4.3: Transformations of variables of the continuous type.

Thm: change of variable (1 variable, continuous type).

let X be a r.v with space $\mathcal{A}$ and p.d.f $f(x)$.

let $y = u(x)$ be a r.v with space B and p.d.f $g(y)$.

such that $u: \mathcal{A} \to B$, $u(x) = y$ is a one to one function then

$$g(y) = \begin{cases} f(w(y)) \cdot |w'(y)| & , y \in B \\ 0 & , \text{elsewhere} \end{cases}$$

Where $w(y) = x = u^{-1}(y)$.

expl: X is a r.v with p.d.f

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

let $y = 8x^3$. Find the p.d.f of Y.

Solution:

r.v X p.d.f $f(x)$

r.v Y : p.d.f $g(y)$

$Y = u(x) = 8x^3 \to y = u(x) = 8x^3 \to$ is one to one function.

$u: \mathcal{A} \to B$

$\mathcal{A}: \{x: 0 < x < 1\}$. $\to$ $0<x<1 \to 0 < x^3 < 1 \to 0 < 8x^3 < 8$

$0 < y < 8$

A صورة $B: \{y: 0 < y < 8\}$

$u: 1\text{-}1$

$y = 8x^3 \to \frac{y}{8} = x^3 \to x = \frac{1}{2} y^{\frac{1}{3}}$

$\Rightarrow u^{-1}(y) = w(y) = x = \frac{1}{2} y^{\frac{1}{3}}$.

$$\Rightarrow g(y) = \begin{cases} f(w(y))\,|\bar{w}(y)| & , y \in B \\ 0 & , \text{elsewhere.} \end{cases}$$

$\bar{w}(y) = \frac{1}{2} \cdot \frac{1}{3} y^{\frac{1}{3}-1} = \frac{1}{6y^{\frac{2}{3}}}$

$$= \begin{cases} f\left(\frac{1}{2} y^{\frac{1}{3}}\right) \left|\frac{1}{6} y^{\frac{-2}{3}}\right| & , 0 < y < 8 \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} y^{\frac{1}{3}} \cdot \frac{1}{6} y^{-\frac{2}{3}} & , 0 < y < 8 \\ 0 & , \text{else} \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{6y^{\frac{1}{3}}} & , 0 < y < 8 \\ 0 & , \text{else} \end{cases}$$

Thm: Change of variable (2 variables, continuous type).

let X_1, X_2 be r.v's with space A and joint p.d.f $f(x_1, x_2)$

let y_1, y_2 be r.v's with space B and joint p.d.f $g(y_1, y_2)$.

such that $u: A \to B$ where u is one to one and

$u(x_1, x_2) = (Y_1, Y_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ then

$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) \cdot |J| & , y_1, y_2 \in B \\ 0 & , \text{elsewhere.} \end{cases}$$

where $w_1(y_1, y_2) = x_1 = u_1^{-1}(y_1, y_2)$ and $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

$w_2(y_1, y_2) = x_2 = u_2^{-1}(y_1, y_2)$

exp 5: X_1, X_2 random sample of size $n=2$ from standard Normal dist.

$X_1 \sim N(0,1)$

$X_2 \sim N(0,1)$

X_1, X_2 indep.

Find dist. of $Y_1 = \frac{X_1}{X_2}$.

solution:

$\rightarrow f(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}}$, $x_1 \in \mathbb{R}$ " p.d.f of X_1 "

$\rightarrow f(x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}$, $x_2 \in \mathbb{R}$ " p.d.f of X_2 "

$\rightarrow h(x_1, x_2) = f(x_1) \cdot f(x_2)$, $(x_1, x_2) \in \mathbb{R}^2$ " joint p.d.f of X_1, X_2 ".

$= \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}}$, $(x_1, x_2) \in \mathbb{R}^2$.

$\rightarrow y_1 = \frac{x_1}{x_2} \iff x_1 = y_1 x_2 = y_1 y_2$

$y_2 = x_2 \iff x_2 = y_2$.

[illegible]

$\rightarrow A = \{(x_1, x_2) : -\infty < x_1 < \infty, -\infty < x_2 < 0 \text{ or } 0 < x_2 < \infty\}$.

$B = \{(y_1, y_2) : -\infty < y_1 < \infty, -\infty < y_2 < 0 \text{ or } 0 < y_2 < \infty\}$.

$$\rightarrow J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

$\rightarrow$

$$\rightarrow g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2))\ |J| & , (y_1, y_2) \in B \\ 0 & , \text{elsewhere.} \end{cases}$$

$$= \begin{cases} f(y_1 y_2, y_2)\ |y_2| & , (y_1, y_2) \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi} e^{-\frac{y_1^2 y_2^2 + y_2^2}{2}} \underline{|y_2|} & , (y_1, y_2) \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{y_2}{2\pi} e^{-\frac{y_1^2 y_2^2 + y_2^2}{2}} & , y_1 \in \mathbb{R}, y_2 > 0 \\ \frac{-y_2}{2\pi} e^{-\frac{y_1^2 y_2^2 + y_2^2}{2}} & , y_1 \in \mathbb{R}, y_2 < 0. \\ 0 & , \text{else} \end{cases}$$

$$\rightarrow g_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2 = \int_0^{\infty} \frac{y_2}{2\pi} e^{-\frac{y_1^2 y_2^2 + y_2^2}{2}} dy_2 + \int_{-\infty}^{0} \frac{-y_2}{2\pi} e^{-\frac{y_1^2 y_2^2 + y_2^2}{2}} dy_2$$

$$g_1(y_1) = \frac{1}{\pi(1 + y_1^2)} \quad , y_1 \in \mathbb{R}.$$

" check the result ".

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