# Ch4 : Applications of Derivatives



**Def** Let $f(x)$ be function defined on interval $I$.
Then $\Rightarrow$

[1] $f$ is increasing on $I$ if whenever

$$x_2 > x_1 \quad \text{gives} \quad f(x_2) > f(x_1) \quad \forall x_1, x_2 \in I$$

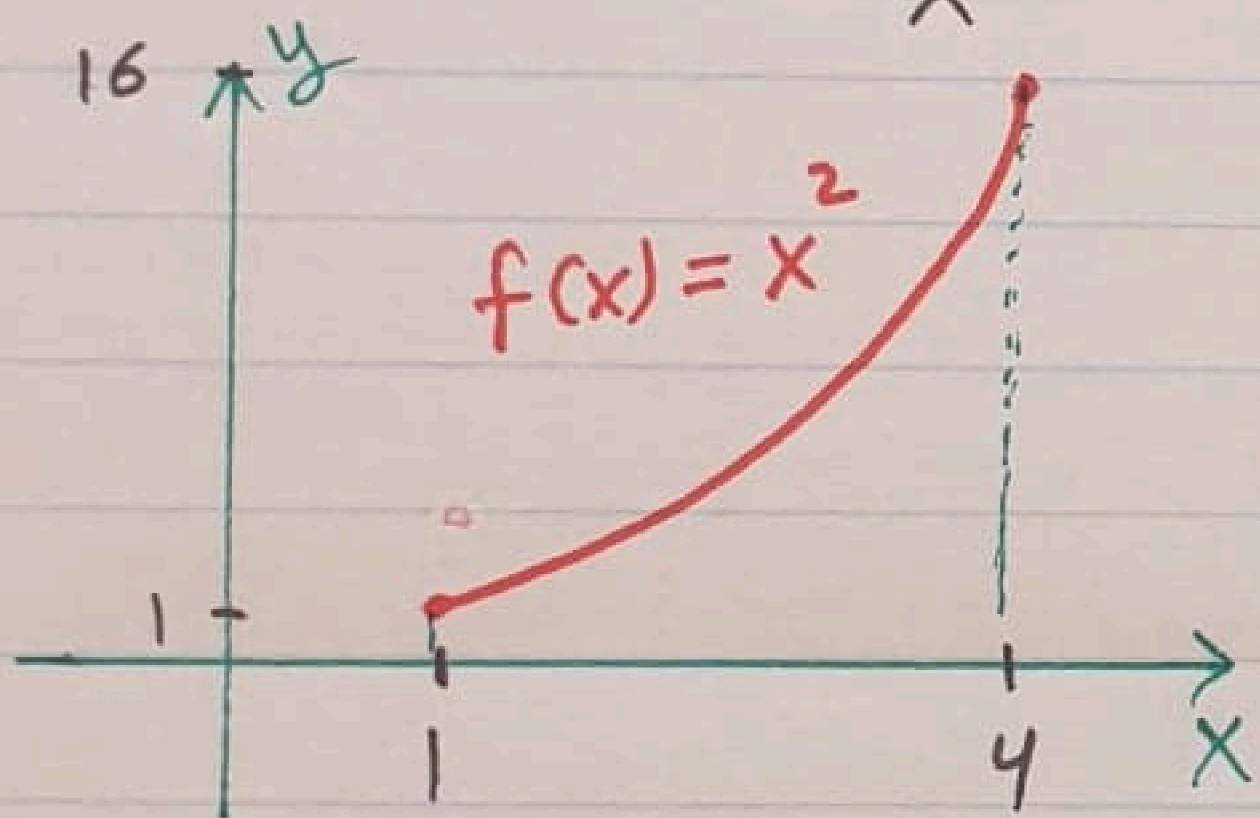
[2] $f$ is decreasing on $I$ if whenever

