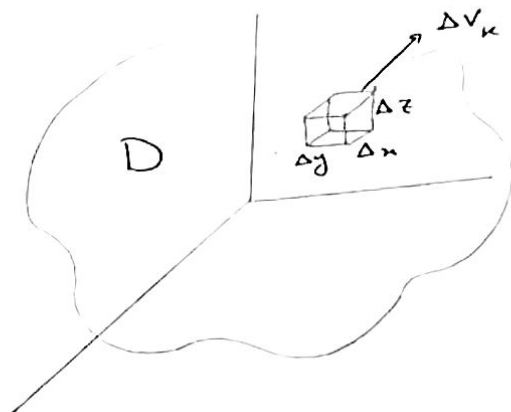


15.5 Triple Integrals in Rectangular Coordinates:

Let D be a closed and bounded region in space.

$$S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k$$

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(x, y, z) dx dy dz$$



• If $F(x, y, z) = 1$, then:

$$S_n = \sum_{k=1}^n \Delta V_k$$

$$\lim_{n \rightarrow \infty} S_n = \iiint_D dV$$

Def: The volume of a closed, bounded region D in space is

$$V = \iiint_D dV = \iiint_D dx dy dz$$

Example: Set up the limits of Integration for evaluating the triple integral of a function $f(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$. Use the order $dydzdx$

1) We draw $M \parallel y$ -axis.

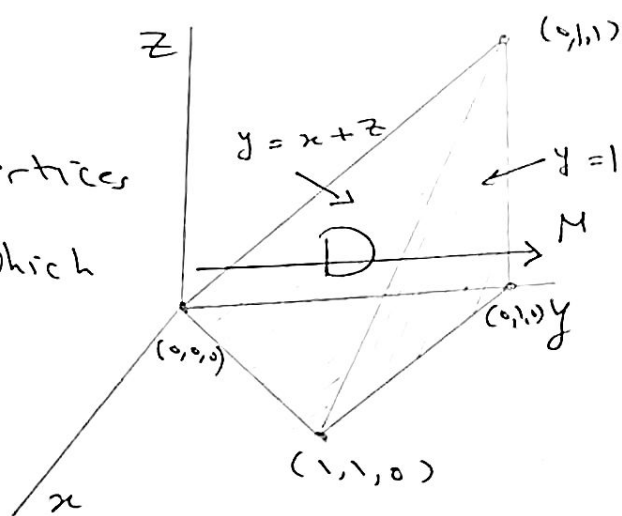
□ M enters D at the plane with vertices $(0, 0, 0)$ & $(1, 1, 0)$, & $(0, 1, 1)$ which is

$$y = x + z$$

How to find equation of a plane?

Equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$a\vec{i} + b\vec{j} + c\vec{k} \perp \text{plane}$ & $(x_0, y_0, z_0) \in \text{plane}$.
Find $\vec{v}_1$ & $\vec{v}_2 \Rightarrow \vec{v}_1 \times \vec{v}_2 = a\vec{i} + b\vec{j} + c\vec{k}$



□ M leaves D at the plane with vertices: $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$

which is $y = 1$

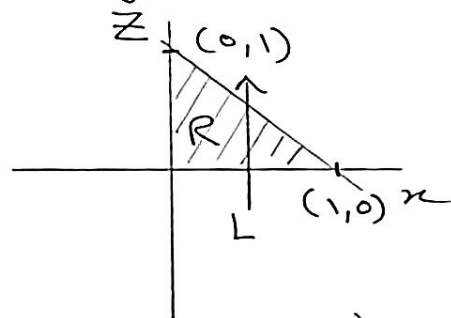
(vertical projection in xz plane)

2) Now assume $y = 0$, to find the limits of z in

the xz plane: Draw $L \parallel z$ axis:

then L enters R at $z = 0$

& leaves at $z = 1 - x$
equation of line



(12)

3) Need to find New x limits: $x = 0$ to $x = 1$

Then:

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x,y,z) dy dz dx$$

Example: Integrate $F(x,y,z) = 1$, over the tetrahedron

D in the previous example in the order $dz dy dx$

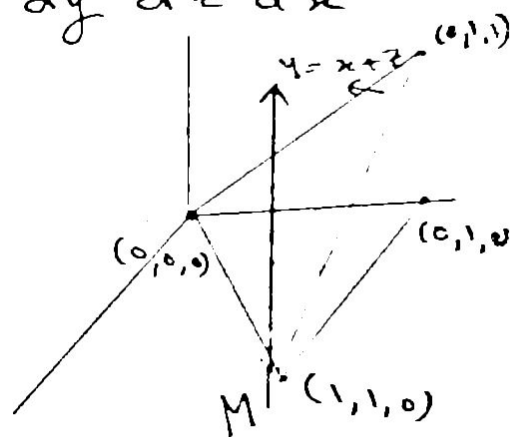
and then Integrate in the order $dy dz dx$

1) $M \parallel z$ -axis:

