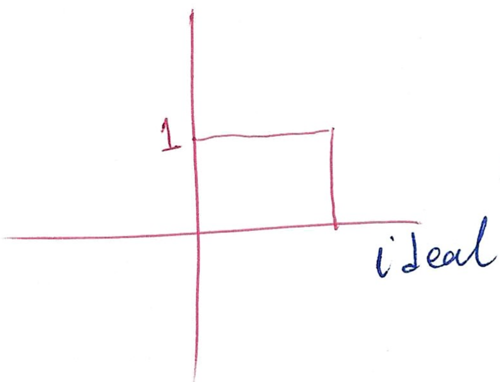


Example :- non linear phase (Example 5.1, page 243



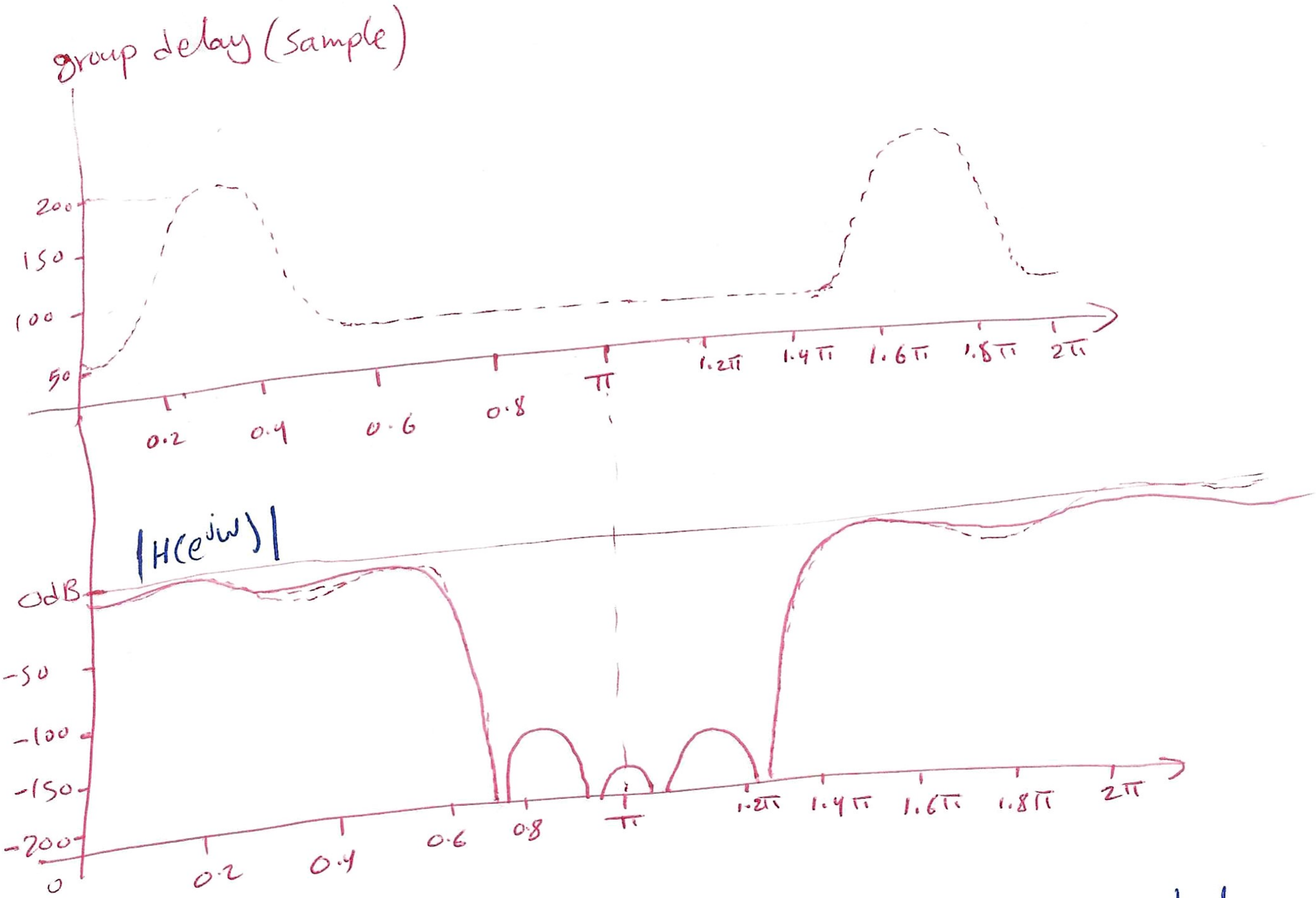
$$\text{decible (dB)} = 20 \log_{10}(|\text{magnitude}|) \quad H(e^{j\omega})$$

$$= 10 \log_{10}(|H(e^{j\omega})|^2)$$



$20 \log_{10}(1)$

$= 0 \text{ dB}$



0dB → means that no change in the magnitude at the output

The cut off frequency is at 0.707 of the magnitude

$20 \log_{10}(0.707) = -3dB$

any signal (frequency) has a magnitude equals or less than ~~-~~3dB will be attenuated

This is why the High frequency of $X[n]$ at 0.85π will be attenuated. only the mid and the low frequenly will be at the output. The magnitude at the two frequencies will not change The low frequency will be delayed by almost 200 samples and the mid will be delayed by 50 samples so a distortion will occure.