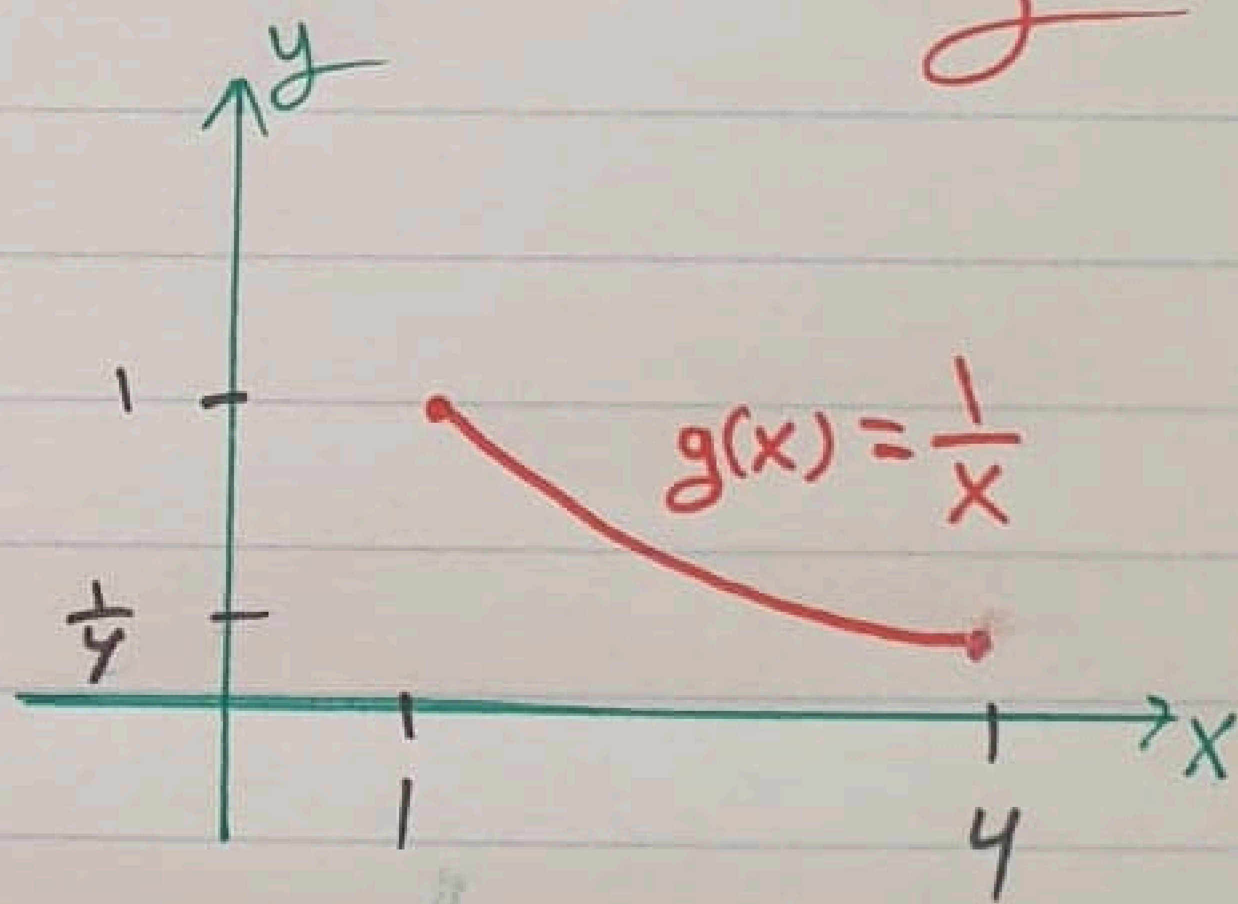
$$x_2 > x_1 \quad \text{gives} \quad f(x_2) < f(x_1) \quad \forall x_1, x_2 \in I$$

---

**Exp** ① $f(x) = x^2$ on $[1, 4]$ is increasing

② $g(x) = \frac{1}{x}$ on $[1, 4]$ is decreasing




---

We can use the first derivative to determine whether the function $f$ is increasing or decreasing