M enters at the plane with vertices $(0,0,0)$, $(1,1,0)$ & $(0,1,0)$ which is

$\boxed{z = 0}$

& leaves at the plane $\boxed{z = y - x}$

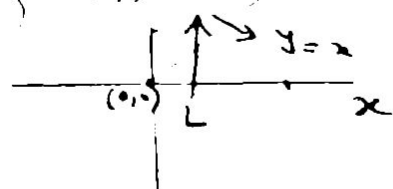


2) Assume $z = 0$ to find vertical projection in xy -plane

$L \parallel y$ -axis, then L enters at $\boxed{y = x}$

& leaves at $\boxed{y = 1}$

$y = 1 \rightarrow (1,1)$



3) x changes from $\overset{\text{when } y=0}{0}$ to 1

$\rightarrow$

$$\int_0^1 \int_x^{y-x} \int_0^{y-x} F(x,y,z) dz dy dx$$

(122)

When $F(x, y, z) = 1$, then $V = \iiint dx dy dz$

So $V = \int_0^1 \int_x^1 \int_0^{y-x} dz dy dx$

$$= \int_0^1 \int_x^1 z \Big|_0^{y-x} dy dx = \int_0^1 \int_x^1 (y-x) dy dx$$

$$= \int_0^1 \left(\frac{y^2}{2} - xy \right) \Big|_x^1 dx = \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2} x^2 \right) dx$$

$$= \boxed{\frac{1}{6}}$$

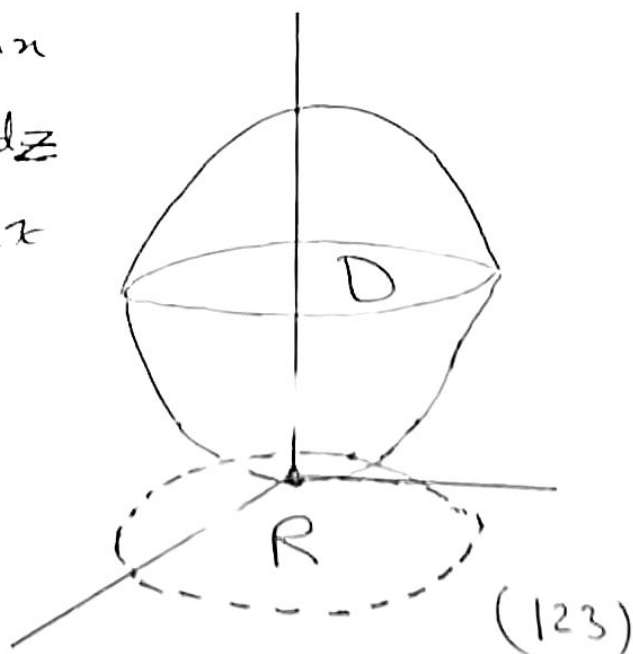
Similarly: $V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx = \boxed{\frac{1}{6}}$

Example: Find the Volume of the Region D enclosed

by the surfaces $z = x^2 + 3y^2$ & $z = 8 - x^2 - y^2$

1) Using the order $dz dy dx$

2) Using the order $dx dy dz$

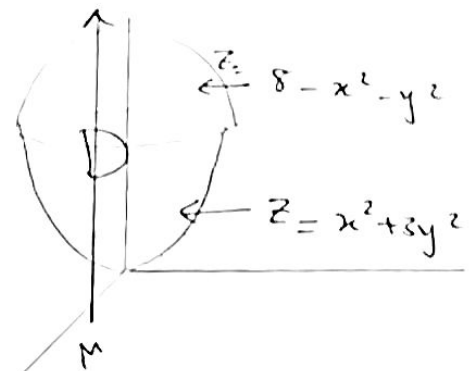


(123)

1) Limits of z :

$M \parallel z$ -axis: then M enters D

at $z = x^2 + 3y^2$ & leaves at $z = 8 - x^2 - y^2$



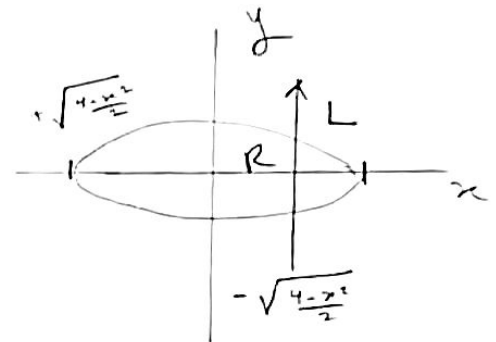
2) Limits of y : Find the vertical projection when $z = 0$ in the xy -plane.

$$\Rightarrow 8 - x^2 - y^2 = x^2 + 3y^2$$

$$\Leftrightarrow \boxed{x^2 + 2y^2 = 4} \text{ ellipse}$$

$$L \parallel y\text{-axis}: y = \pm \sqrt{\frac{4-x^2}{2}}$$

L enters R at $y = -\sqrt{\frac{4-x^2}{2}}$ & leaves at $y = \sqrt{\frac{4-x^2}{2}}$



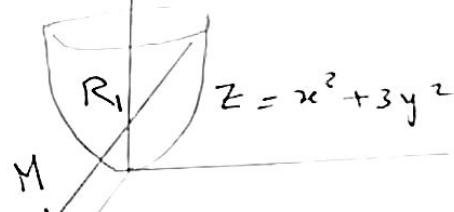
3) Limits of x : when $y = 0$, then $x = \pm 2$

$$\therefore V = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx$$

$$= 8\pi\sqrt{2}$$

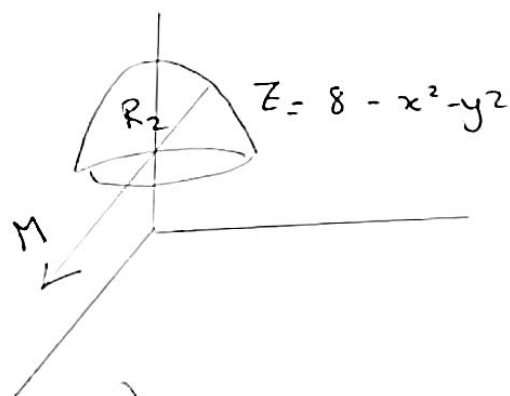
2) (i) $M \parallel x$ -axis (R_1) : $x = \pm \sqrt{3y^2 + z}$

