ch 5
Integration

Def A Function F is Antiderivative of function f on interval I if $F'(x) = f(x)$ for all $x \in I$.

Exp Find 4 antiderivatives of $f(x) = 2x$

$F_1(x) = x^2 + 1$
$F_2(x) = x^2 - \sqrt{2}$
$F_3(x) = x^2 + \pi$
$F_4(x) = x^2 - 3$
$\vdots$

$F(x) = \int f(x)\,dx = \int 2x\,dx = x^2 + c$

Def The set of all antiderivatives is called indefinite integral denoted by $\int f(x)\,dx$

Indefinite Integrals of Polynomials and Trigonometric functions

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \csc^2 x\,dx = -\cot x + c$
- $\int \sec x \tan x\,dx = \sec x + c$
- $\int \csc x \cot x\,dx = -\csc x + c$

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Exp Find the following integrals

① $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$

$= \frac{1}{2}x + \frac{1}{2}\,\frac{\sin 2x}{2} + c$

$= \frac{x}{2} + \frac{\sin 2x}{4} + c$

② $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

③ $\int (x^{-3} - 3x^2 + \sqrt{2}) \, dx = \frac{x^{-2}}{-2} - 3\,\frac{x^3}{3} + \sqrt{2}\,x + c$

$= -\frac{1}{2x^2} - x^3 + \sqrt{2}\,x + c$

④ $\int \frac{\csc\theta}{\csc\theta - \sin\theta} \, d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} \, d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}} \, d\theta$

$= \int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \, d\theta$

$= \tan\theta + c$

⑤ $\int \frac{\sec t \tan t}{\sqrt{\sec t}} \, dt = \int \frac{du}{\sqrt{u}} = 2\int \frac{du}{2\sqrt{u}}$ $u = \sec t$

$du = \sec t \tan t \, dt$

$= 2\sqrt{u} + c$

$= 2\sqrt{\sec t} + c$

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Definite Integral $\Rightarrow \int_a^b f(x)\,dx$ is a number

Exp Find $\int_0^1 (6x^2 - 4x + 2)\,dx = 6\frac{x^3}{3} - 4\frac{x^2}{2} + 2x \Big|_0^1$

$= (2x^3 - 2x^2 + 2x)\Big|_0^1 = (2-2+2) - (0-0+0) = 2$

Th (Fundamental Theorem of Calculus)

Assume $f(x)$ is cont. on $[a,b]$

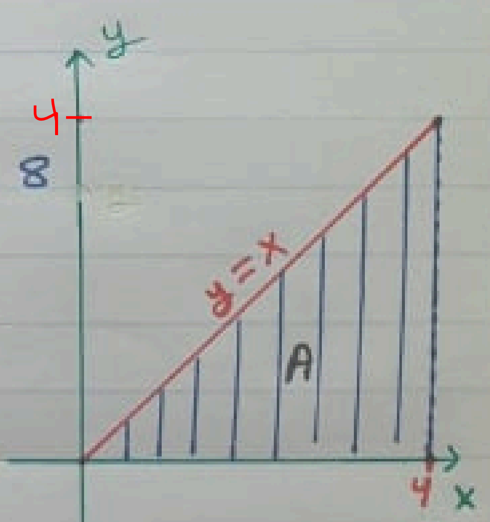
- If $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If $F(x) = \int_a^x f(t)\,dt$ is cont. on $[a,b]$ and diff on (a,b) then $F'(x) = f(x)$

Exp Find ① $\int_0^4 x\,dx$

$\int_0^4 x\,dx = \frac{x^2}{2}\Big|_0^4 = \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8$



② Area $A = \int_0^4 x\,dx = 8$

③ what is the conclusion

If $f(x) \geq 0$ is integrable on $[a,b]$, then $\int_a^b f(x)\,dx$ is the area enclosed by the curve $f(x)$ and the x-axis