---

**Th** Suppose $f$ is cont. on $[a,b]$ and diff on $(a,b)$. Then

[1] $f$ is increasing on $[a,b]$ if $f'(x) > 0 \quad \forall x \in (a,b)$ and

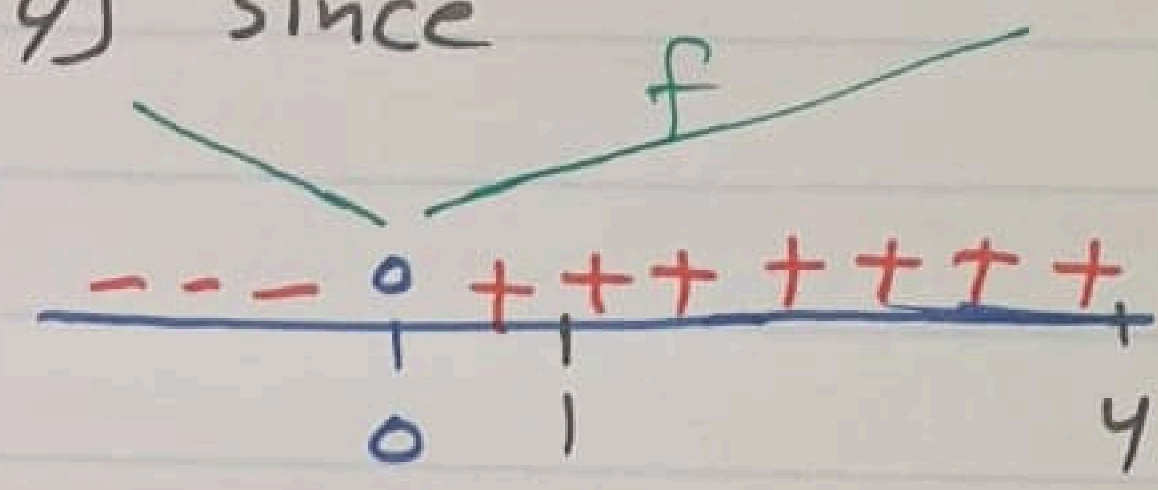
[2] $f$ is decreasing on $[a,b]$ if $f'(x) < 0 \quad \forall x \in (a,b)$

---

increasing $\uparrow$ , decreasing $\downarrow$ , Maximum - Max , Minimum - Min

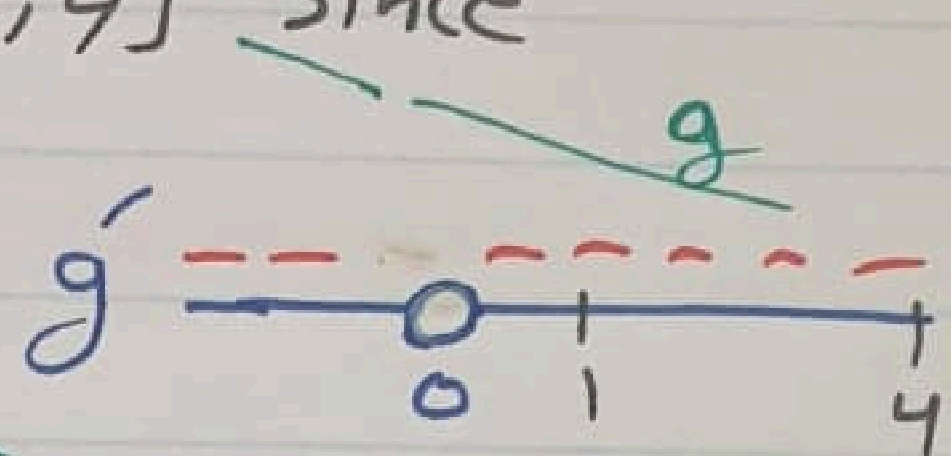
Exp ① $f(x) = x^2$ is increasing on $[1,4]$ since

