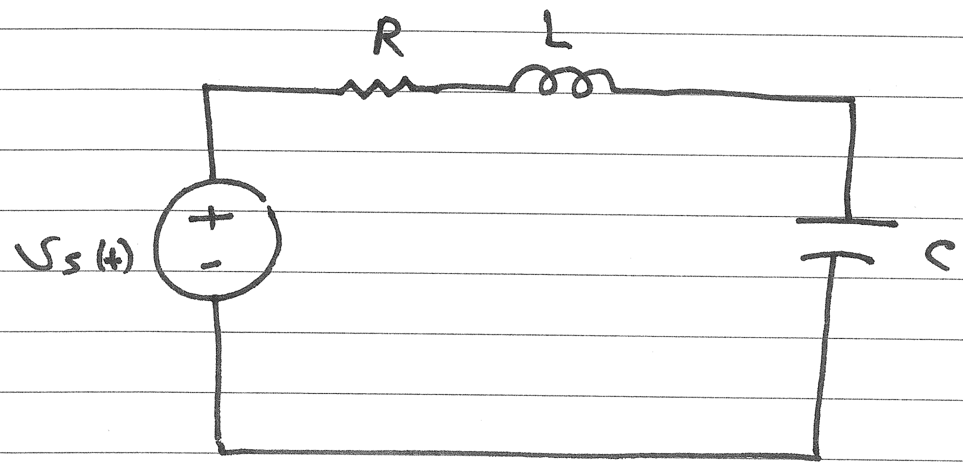
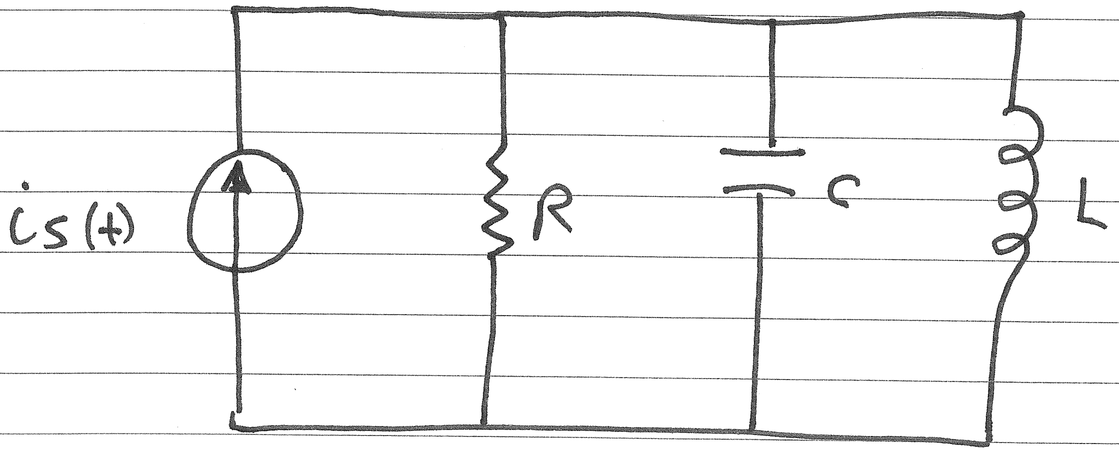
Chapter 8

Natural and Step Response of RLC Circuits

What is a 2nd order Circuit ?

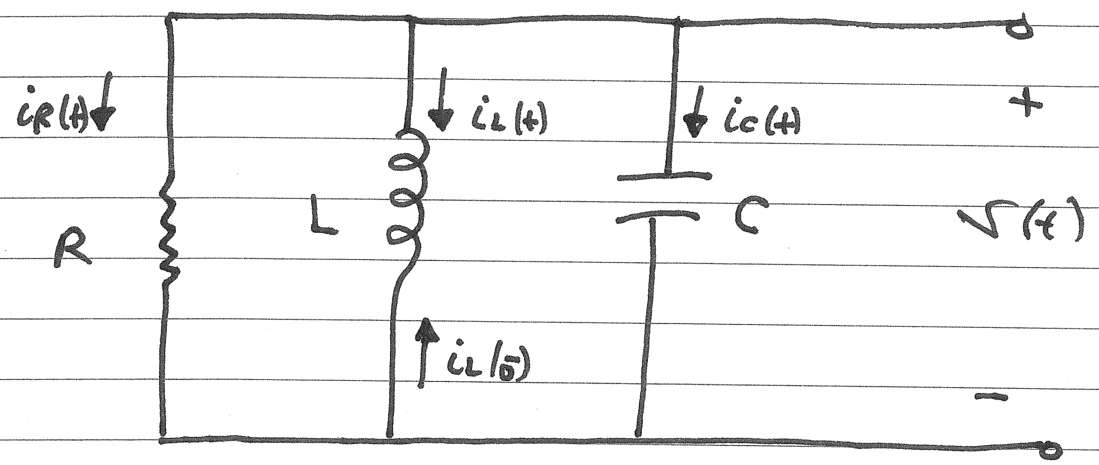
A second-order circuit is characterized by a second-order differential equation.





Natural Response of Parallel RLC Circuit

For $t > 0$



$v_C(0^-) = 0 \quad ; \quad i_L(0^-) = 10A$

KCL :

$$i_R(t) + i_L(t) + i_C(t) = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt - i_L(0^-) + C\frac{dv(t)}{dt} = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt + C\frac{dv(t)}{dt} = i_L(0^-) \quad \text{— (1)}$$

Differentiate (1)

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

second-order homogeneouse differential equation

$$\therefore \; v(t) = A e^{st} \quad \text{for} \quad t > 0$$

$$CAs^2 e^{st} + \frac{1}{R} s A e^{st} + \frac{1}{L} A e^{st} = 0$$

$$A e^{st} \left(Cs^2 + \frac{1}{R} s + \frac{1}{L} \right) = 0$$

$$\therefore \; Cs^2 + \frac{1}{R} s + \frac{1}{L} = 0$$

Characteristic equation

$$s_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Let $w_0 = \dfrac{1}{\sqrt{LC}}$

$w_0 \equiv$ resonant frequency

and

$$\alpha = \frac{1}{2RC}$$

$\alpha \equiv$ damping Coefficient

$\therefore$

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

1) If $\alpha > \omega_0$, the solutions are real, unequal and the response is termed overdamped.

2) If $\alpha < \omega_0$, the solutions are complex conjugates and the response is termed Underdamped.

3) If $\alpha = \omega_0$, the solutions are real and equal and the response is termed Critically damped.

1) The overdamped Case

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

If $\alpha > \omega_0$

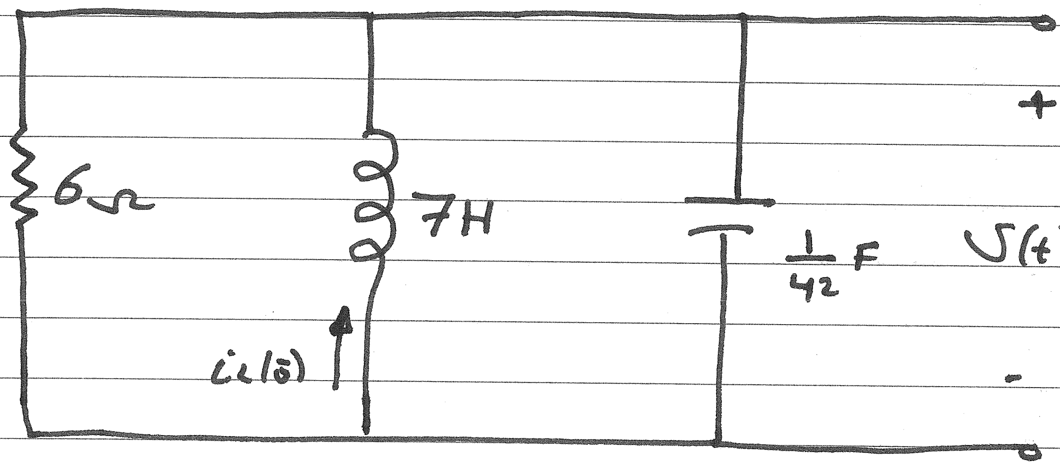
$$\therefore \quad S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} < 0$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} < 0$$

S_1, S_2 are real, unequal

$$\therefore \quad v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} \qquad t > 0$$

Overdamped parallel RLC



$v_c(0^-) = 0$, and $i_L(0^-) = 10A$

$$\alpha = \frac{1}{2RC} = 3.5$$

$$W_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} = 2.45$$

$\therefore \alpha > W_0$ overdamped

$$S_1 = -\alpha + \sqrt{\alpha^2 - W_0^2} = -1$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - W_0^2} = -6$$