System Functions for systems characterized by LCCDE

Linear constant coefficient difference equation

degree

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

degree

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} \overbrace{(1 - c_k z^{-1})}^{\text{zeros}}}{\prod_{k=1}^{N} \underbrace{(1 - d_k z^{-1})}_{\text{Poles}}}$$

product of factors

Examples- $H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})}$, find the DE ??

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}} = \frac{Y(z)}{X(z)}$$

$$\left(1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}\right) Y(z) = 1 + 2z^{-1} + z^{-2} X(z)$$

$$Y(z) + \frac{1}{4} z^{-1} Y(z) - \frac{3}{8} z^{-2} Y(z) = X(z) + 2 z^{-1} X(z) + z^{-2} X(z)$$

The difference equation is

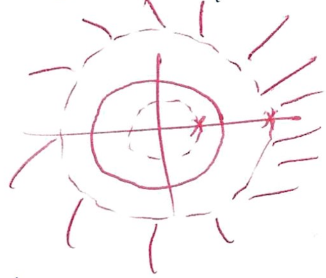
$$y[n] + \frac{1}{4} y[n-1] - \frac{3}{8} y[n-2] = x[n] + 2x[n-1] + x[n-2]$$

* stability and Causality :-

from RoC not from $H(z)$

Causal :- $h[n]$ must be Right sided

RoC of $H(z)$ must be outside the outermost pole

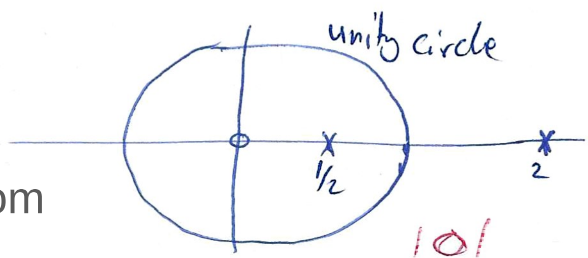


stable: RoC must include the unit circle.

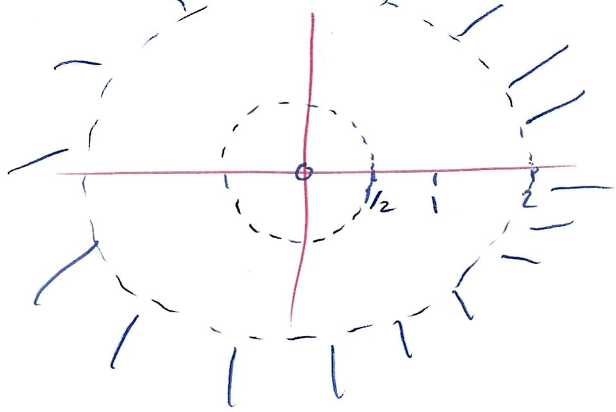
Example:- Determine RoC of the system described by

$$y[n] = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} \qquad y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$