$f'(x) = 2x > 0$ on $[1,4] \Rightarrow f'$




② $g(x) = \frac{1}{x}$ is decreasing on $[1,4]$ since

$g'(x) = -\frac{1}{x^2} < 0$ on $[1,4] \Rightarrow g'$



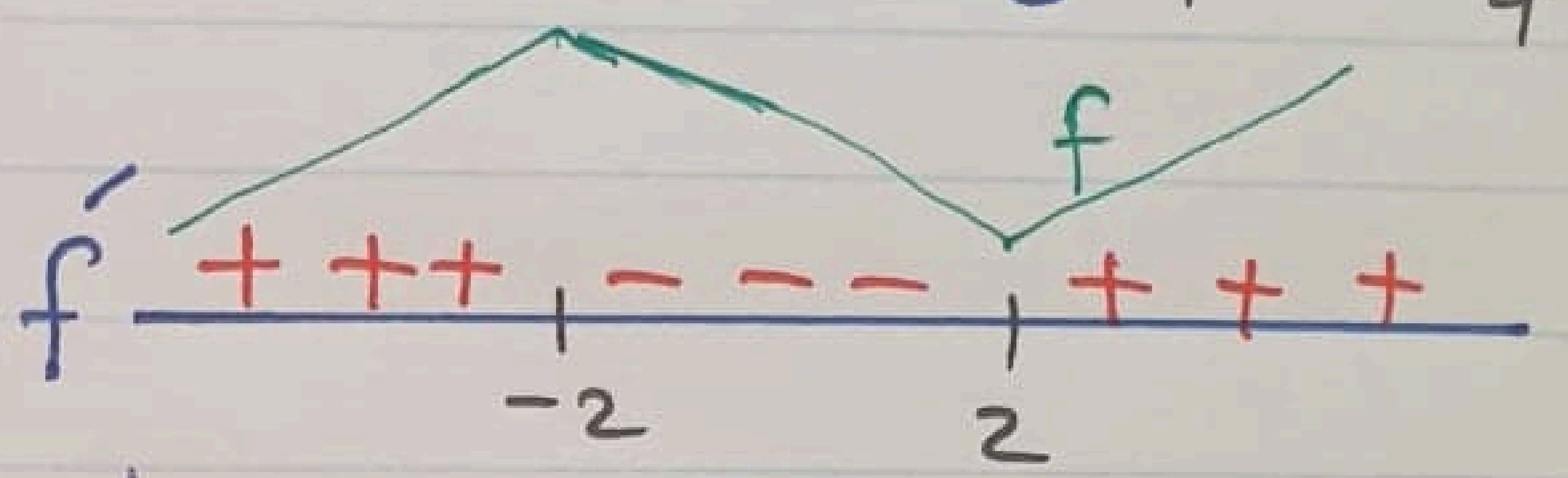

③ $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$f'(x) = 0 \quad \Rightarrow x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2, \ x = -2$




$f \uparrow$ on $(-\infty, -2] \cup [2, \infty)$

$f \downarrow$ on $[-2, 2]$

---

Def (Critical point)

Critical points are interior points of $f$ s.t $f' = 0$ or $f'$ is undefined

---

Exp ① $f(x) = x^2$ on $[1,4]$ has no critical points

but $f(x) = x^2$ on $\mathbb{R}$ has one critical point $(0,0)$ since $f' = 2x = 0 \Rightarrow x = 0$

② $g(x) = \frac{1}{x}$ has no critical points

$g'(x) = -\frac{1}{x^2} \Rightarrow g'$ is undefined at $x = 0$

but $x = 0 \notin D(g) = \mathbb{R} \setminus \{0\}$

---

③ $f(x) = x^3 - 12x - 5$ has two critical points:

$(2, f(2)) = (2, -21)$ and $(-2, 11)$ since $f'(2) = f'(-2) = 0$

and $2, -2 \in D(f) = \mathbb{R}$

# Extreme Values (EV's)



Absolute: Abs.