$\therefore v(t) = A_1 e^{-t} + A_2 e^{-6t}$ for $t > 0$

To find A_1, and A_2, we need

$v(0^+)$ and $\frac{dv(0^+)}{dt}$

For $t > 0$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt + C\frac{dv(t)}{dt} = i_L(0^-)$$

at $t = 0^+$

$$\frac{v(0^+)}{R} + \frac{1}{L}\int_{0^-}^{0^+} v(t)\,dt + C\frac{dv(0^+)}{dt} = i_L(0^-)$$

$$v(0^+) = v_c(0^+) = v_c(0^-) = 0$$

$$\int_{0^-}^{0^+} v(t)\,dt = 0$$

$\therefore$ $$i_L(0^-) = C\frac{dv(0^+)}{dt}$$

$$\frac{dv(0^+)}{dt} = \frac{i_L(0^-)}{C} = 420$$

also $v(0^+) = 0 = v_c(0^-)$

$$v(t) = A_1 e^{-t} + A_2 e^{-6t} \quad \text{for } t>0$$

$$v(0^+) = A_1 + A_2 = v(0^-) = 0$$

$$\therefore A_1 + A_2 = 0 \quad \text{——— } ②$$

$$v(t) = A_1 e^{-t} + A_2 e^{-6t}$$

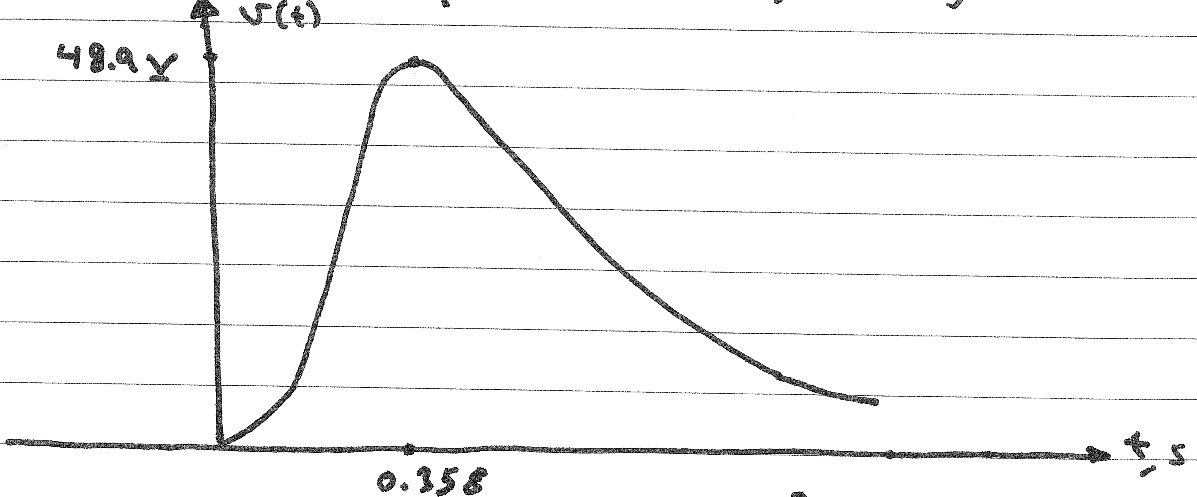
$$\frac{dv(t)}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$$

$$\frac{dv(0^+)}{dt} = -A_1 - 6A_2 = 420 \quad \text{——— } ③$$

Solving ② and ③, we get

$$A_1 = 84 \quad \text{and} \quad A_2 = -84$$

$$\therefore v(t) = 84\left(e^{-t} - e^{-6t}\right) \text{ V} \quad \text{for } t>0$$



2) Critical Damping Case

$$\alpha = \omega_0$$

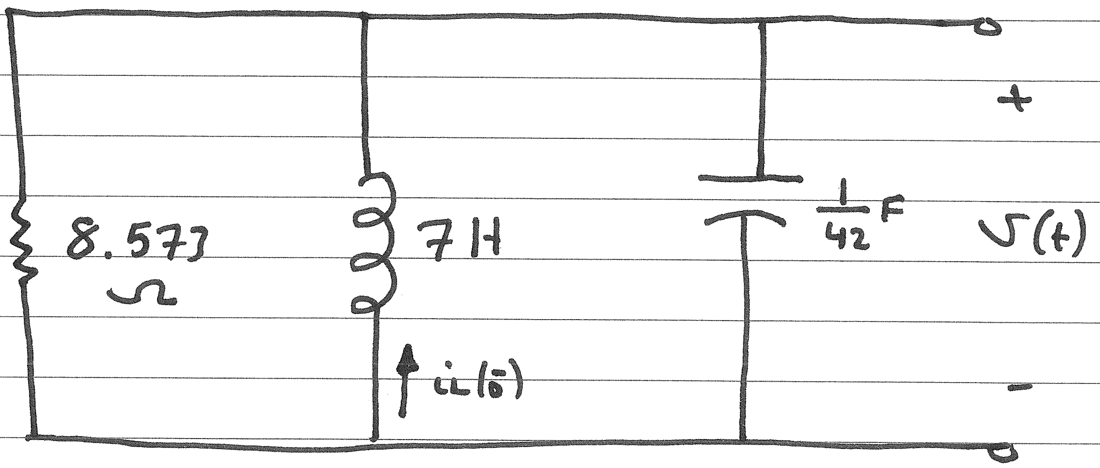
$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha$$

$S_1 = S_2$ real and equal

$$\therefore\ v(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t} \quad \text{for } t>0$$

Critical Damped Parallel RLC



$v_c(0^-) = 0$, and $i_L(0^-) = 10\,A$

$$\alpha = \frac{1}{2RC} = \sqrt{6}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$s_1 = -\sqrt{6}$$

$$s_2 = -\sqrt{6}$$

$$\therefore \; v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \quad \text{for } t>0$$

$$v(0^+) = 0$$

$$\frac{dv(0^+)}{dt} = 420$$

$$v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \quad \text{for } t > 0$$

$$v(0^+) = A_2 = 0$$

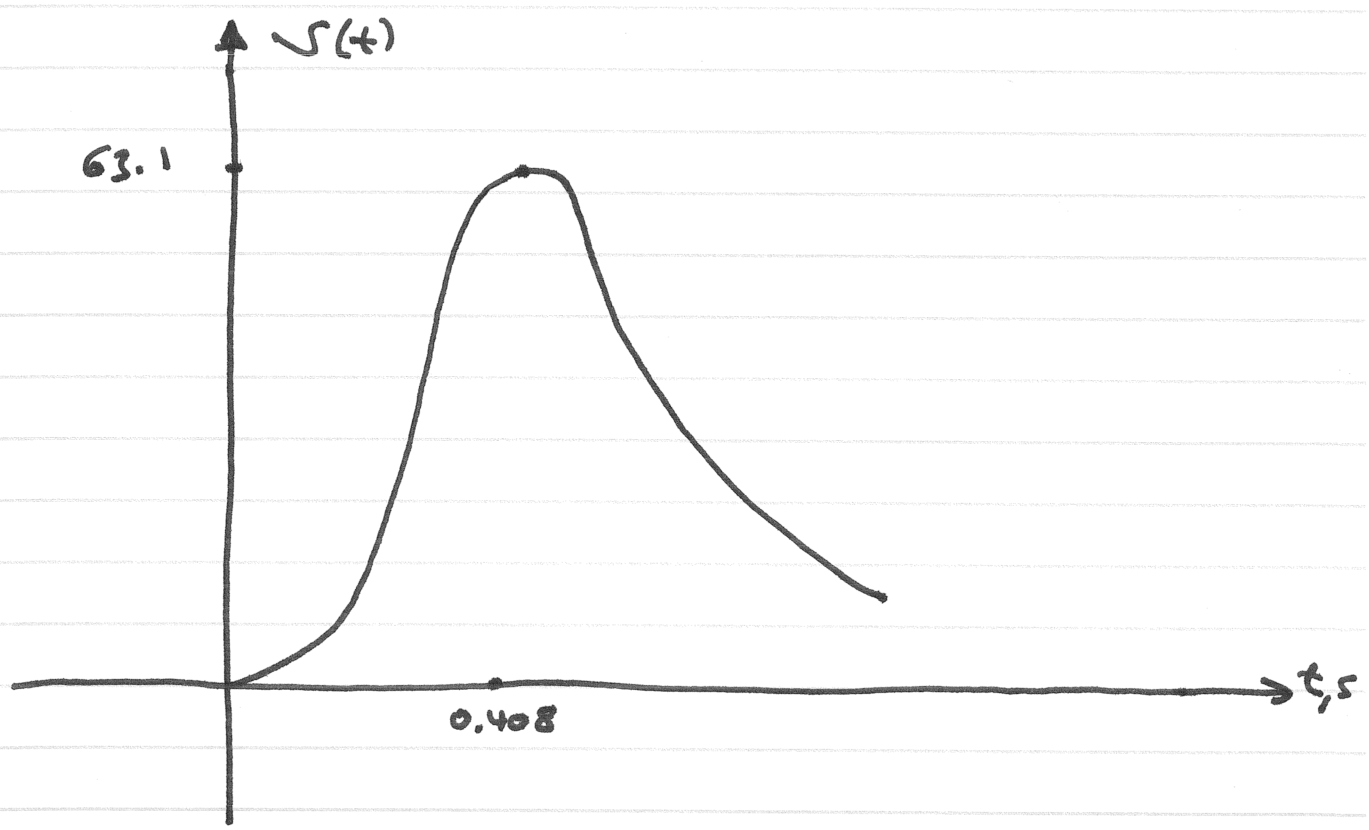
$$\therefore v(t) = A_1 t e^{-\sqrt{6}t} \quad \text{for } t > 0$$

$$\frac{dv(t)}{dt} = (A_1 t)(-\sqrt{6}) e^{-\sqrt{6}t} + A_1 e^{-\sqrt{6}t}$$

$$\frac{dv(0^+)}{dt} = 0 + A_1 = 420$$

$$\therefore A_1 = 420$$

$$\therefore v(t) = 420 t e^{-\sqrt{6}t} \ \underline{V} \quad \text{for } t > 0$$