$$H(z) = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - 2 z^{-1})}$$

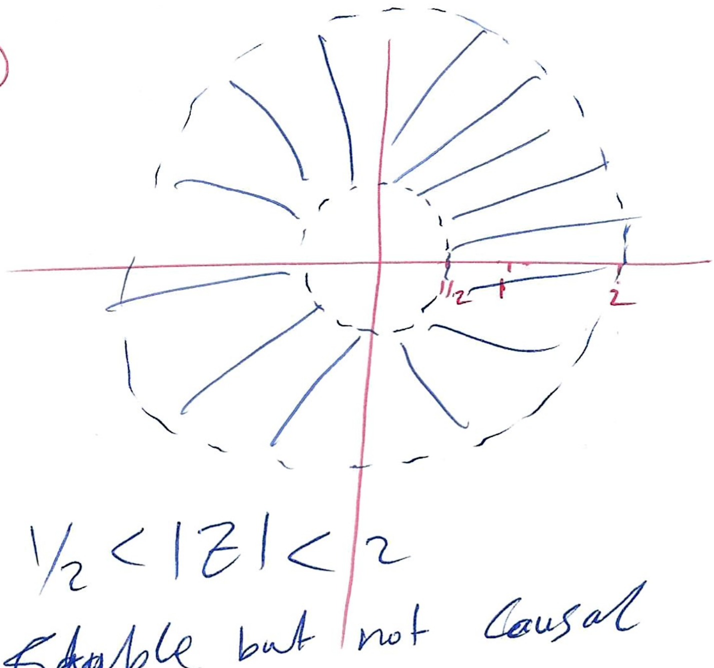


RoC ① Causal but not stable



$|Z| > 2$

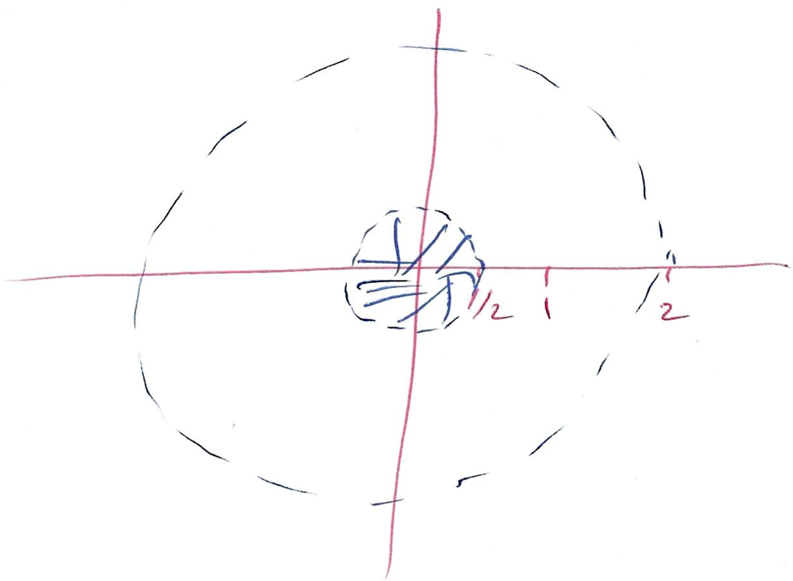
②



$1/2 < |Z| < 2$

Stable but not Causal

③ $|Z| < 1/2$ not Causal and not stable



* Inverse system:

$$H_i(z) \qquad x[n] \to \underbrace{\boxed{h_1[n]} \to \boxed{h_2[n]}}_{h_1 * h_2 = \delta[n]} \to y[n]$$

$$H(z) \cdot H_i(z) = 1$$

$$h[n] * h_i[n] = \delta(n)$$

$$H_i(z) = \frac{1}{H(z)} \quad \text{and} \quad H_i(e^{jw}) = \frac{1}{H(e^{jw})}$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-k})}{\prod_{k=1}^{N} (1 - d_k z^{-k})}$$

$$H_i(z) = \frac{a_0}{b_0} \frac{\prod_{k=1}^{N} (1 - d_k z^{-k})}{\prod_{k=1}^{M} (1 - c_k z^{-k})}$$

poles of $H(z)$ are zeros for $H_i(z)$ and vice versa

what about RoC ??

$$RoC: \; RoC_{system} \cap RoC_{inverse}$$

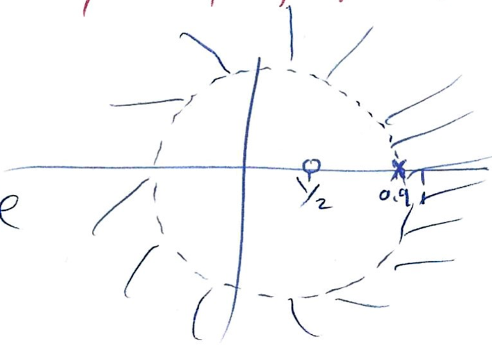
RoC of $H_i(z)$ must overlap with RoC of $H(z)$

If RoC of $H_i(z)$ is not overlapping with RoC of $H(z)$ then $H_i(z)$ deos not exist (the system does not have any inverse)

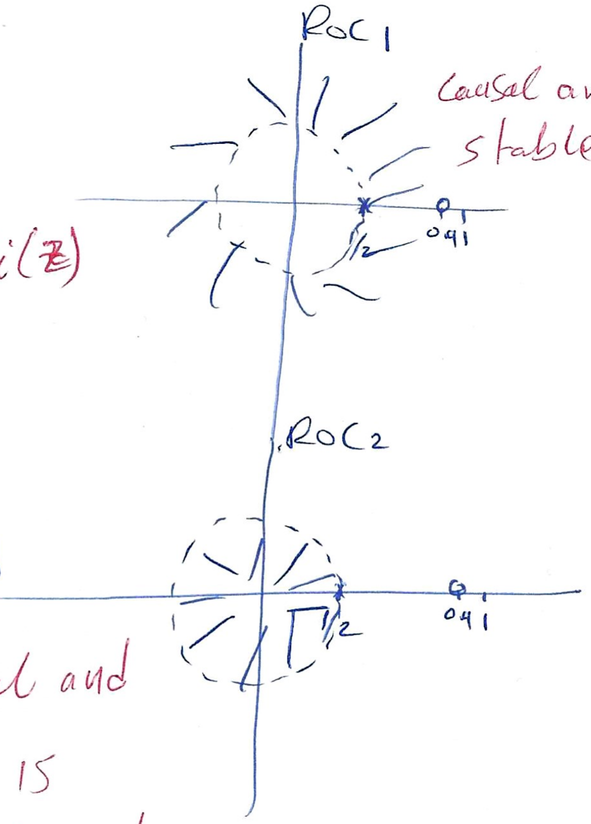
Examples- Find the inverse and Roc of the following system