- Max
  - Abs. ⟶ Local
- Min
  - Abs. ⟶ Local

---

**Def** Assume $f$ is defined on domain $D$. Then

① $f$ has Abs. Max $M$ at point $c \in D$ if

$$M = f(c) \geq f(x) \quad \forall x \in D$$

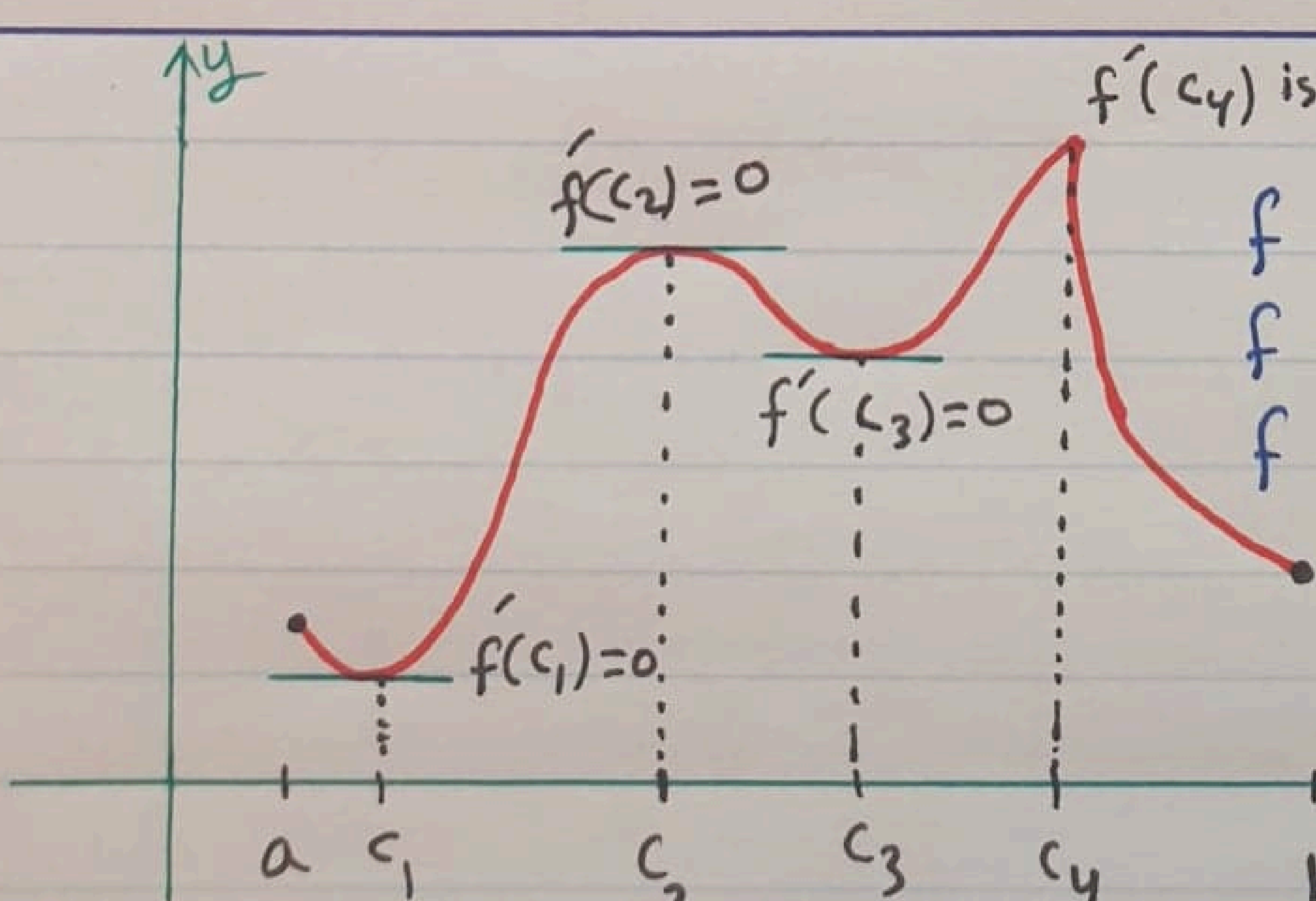
② $f$ has Abs. Min $m$ at point $c \in D$ if