3) The underdamped Case

$$\alpha < \omega_0$$

$$\therefore \alpha^2 - \omega_0^2 < 0$$

$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{(-1)(\omega_0^2 - \alpha^2)}$$

$$\sqrt{\alpha^2 - \omega_0^2} = j\sqrt{\omega_0^2 - \alpha^2}$$

$$\sqrt{\alpha^2 - \omega_0^2} = j\,\omega_d$$

$\omega_d \equiv$ damped radian frequency

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -\alpha + j\,\omega_d$$

$$s_2 = -\alpha - j\,\omega_d$$

s_1, and s_2 are Complex Conjucate

$$S_1 = -\alpha + jW_d$$

$$S_2 = -\alpha - jW_d$$

$$\therefore V(t) = A_1 e^{(-\alpha + jW_d)t} + A_2 e^{(-\alpha - jW_d)t}$$

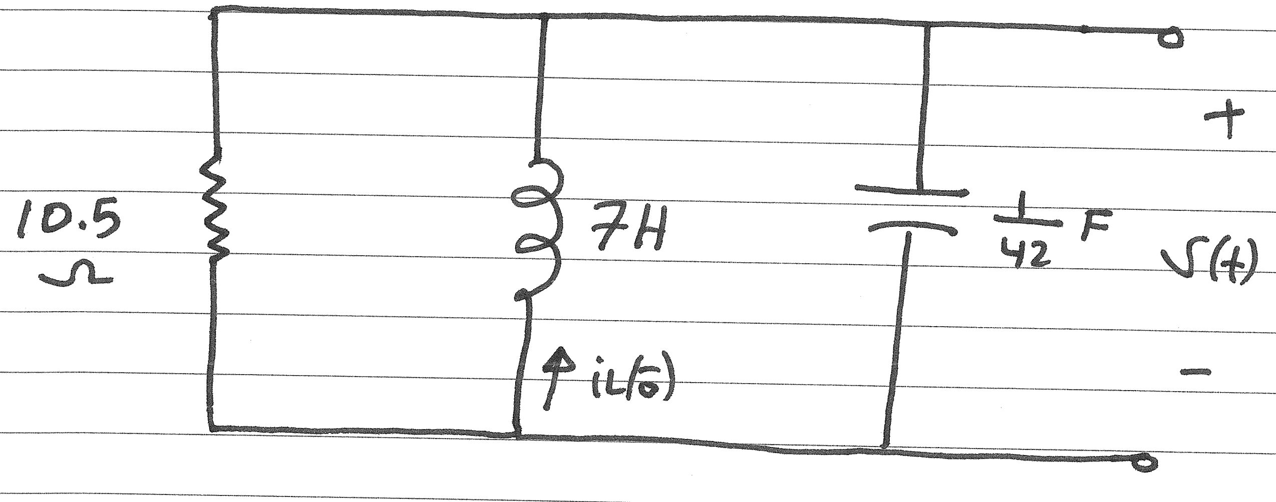
$$e^{jW_d t} = \cos W_d t + j\sin W_d t$$

$$e^{-jW_d t} = \cos W_d t - j\sin W_d t$$

$$V(t) = e^{-\alpha t}\left[(A_1 + A_2)\cos W_d t + j(A_1 - A_2)\sin W_d t\right]$$

$$\therefore V(t) = e^{-\alpha t}\left[B_1 \cos W_d t + B_2 \sin W_d t\right]$$

Underdamped Parallel RLC



$v_c(0^-) = 0$, and $i_L(0^-) = 10A$

$$\alpha = \frac{1}{2RC} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} =$$

$\therefore \quad \alpha < \omega_0$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}$$

$$\therefore v(t) = e^{-\alpha t}\left(\beta_1 \cos \omega_d t + \beta_2 \sin \omega_d t\right) \quad for\ t>0$$

$$\therefore v(t) = e^{-2t}\left(\beta_1 \cos\sqrt{2}\,t + \beta_2 \sin\sqrt{2}\,t\right) \quad for\ t>0$$

$$v(0^+) = 0$$

$$\frac{dv(0^+)}{dt} = 420$$

$$v(t) = e^{-2t}\left(B_1 \cos\sqrt{2}\, t + B_2 \sin\sqrt{2}\, t\right)$$

$$v(0^+) = B_1 = 0$$

$$\therefore B_1 = 0$$

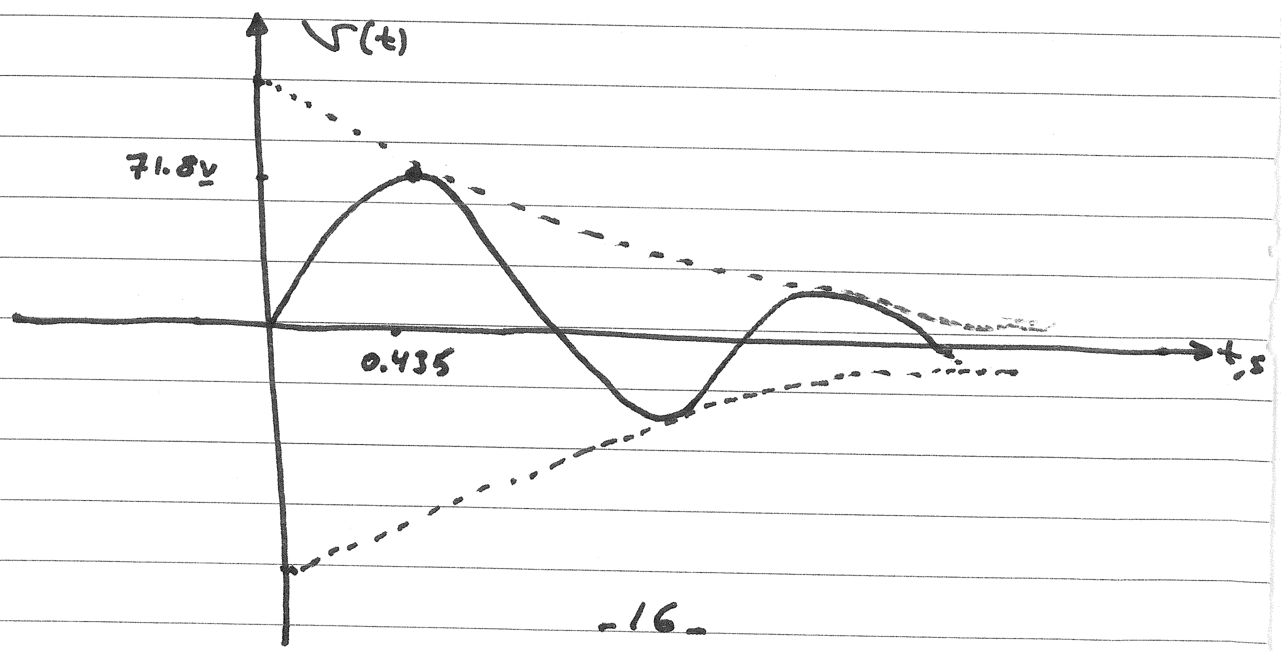
$$\therefore v(t) = e^{-2t} B_2 \sin\sqrt{6}\, t \ \underline{V} \quad \text{for } t>0$$

$$\frac{dv(t)}{dt} = \left(B_2 e^{-2t}\right)\left(\sqrt{2}\cos\sqrt{2}\, t\right) + \left(\sin\sqrt{2}\right)\left(-2 e^{-2t} B_2\right)$$