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Exp Find $\int_0^{\pi} \left(\cos x + |\cos x|\right) dx$

$$= \int_0^{\pi} \cos x \, dx + \int_0^{\pi} |\cos x| \, dx$$

$$= \sin x \Big|_0^{\pi} + \int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x \, dx$$

$$= \sin \pi - \sin 0 + \sin x \Big|_0^{\frac{\pi}{2}} - \left(\sin x\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 0 - 0 + \sin \frac{\pi}{2} - \sin 0 - \left(\sin \pi - \sin \frac{\pi}{2}\right)$$

$$= 1 - 0 - (0 - 1)$$
$$= 2$$

Exp Find the following derivatives:

1. $\frac{d}{dx} \int_0^{x} \cos t \, dt = \cos x$

2. $\frac{d}{dx} \left(\int_{\frac{\pi}{2}}^{\pi} \frac{\csc \theta}{1 - \cos \theta} \, d\theta \right) = 0$

3. $\left(\int_{\tan x}^{\frac{\pi}{4}} \frac{dt}{1+t^2} \right)' = \left(- \int_{\frac{\pi}{4}}^{\tan x} \frac{dt}{1+t^2} \right)' = - \frac{\sec^2 x}{1 + \tan^2 x}$

$$= - \frac{\sec^2 x}{\sec^2 x} = -1$$

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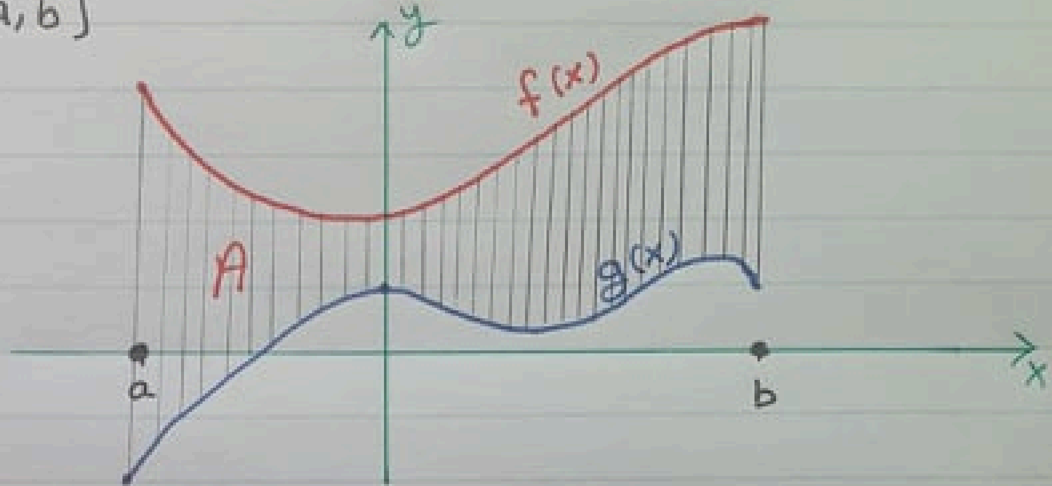
Remark: If $f(x)$ and $g(x)$ are integrable on $[a,b]$ s.t $f(x) \geq g(x)$ $\forall x \in [a,b]$ then

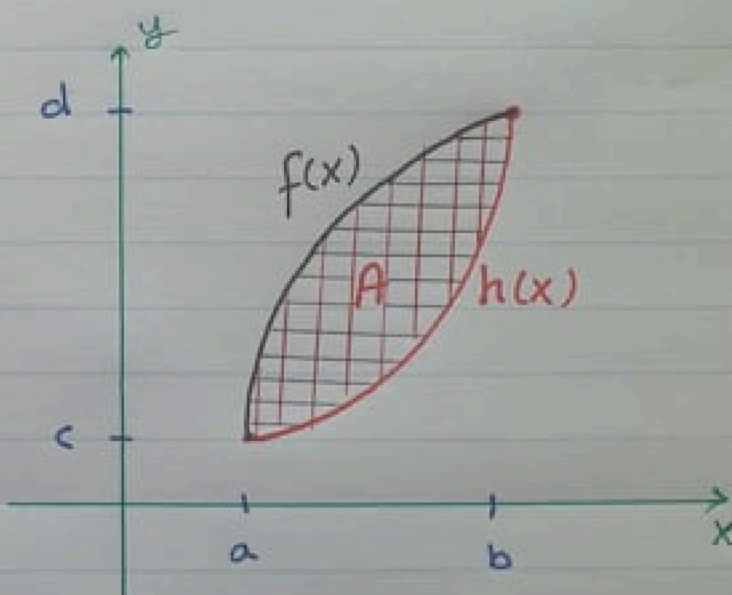
$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

