3.2 The Derivative as a function

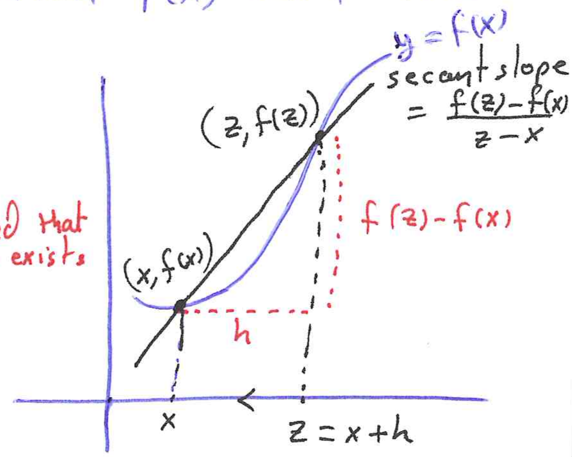
Def: The derivative of the function $f(x)$ w.r.t the variable x is

prime

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that limit exists

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$



- If f' exists at x, we say that f is differentiable (has derivative) at x.
- If f' exists at every point in the domain of f, we call f is differentiable.

- There are many ways to denote the derivative of $y = f(x)$

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

- The derivative at specified number $x = a$ is

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

Example: Using the definition (a) find the derivative $f(x) = \sqrt{x}$ for $x > 0$

(a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(b) $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

(c) tangent line at $x = 4$

$m = f'(4) = \frac{1}{4}$, point $(4, 2)$

$y - 2 = \frac{1}{4}(x - 4) = \frac{1}{4}x - 1$

$$\boxed{y = \frac{1}{4}x + 1}$$

- A function $y=f(x)$ is differentiable on an open interval (finite or infinite) if it has derivative at each point of the interval.

- A function $y=f(x)$ is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and

(a) • $\lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ exists and Right-hand derivative at a

(b) • $\lim\limits_{h\to 0^-} \dfrac{f(b+h)-f(b)}{h}$ exists left-hand derivative at b

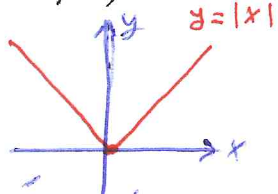
Example: show that $y=|x|$ is differentiable on $(-\infty,0)$ and $(0,\infty)$

For $x>0 \Rightarrow |x|=x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{x+h-x}{h}=1$$

For $x<0 \Rightarrow |x|=-x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{-x-h+x}{h}=-1$$

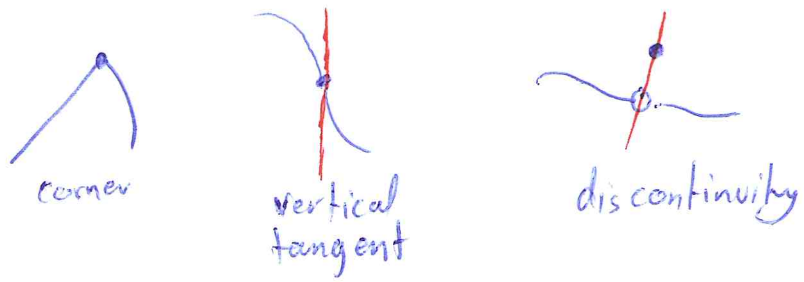


y' is not defined at $x=0$ (corner) because (a) $\neq$ (b)

At $x=0$, there is no derivative because (a) $\neq$ (b)

- Right-hand derivative at zero $=\lim\limits_{h\to 0^+}\dfrac{|0+h|-|0|}{h}=\lim\limits_{h\to 0^+}\dfrac{|h|}{h}=1$
- left-hand derivative at zero $=\lim\limits_{h\to 0^-}\dfrac{|0+h|-|0|}{h}=\lim\limits_{h\to 0^-}\dfrac{|h|}{h}=-1$

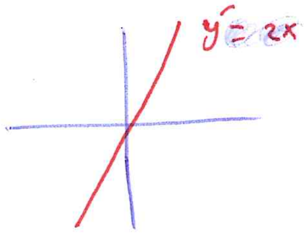
The function has no derivative at corners, vertical tangent and discontinuity.

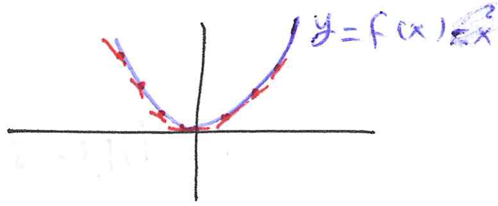


Theorem 1: If f has a derivative at $x=c$, then f is continuous at $x=c$

Example: Given the graph of $y=f(x)$. Graph the derivative

$\Rightarrow$





Proof: Assume that $f'(c)$ exists.

we need to show $\lim_{x\to c} f(x) = f(c) \quad \Leftrightarrow \quad x = c+h$

$$\lim_{h\to 0} f(c+h) = f(c)$$

If $h \neq 0$, then

$$f(c+h) = \cancel{f(c)+f(h)+f(c)-f(c)}$$
$$= f(c) + f(c+h) - f(c)$$
$$= f(c) + \frac{f(c+h)-f(c)}{h} \cdot h$$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot h$$
$$= f(c) + f'(c) \lim_{h\to 0} h$$
$$= f(c) + f'(c)\,(0)$$

$$\lim_{h\to 0} f(c+h) = f(c)$$