$$m = f(c) \leq f(x) \quad \forall x \in D$$

③ $f$ has Local Max at point $c \in D$ if
① holds on small interval around $c$

④ $f$ has local Min at point $c \in D$ if
② holds on small interval around $c$

---



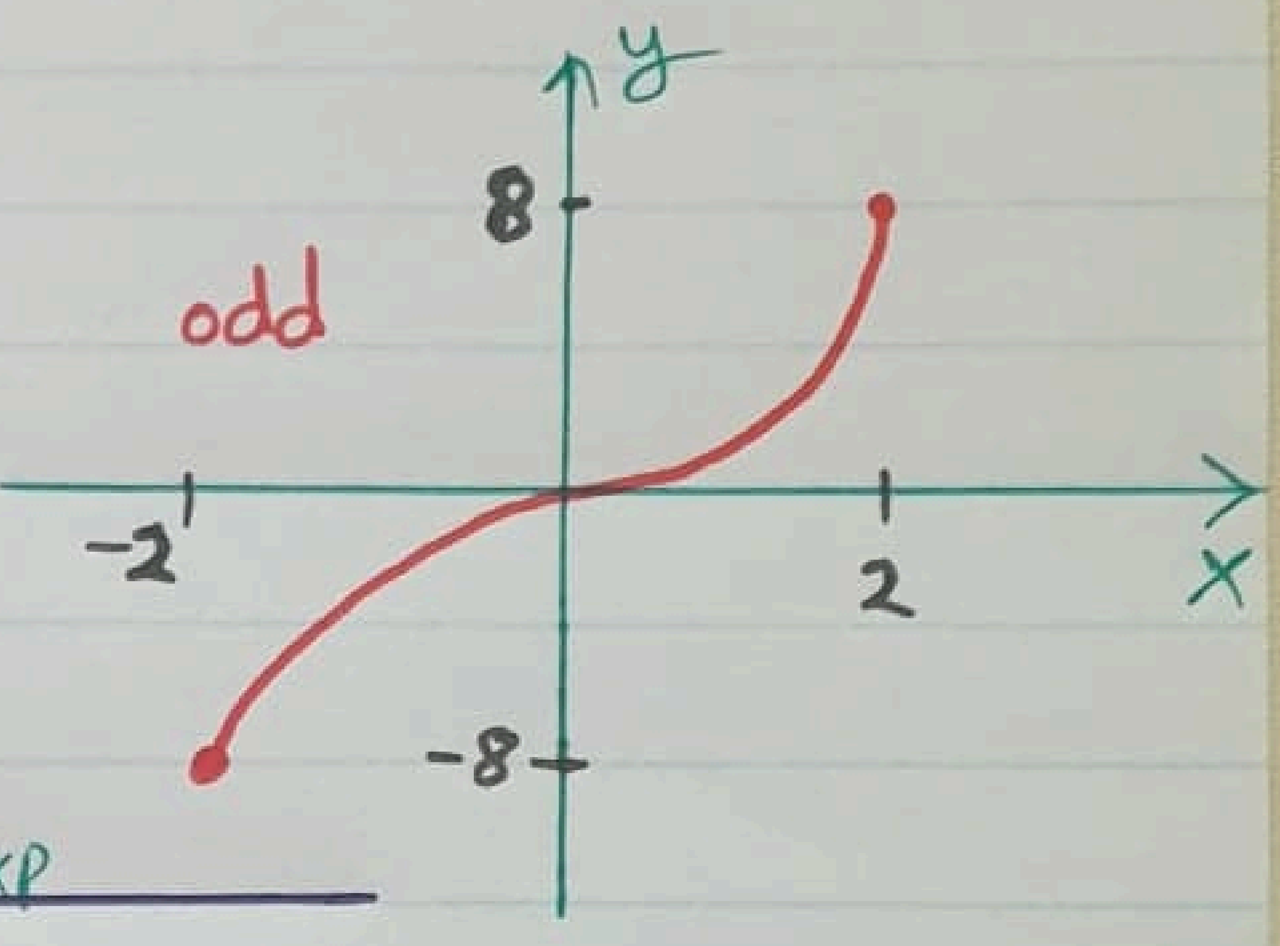

$f$ has Abs. Max at $x = c_4$
$f$ has Abs. Min at $x = c_1$
$f$ has Local Max at $x = a, c_2, c_4$
$f$ has L.Min at $x = c_1, c_3, b$

EV's occur at critical points or end points

$(c_1, f(c_1)), (c_2, f(c_2)), (c_3, f(c_3)), (c_4, f(c_4))$ are critical points

**Exp** Find Extreme Values of $f(x) = x^3$ on $[-2,2]$




f has Abs. Max of 8 at $x=2$
f has Abs. Min of $-8$ at $x=-2$

f has L. Max of 8 at $x=2$
f has L. Min of $-8$ at $x=-2$
EV's occur at the endpoints in this exp