M enters D at $-\sqrt{z-3y^2}$ & leaves at $+\sqrt{z-3y^2}$



(ii) $M \parallel x$ -axis (R_2) : $x = \pm \sqrt{8 - y^2 - z}$

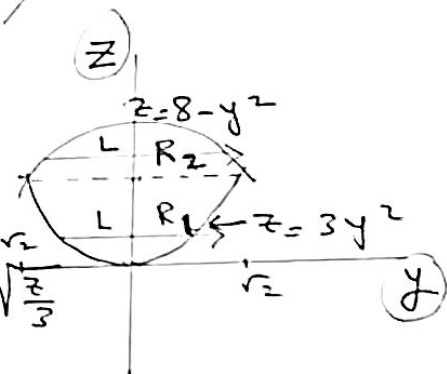
M enters D at $-\sqrt{8 - y^2 - z}$ & leaves at $+\sqrt{8 - y^2 - z}$



Now: assume $x = 0$ to find the vertical projection in yZ plane.

$\therefore$ ① $L \parallel y$ -axis (R_1) : $y = \pm \sqrt{\frac{z}{3}}$

$\therefore L$ enters R at $y = -\sqrt{\frac{z}{3}}$ & leaves at $y = \sqrt{\frac{z}{3}}$



② $L \parallel y$ -axis in (R_2) : $y = \pm \sqrt{8 - z}$

$\therefore L$ enters R at $y = -\sqrt{8 - z}$ & leaves at $y = \sqrt{8 - z}$

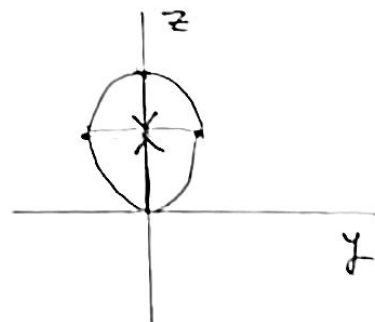
Now: $3y^2 = 8 - y^2 \Rightarrow 4y^2 = 8$

$\therefore y = \pm \sqrt{2}$, then:

z starts at 0 & ends at $3(\sqrt{2})^2 = \boxed{6}$

in $\boxed{R_1}$

& z starts at 6 & ends at 8 in $\boxed{R_2}$



Therefore,

(12.5)

$$\therefore V = \int_0^6 \int_{-\sqrt{\frac{z}{3}}}^{\sqrt{\frac{z}{3}}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} dx dy dz.$$

$$+ \int_6^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-y^2-z}}^{\sqrt{8-y^2-z}} dx dy dz = 8\pi\sqrt{2}.$$

The Average Value of a Function in Space:

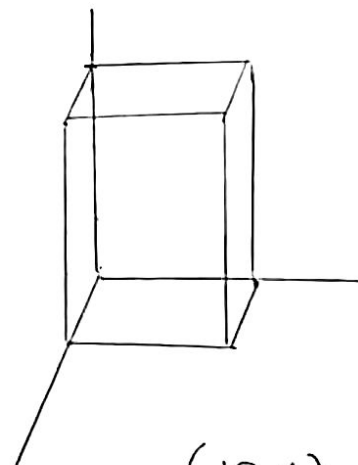
$$av(f) = \frac{1}{\text{volume}(D)} \iiint_D F dv$$

Example: Find the average value of $f(x,y,z) = xyz$

throughout the rectangular Region D in the first octant bounded by the Coordinate planes & the planes $x=1$, $y=2$, $z=3$

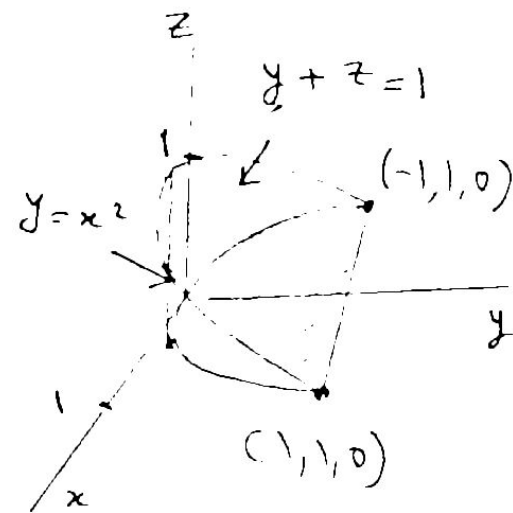
$$V(D) = \int_0^3 \int_0^2 \int_0^1 dx dy dz = 6$$

$$av(f) = \frac{1}{6} \int_0^3 \int_0^2 \int_0^1 xyz dx dy dz = \boxed{\frac{3}{4}}$$



(126)