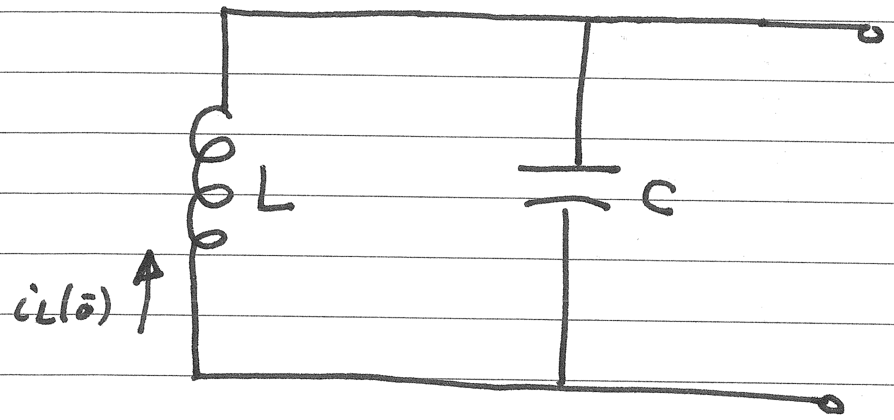
$$\frac{dv(0^+)}{dt} = \sqrt{2}\, B_2 + 0 = 420$$

$$\therefore B_2 = \frac{420}{\sqrt{2}}$$

$$\therefore v(t) = \frac{420}{\sqrt{2}} e^{-2t} \sin\sqrt{2}\, t \ \underline{V} \quad \text{for } t>0$$



The Lossless LC Circuit



$L = 7H$, $C = \frac{1}{42}F$, and $R = \infty$

$$\alpha = \frac{1}{2RC} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$\alpha < \omega_0$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{6}$$

$\therefore\ v(t) = B_1 \cos\sqrt{6}\,t + B_2 \sin\sqrt{6}\,t$ for $t > 0$

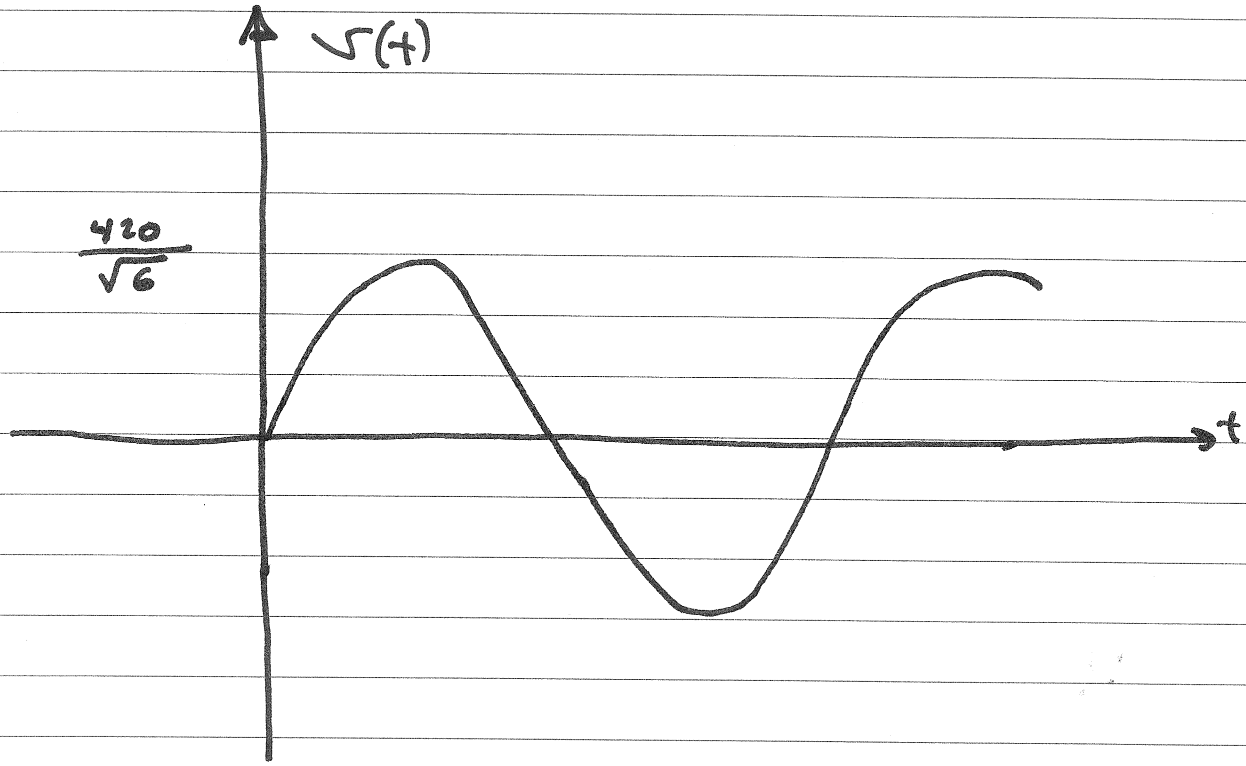
$v(0^+) = 0$, and $\frac{dv(0^+)}{dt} = 420$

$$v(0^+) = B_1 = 0$$

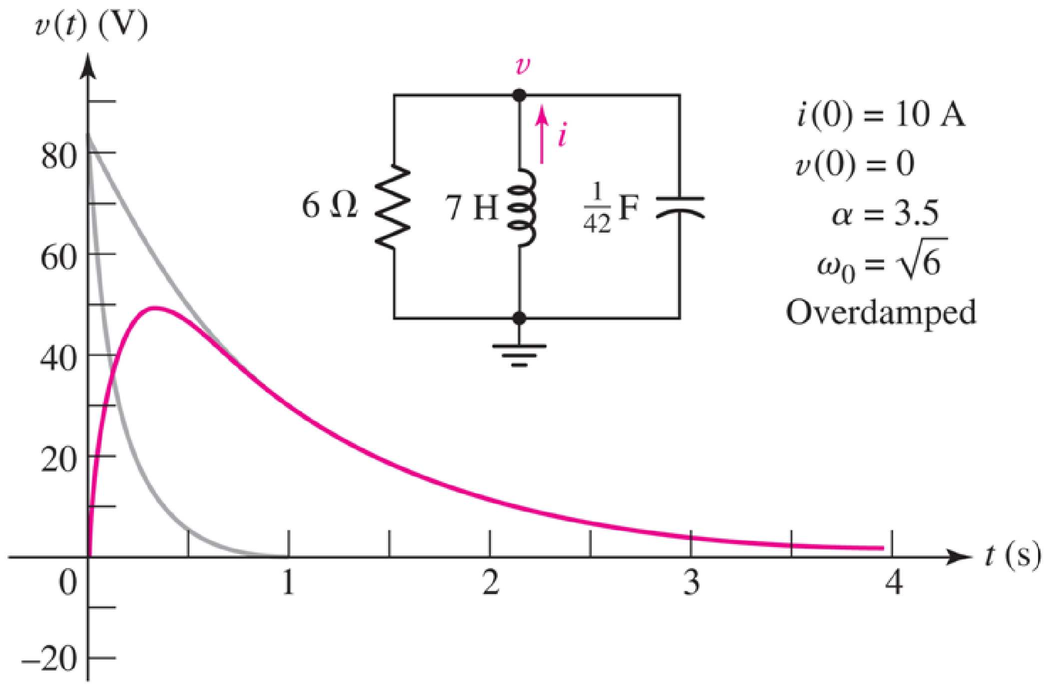
$$\frac{dv(0^+)}{dt} = \sqrt{6}\,B_2 = 420$$

$$\therefore\ B_2 = \frac{420}{\sqrt{6}}$$

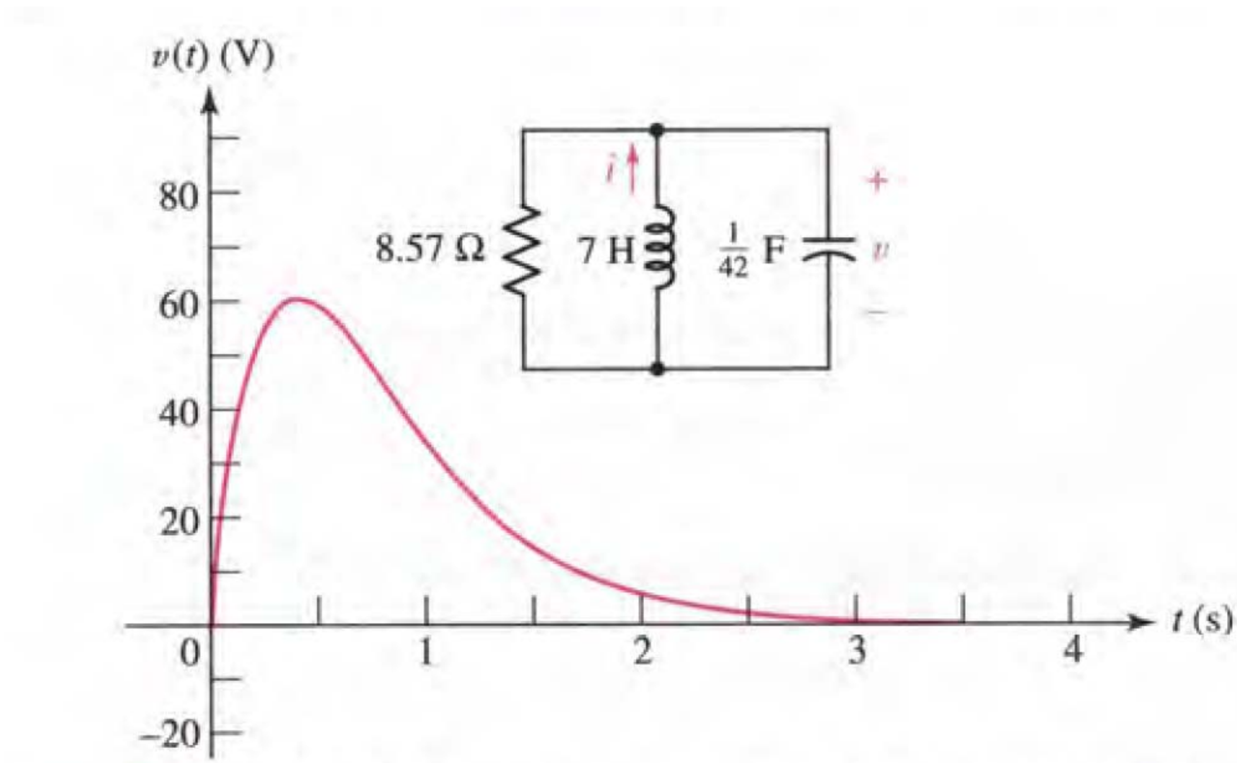
$$v(t) = \frac{420}{\sqrt{6}} \sin \sqrt{6}\, t \ \text{V} \quad \text{for } t>0$$