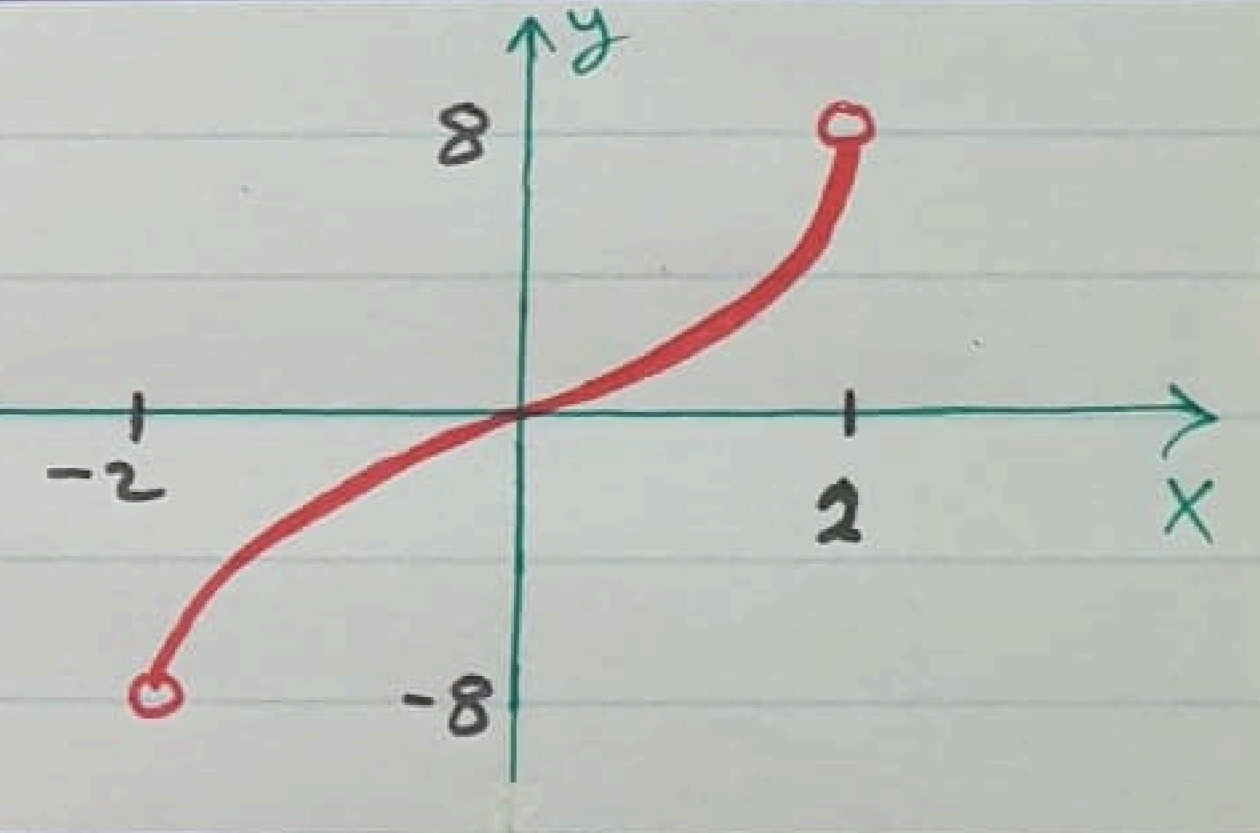
**Remark:** Abs. $\Rightarrow$ Local

---

**Th (Extreme Value Theorem _ EVT)**
If f is cont. on $[a,b]$ then f has Abs. Max and Abs. Min

---

**Exp** $f(x) = x^3$ on $(-2,2)$

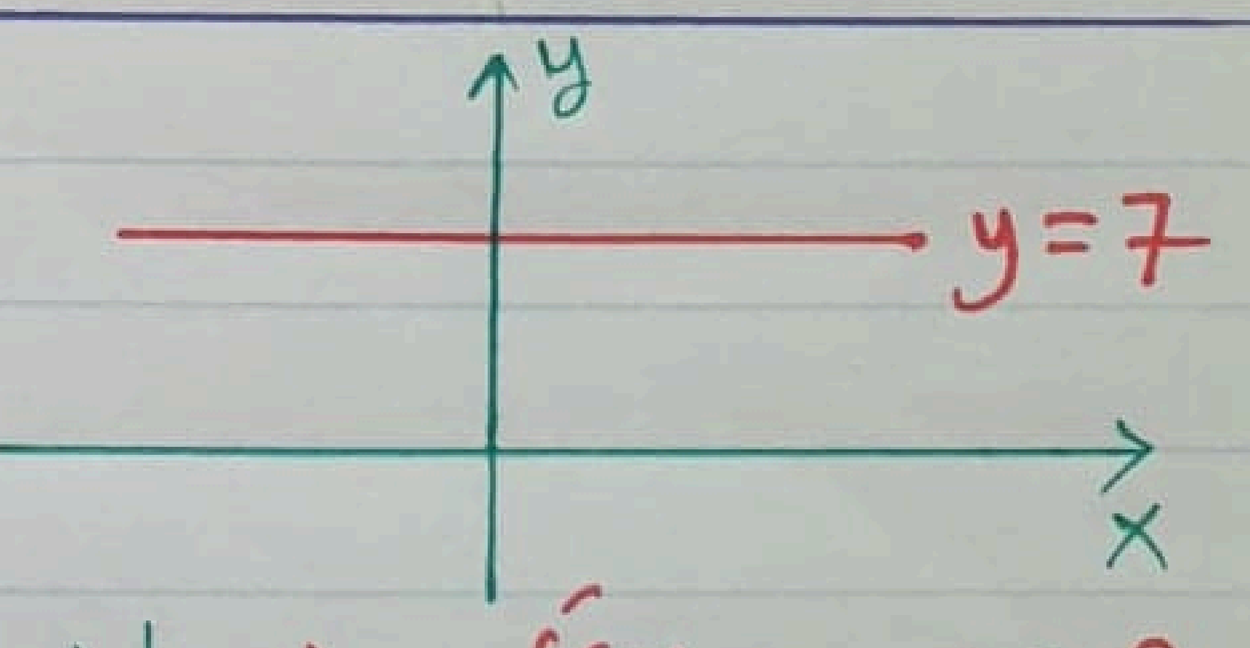



f has no extreme values

Note $f'(x) = 3x^2 = 0 \Leftrightarrow x = 0$ $f'$ $\underset{0}{++\,++}$ $f$
$x=0 \Rightarrow y=0 \Rightarrow (0,0)$ critical point