(21) $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz dy dx$



(a) $dy dz dx$:
 $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy dz dx$

(b) $dy dx dz$: $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy dx dz$

(c) $dx dy dz$: $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx dy dz$

(d) $dx dz dy$: $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx dz dy$

(e) $dz dx dy$: $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz dx dy$

(127)

(24) The Region in the first octant bounded by the coordinate planes and the planes $x+z=1$, $y+2z=2$. Find the volume? $dy dz dx$

1) y -limits: $M \parallel y$ -axis:

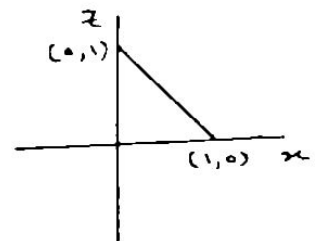
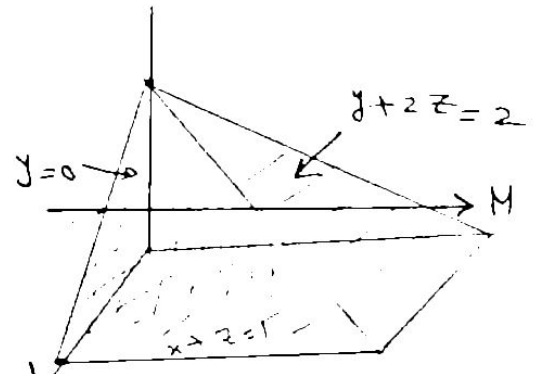
$$y = 0 \text{ to } y = 2 - 2z$$

2) z -limits: $L \parallel z$ -axis. (xz -plane)

$$z = 0 \text{ to } z = 1 - x$$

3) x -limits: $x = 0$ to $x = 1$

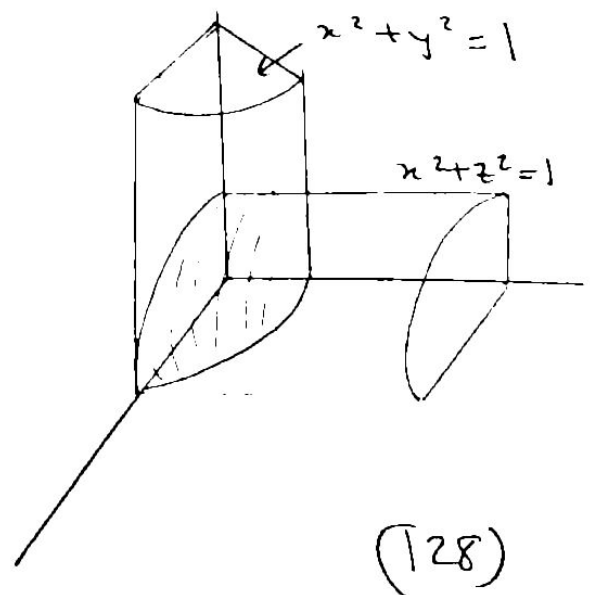
$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} 1 \, dy \, dz \, dx = \boxed{\frac{2}{3}}$$



(29) The Region common to the Interiors of the cylinders $x^2+y^2=1$ and $x^2+z^2=1$, one eighth of which is shown in the figure, Find V ?

$$V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1 \, dz \, dy \, dx$$

$$= \boxed{\frac{16}{3}}$$



(128)

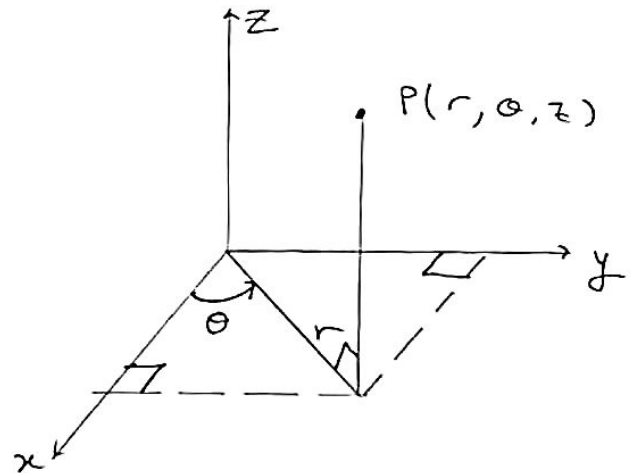
15.7 Triple Integrals in Cylindrical and spherical

Coordinates:

Def: Cylindrical Coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r & θ are polar Coordinates for the vertical projection P on the xy -plane
2. z is the rectangular vertical Coordinate.

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الارتفاع



Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

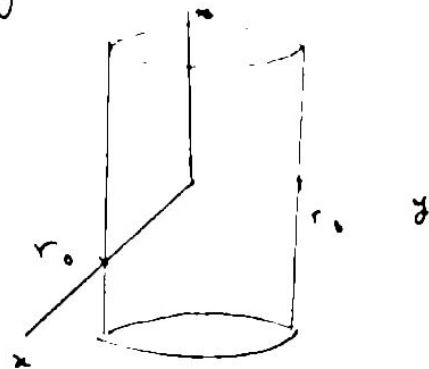
(129)

Remark: In cylindrical Coordinates:

1) $r = r_0$, describes an entire cylinder about

z -axis

θ & z vary.