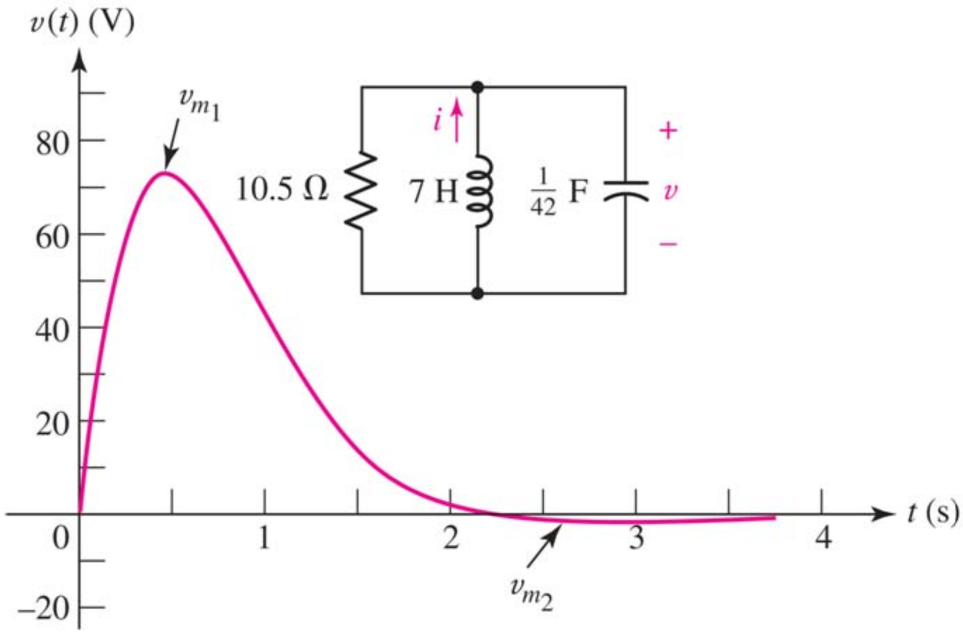
Over damped case



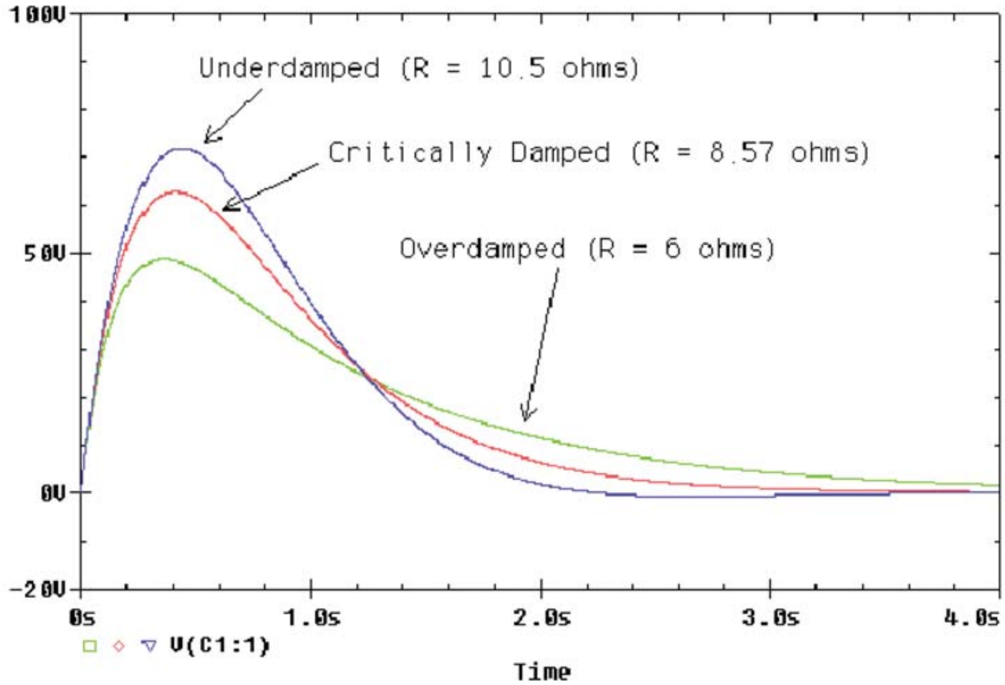
Critical damped case



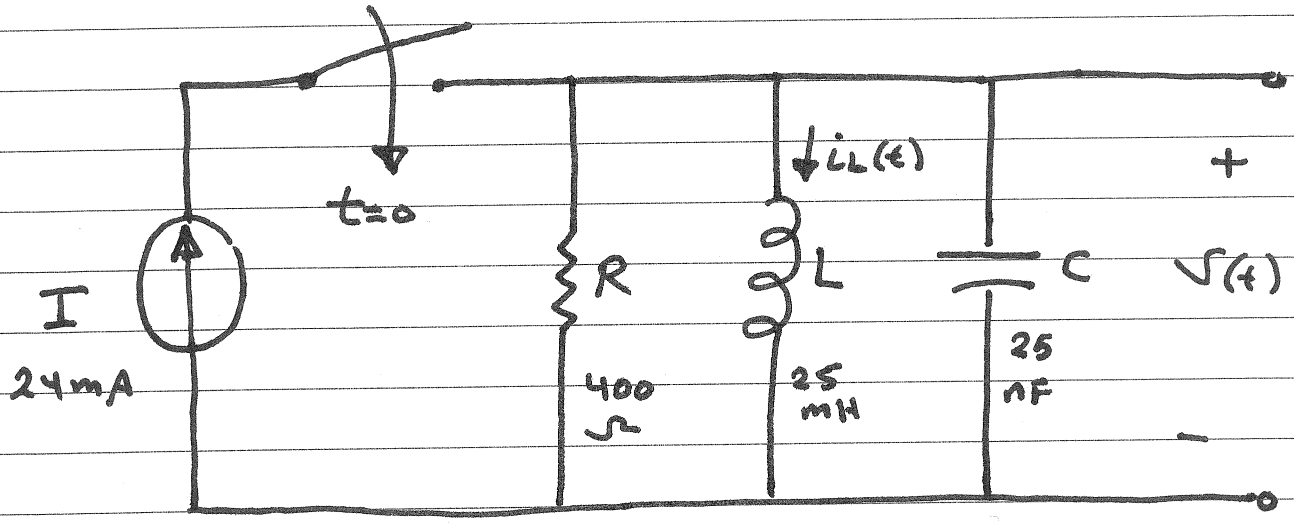
Under damped case



Comparing the Responses



Step response of Parallel RLC Circuit

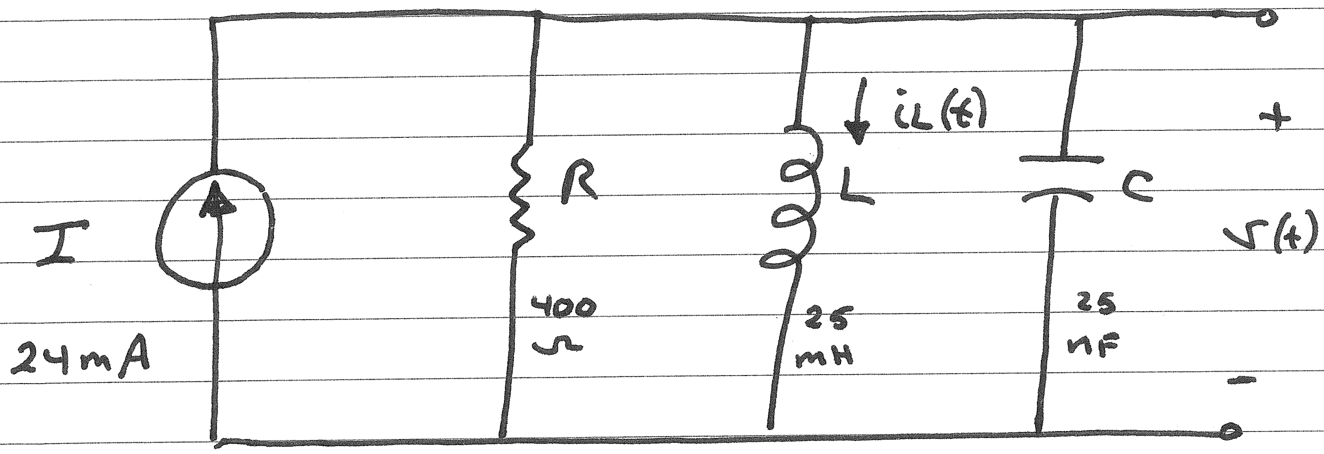


$i_L(0^-) = 0$, and $v_C(0^-) = 0$

Find $v(t)$ for $t > 0$

Find $i_L(t)$ for $t > 0$

For $t > 0$



$i_L(0^-) = 0$, and $v_C(0^-) = 0$, Find $i_L(t)$

KCL :