is the area enclosed between $f(x)$ and $g(x)$ on $[a,b]$

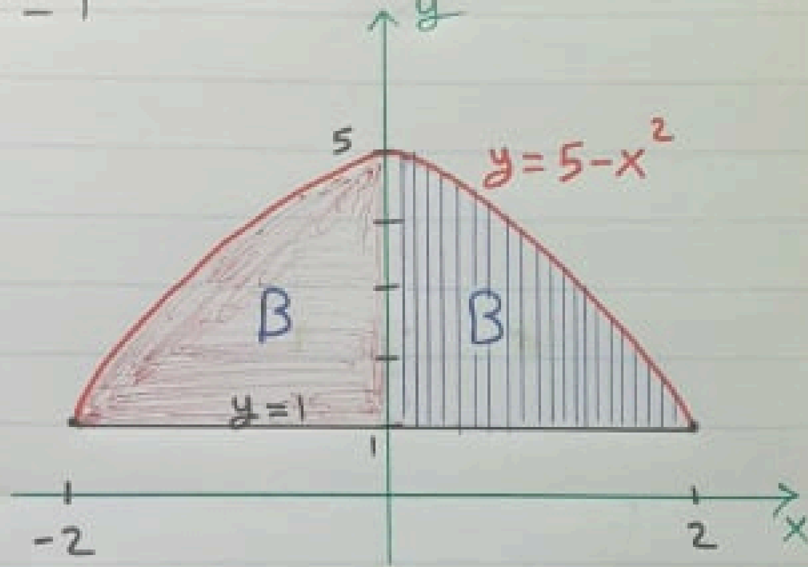


$$A = \int_a^b \left(f(x) - h(x)\right) dx$$

$$= \int_c^d \left(h(y) - f(y)\right) dy$$



Exp Find the area enclosed by the curve $y = 5 - x^2$ and $y = 1$



$$\text{Area} = 2B$$

$$B = \int_0^2 \left(5 - x^2 - 1\right) dx$$

$$= \int_0^2 \left(4 - x^2\right) dx$$

$$= 4x - \frac{x^3}{3} \Big|_0^2$$

$$= 8 - \frac{8}{3} - (0 - 0) = \frac{16}{3}$$

$5 - x^2 = 1$
$x^2 = 4$
$x = \pm 2$

Hence, $\text{Area} = 2B = 2\left(\frac{16}{3}\right) = \frac{32}{3}$

Exp Find the area enclosed by the curve $f(x) = 3x\sqrt{x^2+1}$ and the x-axis on $[0,1]$.

Note that $f(x) \geq 0$ on $[0,1]$ $\Rightarrow$ $\text{Area} = \int_0^1 3x\sqrt{x^2+1}\, dx$

$u = x^2 + 1$
$du = 2x\, dx$
$\frac{du}{2} = x\, dx$

$\frac{3}{2} du = 3x\, dx$

$x = 0 \Rightarrow u = 1$
$x = 1 \Rightarrow u = 2$

$$\text{Area} = \int_1^2 \frac{3}{2}\sqrt{u}\, du$$

$$= \frac{3}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^2 = u^{\frac{3}{2}} \Big|_1^2$$