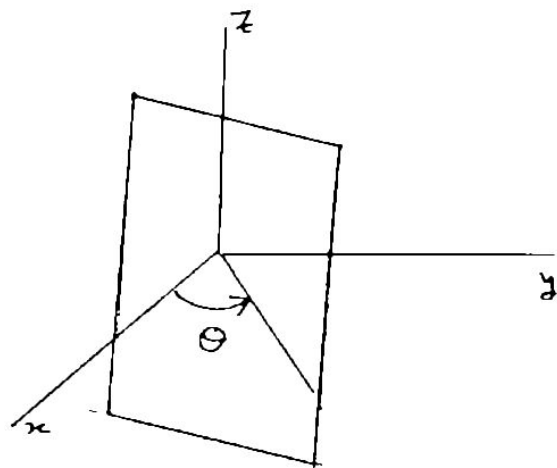


2) $\theta = \theta_0$, describes the plane that contains the

z -axis and makes angle θ

with the positive x -axis

r & z vary.



3) $z = z_0$, describes plane $\perp$ z -axis.

r & θ vary.

Note: $r = 0$ describes z -axis.

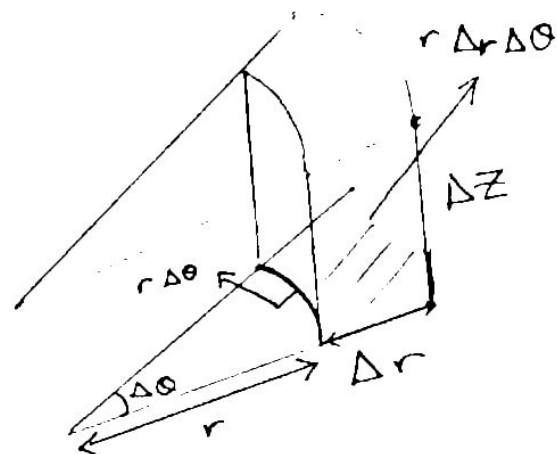
$r = 4$ describes cylinder with $r = 4$ about z -axis

$\theta = \frac{\pi}{3}$ = planes containing z -axis

$z = 2$ = plane $\perp$ z -axis
(130)

To find triple integral over region D in cylindrical coordinates, we partition the region D into small n cylindrical wedges.

- The k th cylindrical wedge r_k, θ_k, z_k changes by $\Delta r_k, \Delta \theta_k, \Delta z_k$.



- The norm of the partition $\max \{ \Delta r_k, \Delta \theta_k, \Delta z_k \}$.

- $\Delta A_k = r_k \Delta r_k \Delta \theta_k$.

- $\Delta V_k = \Delta z_k \Delta A_k = \Delta z_k r_k \Delta \theta_k \Delta r_k$

For a point in the center of the k th wedge, the

Riemann sum of f on D has the form:

$$S_n = \sum_{k=1}^n f(r_k, \theta_k, z_k) \Delta z_k r_k \Delta r_k \Delta \theta_k$$

as $\text{norm} \rightarrow 0 \Rightarrow n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} S_n = \iiint_D f dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r, \theta)}^{g_2(r, \theta)} f dz \, r dr d\theta \quad (131)$$

Example: Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z=0$, & by the circular cylinder $x^2 + (y-1)^2 = 1$ and above by the paraboloid $z = x^2 + y^2$.

• Base of D is the projection R in xy plane

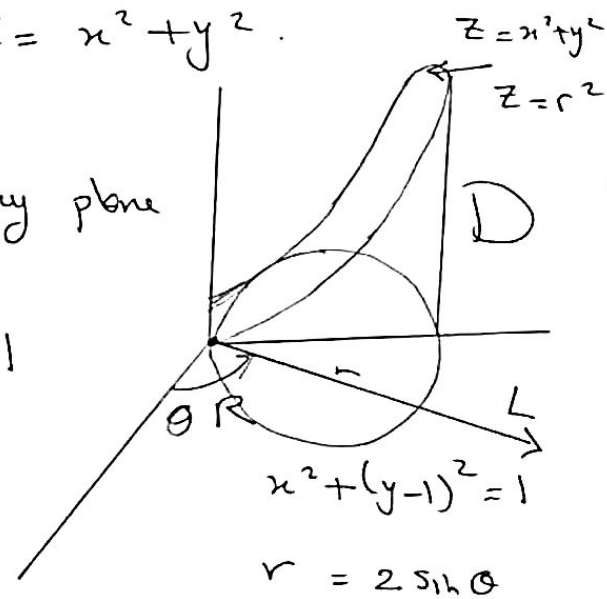
Boundary of R is $x^2 + (y-1)^2 = 1$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin \theta = 0$$

$$r(r - 2 \sin \theta) = 0$$

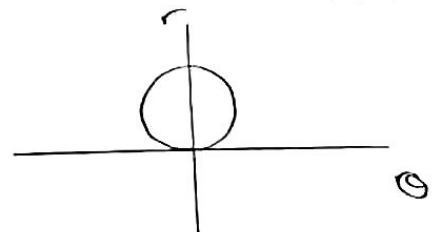
$$\Rightarrow r = 2 \sin \theta.$$



1) z -limits: M enters at $z=0$ & leaves at $z = x^2 + y^2 = r^2$

2) r -limits: L enters at $r=0$ & leaves at $r = 2 \sin \theta$

3) θ -limits: $\theta = 0$ to $\theta = \pi$



$$\iiint_D f(r, \theta, z) dV = \int_0^\pi \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z) dz r dr d\theta$$

Example: ⑭ Convert the Integral:

$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy \text{ to an equivalent}$$

integral in cylindrical Coordinates.