$$I = i_R(t) + i_L(t) + i_C(t)$$

$$I = \frac{v(t)}{R} + i_L(t) + C\frac{dv(t)}{dt}$$

$$v(t) = v_L(t) = L\frac{di_L(t)}{dt}$$

$$\therefore \quad I = LC\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t)$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{I}{LC}$$

second order nonhomogeneouse diff. equation

$i_L(t) = i_n(t) + i_f(t)$

$i_n(t)$ = natural response

$i_f(t)$ = forced response

To find $i_f(t)$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{I}{LC}$$

let $i_f(t) = K$

$$0 + 0 + \frac{1}{LC} i_f(t) = \frac{I}{LC}$$

$$\therefore \quad i_f(t) = I = K$$

To find $i_n(t)$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$S_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$S_1 = -20000$$

$$S_2 = -80000$$

Since S_1, S_2 are real and unequal

$\therefore$ we have overdamped case

$$\therefore \; i_{Ln}(t) = A_1 e^{-20000t} + A_2 e^{-80000t}$$

$$\therefore \; i_L(t) = i_{Lf}(t) + i_{Ln}(t)$$

$$i_L(t) = 24mA + A_1 e^{-20000t} + A_2 e^{-80000t} \qquad t>0$$

To find A_1, and A_2, we need $i_L(0^+)$ and $\frac{di_L(0^+)}{dt}$

$$i_L(0^+) = i_L(0^-) = 0$$

$$v_C(t) = v_L(t) = L\frac{di_L(t)}{dt}$$

$$\therefore v_C(0^+) = L\frac{di_L(0^+)}{dt} = v_C(0^-) = 0$$

$$\therefore \frac{di_L(0^+)}{dt} = 0$$

$$i_L(t) = 24mA + A_1 e^{-20000t} + A_2 e^{-80000t} \quad t>0$$

$$i_L(0^+) = 24mA + A_1 + A_2$$

$$\therefore A_1 + A_2 = -24mA \quad \text{———} \quad (1)$$

$$\frac{di_L(0^+)}{dt} = -20000A_1 - 80000A_2 = 0 \quad \text{———} \quad (2)$$

Solving (1) and (2), we get

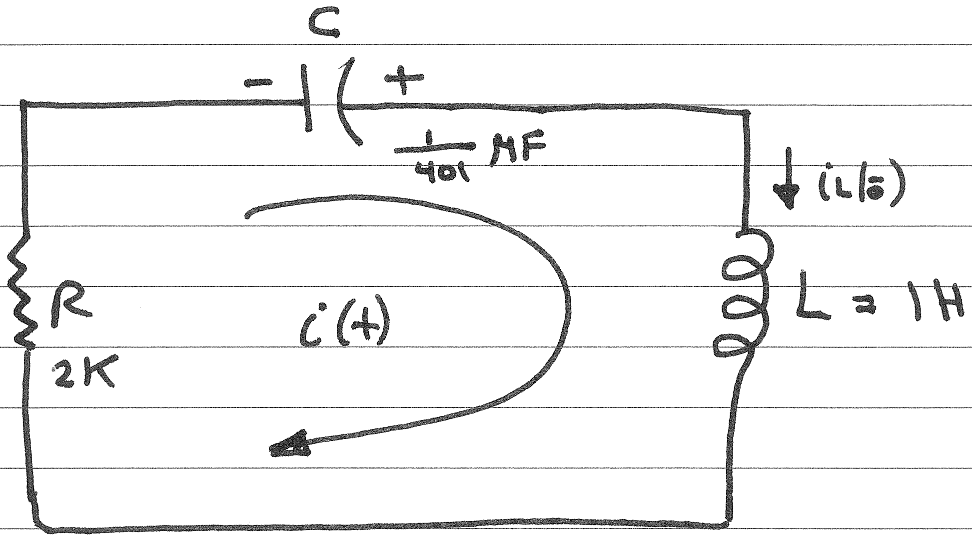
$$A_1 = -32mA$$

$$A_2 = 8mA$$

$$\therefore i_L(t) = \left(24 - 32e^{-20000t} + 8e^{-80000t}\right)mA, \text{ for } t>0$$

Natural Response of Series RLC Circuit

For $t > 0$



$v_C(0^-) = V_0$, and $i_L(0^-) = I_0$

Find $i(t)$ for $t > 0$

KVL :

$$L\frac{di(t)}{dt} + Ri(t) - v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt = 0$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt = v_C(0^-) \qquad \text{— ①}$$

Differentiation of ①

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

second order homogeneouse diff. equation.

$$Ls^2 + Rs + \frac{1}{C} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

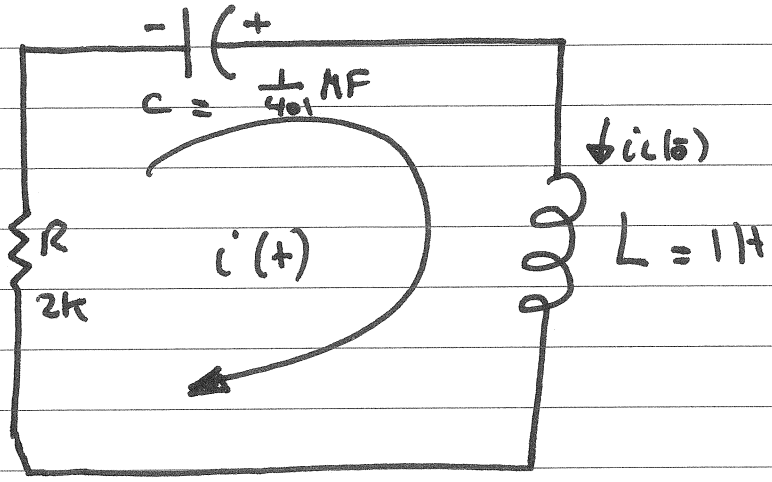
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $\alpha = \frac{R}{2L}$

$w_0 = \frac{1}{\sqrt{LC}}$

$$\therefore \quad s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2}$$



let $v_c(0^-) = V_0 = 2\underline{V}$

$i_L(0^-) = I_0 = 2mA$

$$\alpha = \frac{R}{2L} = 1000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 20025$$

Since $\alpha < \omega_0$

$\therefore$ We have underdamped Case

$\therefore \omega_d = \sqrt{\omega_0^2 - \alpha^2} = 20000$

$\therefore i(t) = e^{-\alpha t}\left(\beta_1 \cos\omega_d t + \beta_2 \sin\omega_d t\right)$ for $t>0$

$\therefore i(t) = e^{-1000t}\left(\beta_1 \cos 20{,}000t + \beta_2 \sin 20{,}000t\right)$ for $t>0$