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}} , \quad |z| > 0.9$$

Causal and stable



$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$



Causal and stable

Roc_1 overlaps with Roc of $H(z)$ so it is the valid Roc of $H_i(z)$

$$H_i(z) = \frac{1}{1-0.5z^{-1}} - \frac{0.9z^{-1}}{1-0.5z^{-1}}$$

$$h_i(n) = (0.5)^n u[n] - 0.9\left[(0.5)^{n-1} u[n-1]\right]$$

If $H(z)$ the system is causal and stable and its inverse is causal and stable, we call such system minimum phase system

poles and zeros inside the unity circle

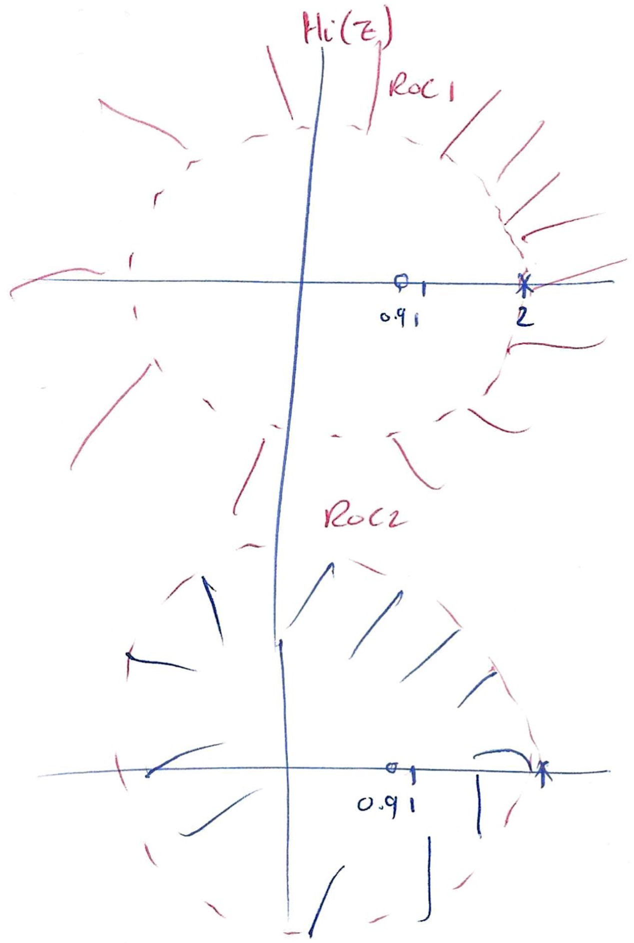
phase response of such system ≈ 0

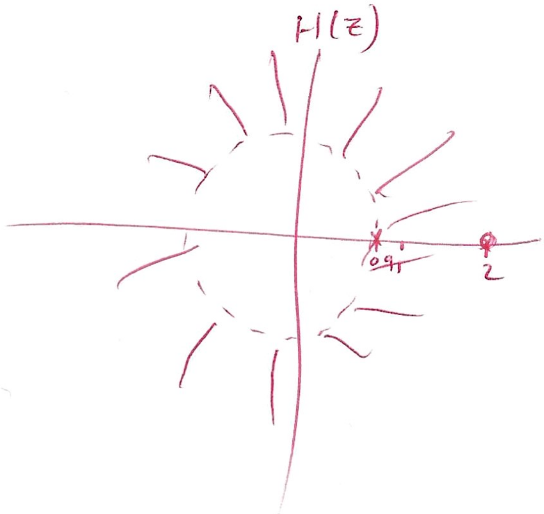
Example :

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}} \quad , \quad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$$

$$\frac{z - 0.9}{1 - 0.5z}$$





Both Roc1 and Roc2 of $H_i(z)$ are overlapped with Roc of $H(z)$, so both are valid solution

$$H_i(z) = \frac{-2}{1 - 2z^{-1}} + \frac{1.8z^{-1}}{1 - 2z^{-1}}$$

if $|z| > 2$

$$h_i[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1]$$

if $|z| < 2$

$$h_i[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-(n-1)-1]$$

$u[-n]$