• z from 0 to $x \Rightarrow z$ from 0 to $r \cos \theta$

• x from 0 to $\sqrt{1-y^2} \Rightarrow r$ from 0 to $x = \sqrt{1-y^2}$
 $x^2 + y^2 = 1$
 $\therefore r^2 = 1$
 $r = 1$

• y from -1 to 1 $\Rightarrow \theta = \sin^{-1}(-1)$ to $\theta = \sin^{-1}(1)$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{r \cos \theta} r^2 dz r dr d\theta.$$

Spherical Coordinates and Integration:

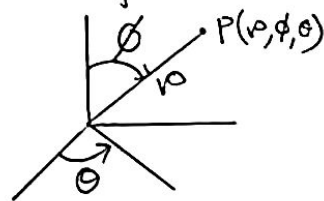
Spherical Coordinates represent a point P in space

by (ρ, ϕ, θ) , in which:

1. ρ is the distance from P to the origin, ($\rho \geq 0$)

2. ϕ is the angle $\vec{OP}$ makes with the positive z -axis ($0 \leq \phi \leq \pi$)

3. θ is the angle from cylindrical Coordinates.
 ($0 \leq \theta \leq 2\pi$).



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates:

$$r = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta = x$$

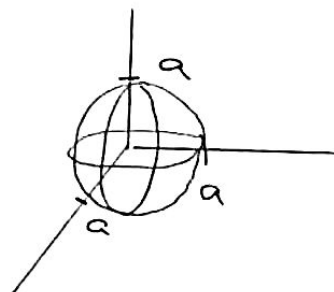
$$z = \rho \cos \phi$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta = y$$

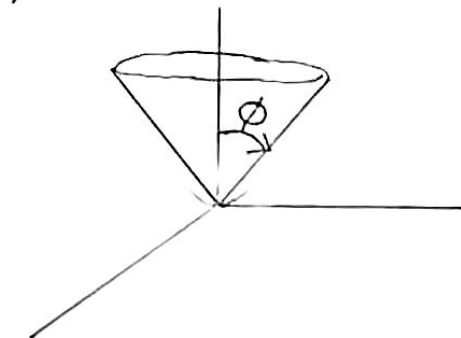
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Remark: In spherical Coordinates:

1) $\rho = a$ is sphere with radius a centered at origin
 ϕ & θ vary.

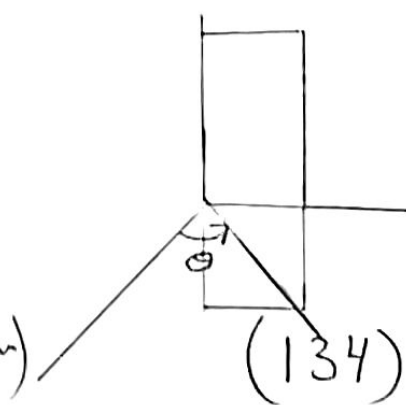


2) $\phi = \phi_0$ is Cone whose vertex at origin and whose axis is the z -axis (ρ & θ vary)



• As $\phi = \frac{\pi}{2}$ $\Rightarrow$ we get xy -plane
• The Cone open down for $\phi > \frac{\pi}{2}$

3) $\theta = \theta_0$ is half plane contains z -axis and makes angle θ_0 with positive x -axis. (Not in negative axis)



To find triple Integral over region D in spherical coordinates:

- we partition D into n small spherical wedges.
- in the kth spherical wedge ρ_k, ϕ_k, θ_k

change by $\Delta \rho_k, \Delta \phi_k, \Delta \theta_k$.

- $\Delta V_k = \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k$

- Riemann Sum:

$$S_n = \sum f(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \iiint_D f(\rho, \phi, \theta) dV$$

$$= \int_{\alpha}^{\beta} \int_{\psi_1(\theta)}^{\psi_2(\theta)} \int_{\rho_1(\phi, \theta)}^{\rho_2(\phi, \theta)} f(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta.$$

Example: Find spherical Coordinate to $x^2 + y^2 + (z-1)^2 = 1$

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\Rightarrow \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 = 1$$

$$\Rightarrow \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1 = 1$$

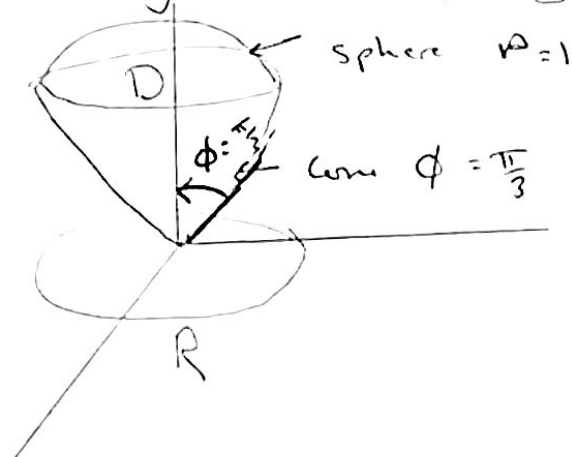
$$\Rightarrow \rho^2 - 2\rho \cos \phi = 0 \Rightarrow \rho^2 = 2\rho \cos \phi$$
$$\rho = 2 \cos \phi, \rho > 0 \quad (135)$$