To find B_1 and B_2, we need to have $i(0^+)$ and $\frac{di(0^+)}{dt}$

$$i(0^+) = i_L(0^+) = i_L(0^-) = 2mA$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{0^-}^{t} i(t)dt - v_c(0^-) = 0$$

at $t = 0^+$

$$L\frac{di(0^+)}{dt} + Ri(0^+) + 0 - v_c(0^-) = 0$$

$$\therefore \frac{di(0^+)}{dt} = \frac{v_c(0^-) - Ri(0^+)}{L} = -2$$

$$i(t) = e^{-1000t}\left(\beta_1 \cos 20000t + \beta_2 \sin 20000t\right)$$

$$i(0^+) = \beta_1 = 2mA$$

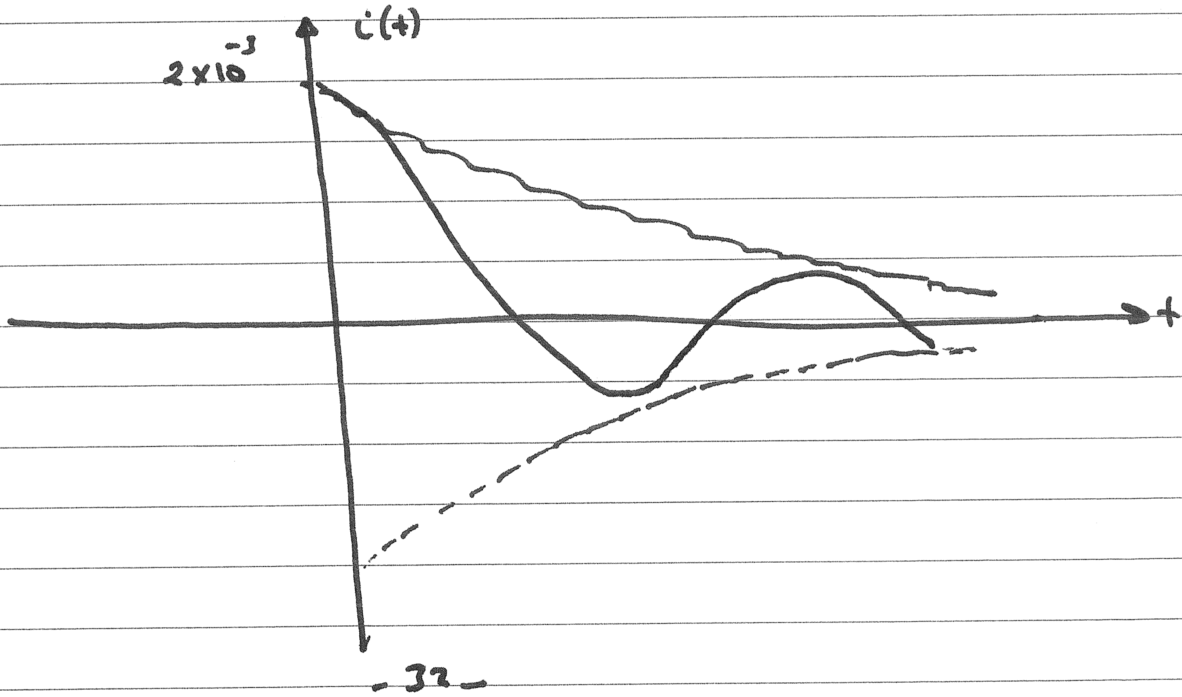
$$\therefore \quad \beta_1 = 2\ mA$$

$$\frac{di(0^+)}{dt} = 20000\,\beta_2 - 2\times10^{-3}(1000)$$

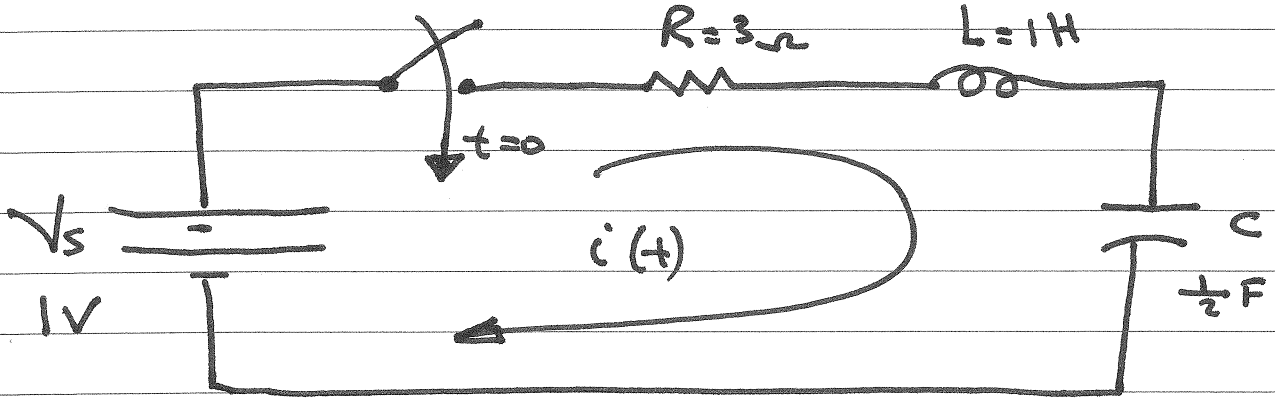
$$\frac{di(0^+)}{dt} = 20000 - 2 = -2$$

$$\therefore \quad \beta_2 = 0$$

$$\therefore \quad i(t) = 2\, e^{-1000t} \cos 20000t \quad mA, \quad for\ t>0$$



Step response of series RLC Circuit



$v_c(0^-) = 0 \quad , \quad i_L(0^-) = 0$

Find $i(t)$ for $t > 0$

KVL :

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + v_c(0^-) + \frac{1}{c}\int_{0^-}^{t} i(t)\,dt$$

$$0 = L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{c}\,i(t)$$

$\therefore\ i(t) = i_n(t)$

$$0 = LS^2 + RS + \frac{1}{c}$$

$$0 = S^2 + 3S + 2$$

$\therefore\ S_1 = -1 \quad , \quad S_2 = -2$

S_1, S_2 are real and unequal

$\therefore$ overdamped Case

$$\therefore\ i(t) = A_1 e^{-t} + A_2 e^{-2t} \quad \text{for } t > 0$$

or

$$\alpha = \frac{R}{2L} = 1.5$$

$$w_0 = \frac{1}{\sqrt{LC}} = \sqrt{2}$$

$$\alpha > w_0 \longrightarrow \text{overdamped Case}$$

To find A_1 and A_2

$$i(0^+) = i(0^-) = 0$$

$$V_s = R i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{0^-}^{t} i(t) + V_c(0^-)$$

<u>at $t=0^+$</u>

$$V_s = R i(0^+) + L\frac{di(0^+)}{dt} + 0 + 0$$

$$\therefore\ \frac{di(0^+)}{dt} = \frac{V_s - R i(0^+)}{L} = \frac{V_s}{L} = \frac{1}{1} = 1$$

$i(0^+) = 0$

$\frac{di(0^+)}{dt} = 1$

$i(t) = A_1 e^{-t} + A_2 e^{-2t}$ for $t > 0$

$i(0^+) = A_1 + A_2 = 0$ ———— ①

$\frac{di(0^+)}{dt} = -A_1 - 2A_2 = 1$ ———— ②

Solving ① and ②

$A_1 = 1 \quad , \quad A_2 = -1$

$\therefore i(t) = \left(e^{-t} - e^{-2t}\right) \underline{\underline{A}}$, for $t > 0$

$v_c(t) = v_c(0^-) + \frac{1}{c}\int_{0^-}^{t} i(t)\,dt$

$v_c(t) = \left(1 - 2e^{-t} + e^{-2t}\right) \underline{\underline{V}}$

for $t > 0$

To find $v_c(t)$

$$V_s = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$$

$$i(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$V_s = RC\frac{dv_c(t)}{dt} + LC\frac{d^2v_c(t)}{dt^2} + v_c(t)$$

second order nonhomogeneouse diff. equation

$$\therefore \quad v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_{cf}(t) = K$$

$$K = V_s$$

$$\therefore \quad v_c(t) = V_s + v_{cn}(t)$$

$$0 = LCs^2 + RCs + 1$$

$$0 = \frac{1}{2}s^2 + \frac{3}{2}s + 1$$

$$\therefore \quad S_1 = -1$$

$$S_2 = -2$$

$$v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_c(t) = A_1 e^{-t} + A_2 e^{-2t} + 1 \qquad \text{for } t > 0$$