---

**Exp** $f(x) = 7$ on $\mathbb{R}$

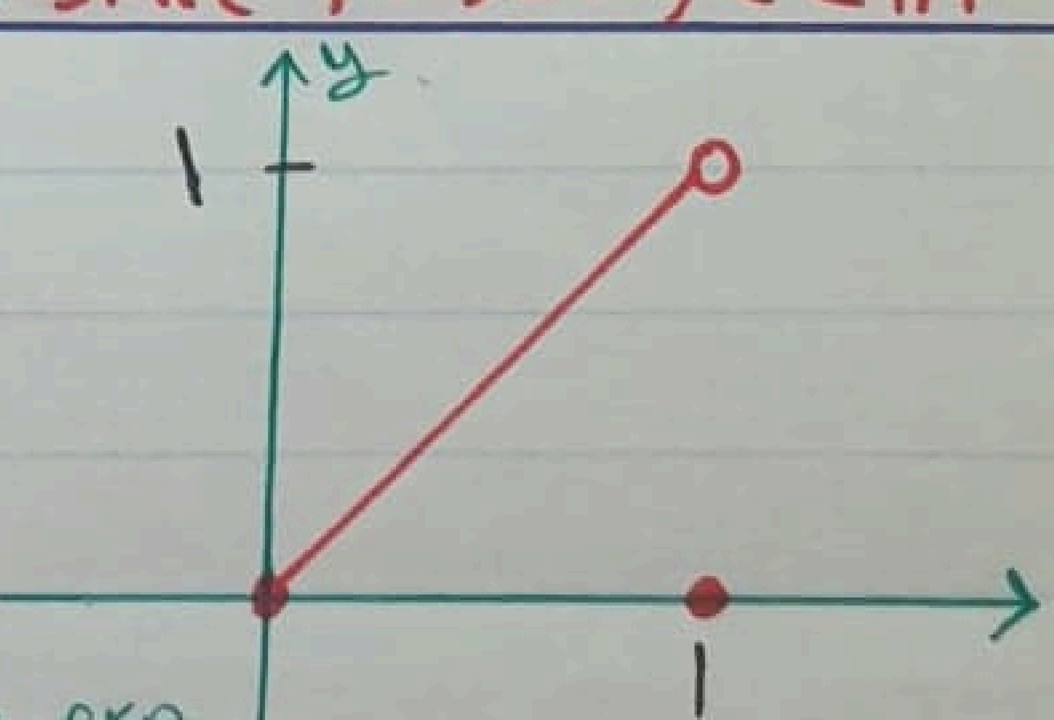



- f has Abs. Min (L. Min) at every point in $\mathbb{R}$
- f has Abs. Max at every point in $\mathbb{R}$ (L. Max)

All points $(c, f(c)) = (c, 7)$ are critical points since $f'(c) = 0$, $c \in \mathbb{R}$

---

**Exp** $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$




f has Abs. Min of 0 at $x = 0, 1$
(L. Min) ✓
EV's occur at the endpoints in this exp

**Remark** To find EV's $\Rightarrow$ check

① Critical points: where $f' = 0$ or $f'$ is undefined
② End points
③ be sure the points are in domain of $f$.

---

**Exp** Find EV's of $f(x) = x^{\frac{1}{3}}$ on $[-1, 8] = D$

• check critical points: $\Rightarrow f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$\Rightarrow f'$ is undefined at $x = 0 \in D$

$\Rightarrow$ critical point $(0, f(0)) = (0, 0)$

• check Endpoints $(-1, f(-1)) = (-1, \sqrt[3]{-1}) = (-1, -1)$

$(8, f(8)) = (8, \sqrt[3]{8}) = (8, 2)$

$f$ has Abs. Max of 2 at $x = 8$ (L. Max)
$f$ has Abs. Min of $-1$ at $x = -1$ (L. Min)

---

**Th*** If $f$ is diff at $c$ and $f$ has EV at $c$ then $f'(c) = 0$

---

**Exp** $f(x) = x^2$ on $[-2, 3]$. Find EV's

$f'(x) = 2x$

$f' = 0 \Rightarrow x = 0 \in [-2, 3] \Rightarrow (0, f(0)) = (0, 0)$ is the only critical value

Abs. Min
(L. Min)
$f'(0) = 0$

Endpoints

$(-2, f(-2)) = (-2, 4)$

$(3, f(3)) = (3, 9)$ Abs. Max (L. Max)