Example: Find the Volume of the ice cream Cone D

cut from the solid sphere $\rho \leq 1$ by the Cone $\phi = \frac{\pi}{3}$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

$$= \boxed{\frac{\pi}{3}}$$



Remark : Coordinate Conversion Formulas:

1) From Cylindrical (r, θ, z) to rectangular (x, y, z) :

$$r \cos \theta = x, \quad r \sin \theta = y, \quad z = z$$

2) From spherical (ρ, ϕ, θ) to rectangular (x, y, z) :

$$\rho \sin \phi \cos \theta = x$$

$$\rho \sin \phi \sin \theta = y$$

$$\rho \cos \phi = z$$

3) From spherical (ρ, ϕ, θ) to cylindrical (r, θ, z) :

$$\rho \sin \phi = r$$

$$\rho \cos \phi = z$$

$$\theta = \theta$$

$$dv = dx \, dy \, dz = dz \, r \, dr \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

(136)

(12) Let D be the region bounded below by the cone

$z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$.

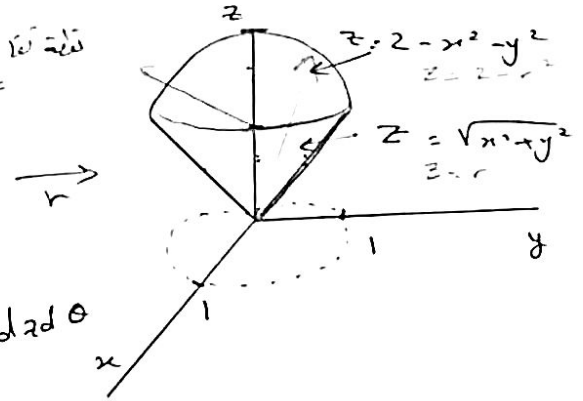
a) $dz dr d\theta$

b) $dr dz d\theta$

c) $d\theta dz dr$

a) $V = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz dr d\theta$

نقطة التقاطع
 $r=1$



b) $V = \int_0^{2\pi} \int_0^1 \int_0^z r dr dz d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r dr dz d\theta$

c) $V = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r d\theta dz dr$

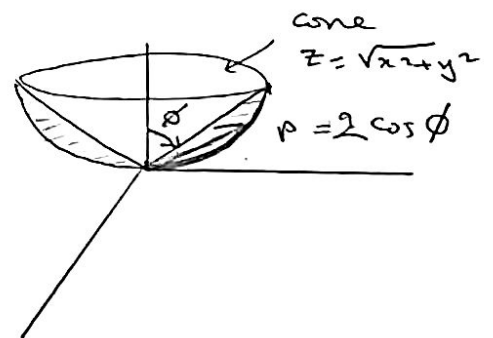
Region under the ray due to disk

(14) $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy = \int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{\cos\theta} r^3 dz dr d\theta$

$x = \sqrt{1-y^2}$
 $x^2 = 1-y^2$
 $x^2 + y^2 = 1$
 $r^2 = 1$

(37) Find the Volume.

$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} r^2 \sin\phi dr d\phi d\theta$

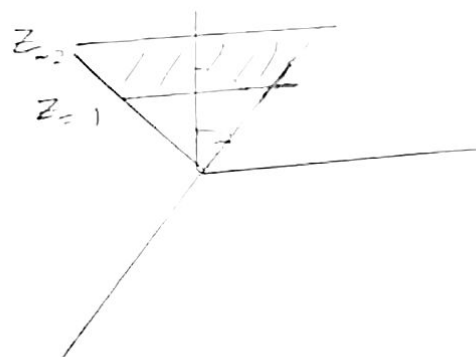


(137)

(52) Find the Volume of the Solid enclosed by the

Cone $Z = \sqrt{x^2 + y^2}$ between $z = 1$ & $z = 2$

$$V = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_{\sec \phi}^{2 \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



$2\pi \times 0 \sim \text{upper}$
 $4 \times \left(\frac{\pi}{2} \times 0 \sim \text{lower} \right)$

$$Z = \rho \cos \phi \Rightarrow \begin{aligned} Z=1 &\Rightarrow \rho = \sec \phi \\ Z=2 &\Rightarrow \rho = 2 \sec \phi \end{aligned}$$

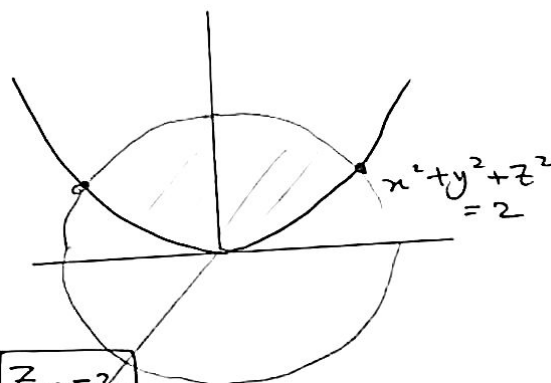
(62) Find the volume of the region bounded by above
 by the sphere $x^2 + y^2 + z^2 = 2$ & below by
 the paraboloid $Z = x^2 + y^2$