To find A_1, A_2

$$v_c(0^+) = v_c(0^-) = 0$$

$$i(t) = i_L(t) = i_C(t) = C\frac{dv_c(t)}{dt}$$

$$\therefore\ i_L(0^+) = i_C(0^+) = C\frac{dv_c(0^+)}{dt} = 0$$

$$\therefore\ \frac{dv_c(0^+)}{dt} = 0$$

$$\therefore\ v_c(0^+) = 0$$

$$\frac{dv_c(0^+)}{dt} = 0$$

$$v_c(t) = 1 + A_1 e^{-t} + A_2 e^{-2t}$$

$$v_c(0^+) = 1 + A_1 + A_2 = 0$$

$$\therefore\ A_1 + A_2 = -1 \qquad \text{———} \ (1)$$

$$\frac{dv_c(t)}{dt} = -A_1 e^{-t} - 2A_2 e^{-2t}$$

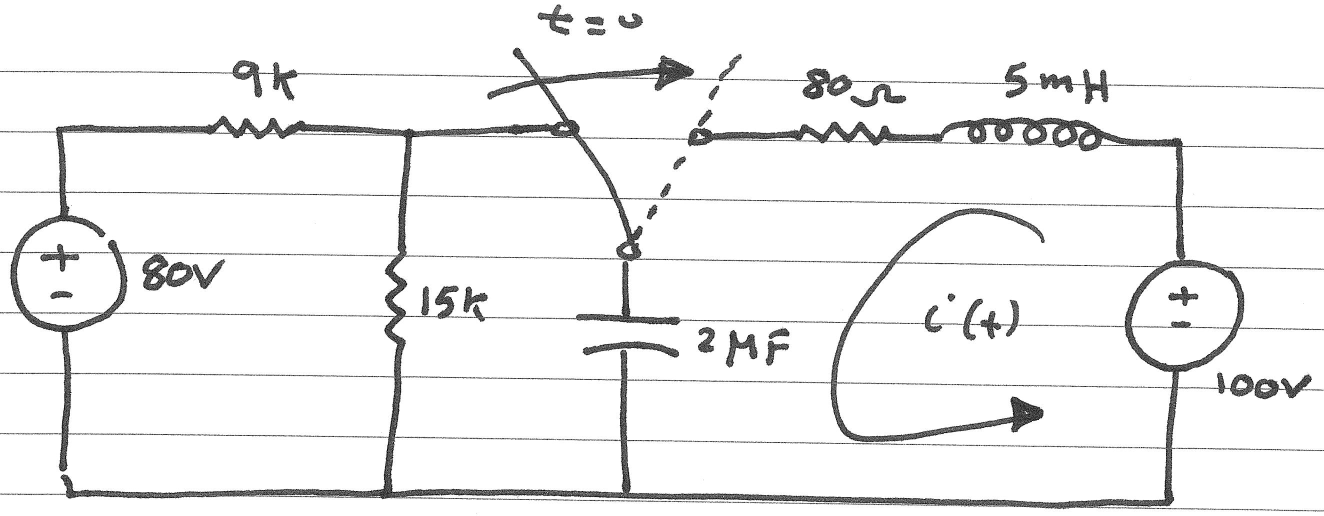
$$\frac{dv_c(0^+)}{dt} = -A_1 - 2A_2 = 0 \qquad \text{———} \ (2)$$

Solving ① and ②

$A_1 = -2$

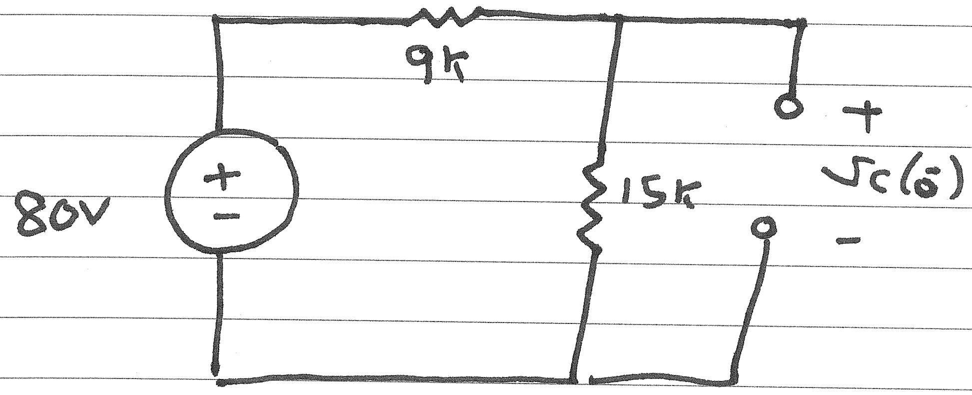
$A_2 = 1$

$\therefore V_c(t) = \left(1 - 2e^{-t} + e^{-2t}\right) \underline{\underline{V}}$ for $t > 0$



1) Find $i(t)$ for $t > 0$

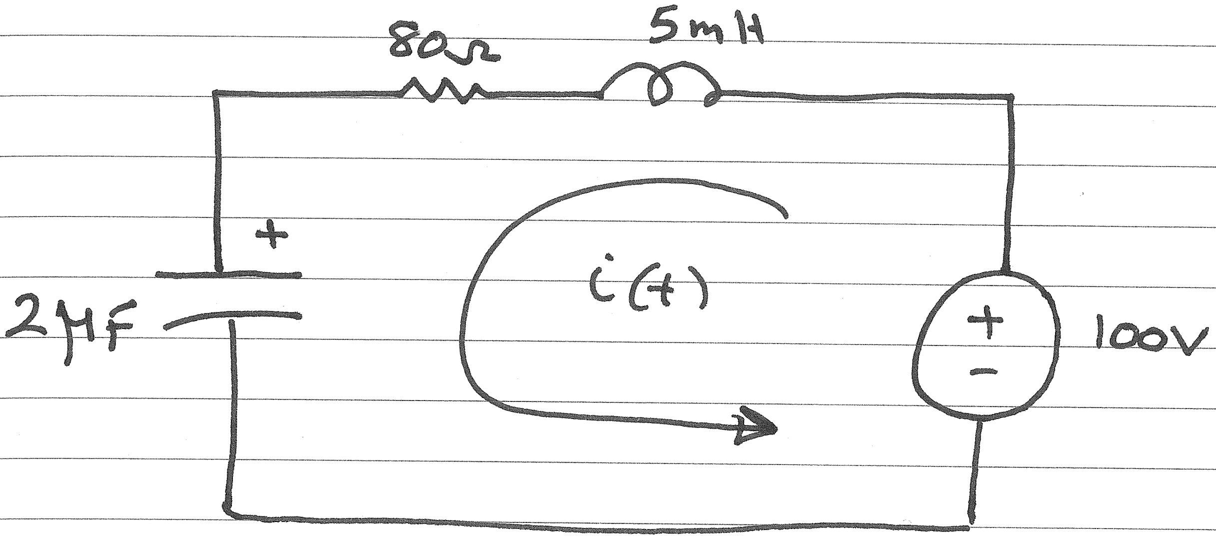
a) For $t < 0$, $t = 0^-$



$$v_c(0^-) = \frac{15k}{15k + 9k} 80 = 50V$$

$$i_L(0^-) = 0$$

b) For $t > 0$



KVL :

$$100 = L\frac{di(t)}{dt} + Ri(t) + v_c(0^-) + \frac{1}{c}\int_{0^-}^{t} i(t)\,dt$$

$$50 = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{c}\int_{0^-}^{t} i(t)\,dt$$

$\frac{d}{dt}$:

$$0 = L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{c}i(t)$$

second order homogeneous diff. equation

$\therefore\ i(t) = i_n(t)$

Let $v_1(t) = 10 \sin(5t - 30^\circ)$

$v_2(t) = 15 \sin(5t + 10^\circ)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by 40°

Let $i_1(t) = 2 \sin(377t + 45^\circ)$

$i_2(t) = 0.5 \cos(377t + 10^\circ)$

$$\cos\alpha = \sin(\alpha + 90^\circ)$$

$$0.5 \cos(377t + 10^\circ) = 0.5 \sin(377t + 100^\circ)$$

$\therefore$ $i_2(t)$ leads $i_1(t)$ by 55°

$0 = LCs^2 + RCs + 1$

$0 = 10\times10^{-9} s^2 + 160\times10^{-6} s + 1$

$\therefore\ s_1 = -8000 + j\,6000$

$s_2 = -8000 - j\,6000$

underdamped Case

$i(t) = e^{-8000t}\left(\beta_1 \cos 6000t + \beta_2 \sin 6000t\right)$

To find β_1 and β_2

$i(0^+) = i(0^-) = 0$

$\dfrac{di(0^+)}{dt} = \dfrac{V_s - V_c(0^-)}{L} = 10{,}000$

$\beta_1 = 0$

$\beta_2 = 1.67$

$i(t) = 1.67\, e^{-8000t} \sin 6000t \ \underline{A}$ for $t>0$

2) Find $v_c(t)$ for $t > 0$

$$V_s = L\frac{di(t)}{dt} + Ri(t) + v_c(t)$$

$$i(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$\therefore \quad V_s = LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t)$$

second order nonhomogeneouse diff. equation

$$\therefore \quad v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_{cf}(t) = K$$

$$V_s = 0 + 0 + K$$

$$\therefore \quad v_{cf} = K$$

To find $v_{cn}(t)$

$$0 = LCs^2 + RCs + 1$$

$$0 = 10\times10^{-9}s^2 + 160\times10^{-6}s + 1$$

$$s_1 = -8000 + j\,6000$$
$$s_2 = -8000 - j\,6000$$

Underdamped Case

$$v_c(t) = 100 + e^{-8000t}\left(\beta_1 \cos 6000t + \beta_2 \sin 6000t\right)$$

To find β_1, and β_2, we need

$v_c(0^+)$ and $\frac{dv_c(0^+)}{dt}$

$$v_c(0^+) = v_c(0^-) = 50\,V$$

$$i_c(0^+) = i_L(0^+) = \frac{C\,dv_c(0^+)}{dt} = 0$$

$$\therefore \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(0^+) = 100 + \beta_1 = 50$$

$$\therefore \beta_1 = -50$$

$$\frac{dv_c(0^+)}{dt} = -8000\beta_1 + 6000\beta_2$$

$$\therefore \beta_2 = -66.67$$

$$\therefore \boxed{v_c(t) = \left[100 + e^{-8000t}\left(-50\cos 6000t - 66.67\sin 6000t\right)\right] V}$$

for $t > 0$