$$= \sqrt{u^3} \Big|_1^2$$

$$= \sqrt{8} - \sqrt{1}$$

$$= 2\sqrt{2} - 1$$

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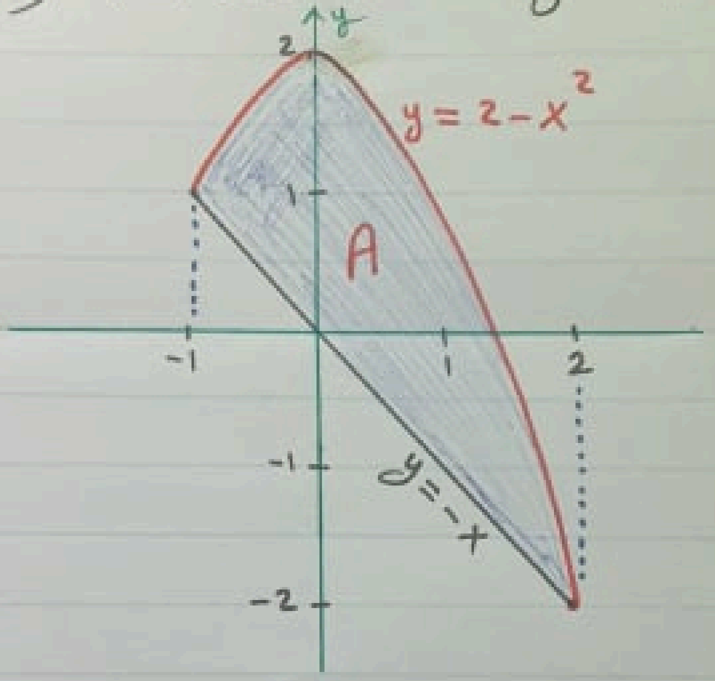
Exp Find the area enclosed by $f(x) = 2 - x^2$ and $y = -x$



$$2 - x^2 = -x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1$$

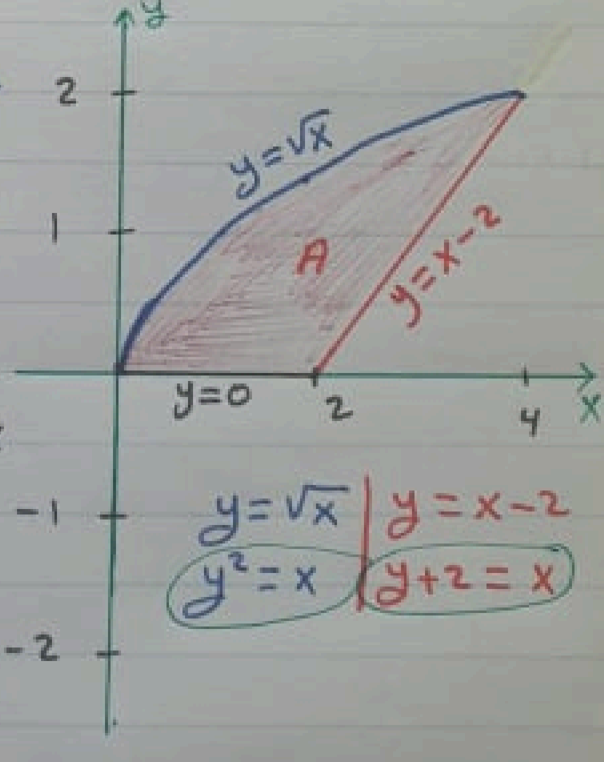
$$A = \int_{-1}^{2} (2 - x^2 - -x)\, dx$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) = 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2} = \frac{9}{2}$$

Exp Find the area enclosed by the curve $y = \sqrt{x}$, the x-axis and the line $y = x - 2$ by with respect to

[1] x-axis [2] y-axis



$y = \sqrt{x}$ | $y = x - 2$

$y^2 = x$ | $y + 2 = x$

[2] $A = \int_0^2 (y + 2 - y^2)\, dy = \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$

$$= \left(2 + 4 - \frac{8}{3}\right) - (0 + 0 - 0) = \frac{10}{3}$$

[1] $A = \int_0^2 (\sqrt{x} - 0)\, dx + \int_2^4 (\sqrt{x} - (x-2))\, dx$

$$= \frac{x^{3/2}}{3/2} \Big|_0^2 + \left(\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right) \Big|_2^4$$

$$= \frac{2}{3}\sqrt{x^3} \Big|_0^2 + \left(\frac{2}{3}\sqrt{x^3} - \frac{x^2}{2} + 2x\right) \Big|_2^4$$

$$= \cancel{\frac{2}{3}\sqrt{8}} - 0 + \frac{2}{3}\sqrt{64} \cancel{- 8} \cancel{+ 8} - \left(\cancel{\frac{2}{3}\sqrt{8}} - 2 + 4\right) = \frac{16}{3} - 2 = \frac{10}{3}$$

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