The y Intersect when:

$$x^2 + y^2 + z^2 = 2 \quad \& \quad x^2 + y^2 = z$$

$$\Rightarrow z^2 + z - 2 = 0 \Rightarrow \boxed{z=1} \text{ or } \boxed{z=-2}$$

Now since $z \geq 0$ $\Rightarrow \boxed{z=1} \Rightarrow x^2 + y^2 = 1 \Rightarrow \boxed{r^2=1}$



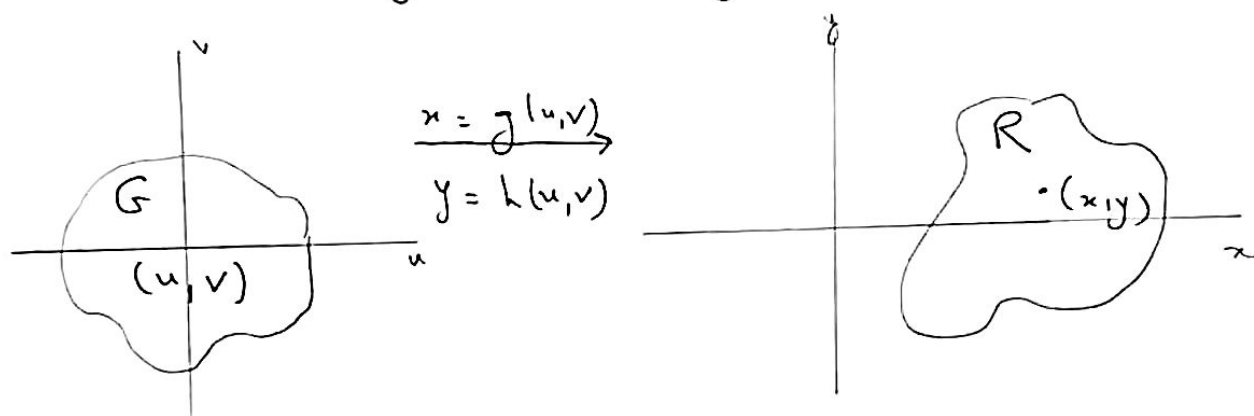
$$\Rightarrow V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$$

(138)

15.8 Substitution in Multiple Integrals:

1) Substitution in Double Integrals:

Suppose that a region G in uv -plane is transformed (1-1) into a region R in the xy -plane through the equations

$$x = g(u, v) \quad , \quad y = h(u, v)$$


- If $f(x, y)$ is defined on R , then $f(x, y)$ is defined on the region G by $f(g(u, v), h(u, v))$
- If g, h and f have continuous partial derivatives, then:

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| \, du \, dv$$

$$\text{where } J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$$

$J(u, v)$ is called the Jacobian of x & y .

(139)

Example: Evaluate

$$\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2} dx dy$$

by applying the transformation $u = \frac{2x-y}{2}$ & $v = \frac{y}{2}$
and Integrate w.r. to u and v

$$\bullet \quad u = x - \frac{y}{2} = x - v \iff \begin{cases} x = u + v \\ y = 2v \end{cases}$$

$$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\text{when } y = 0 \Rightarrow v = 0$$

$$y = 4 \Rightarrow v = 2$$

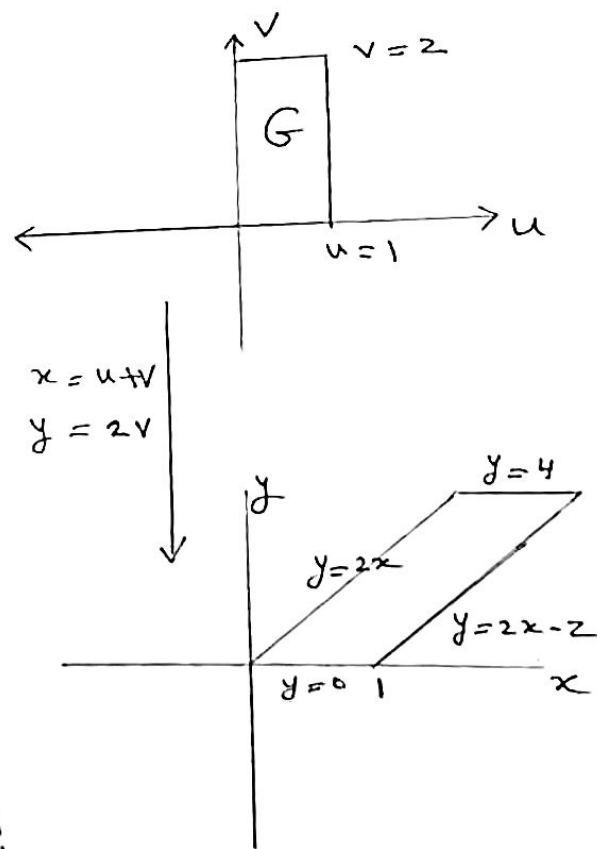
$$\text{when } x = \frac{y}{2} \Rightarrow u = 0$$

$$x = u + v \Rightarrow v = u + v$$

$$x = \frac{y}{2} + 1 \Rightarrow u = 1$$

$$\Rightarrow \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} dx dy$$

$$= \int_0^2 \int_0^1 u(2) du dv = \int_0^2 u^2 \Big|_0^1 dv = \int_0^2 dv = \boxed{2}$$



Example: Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx$

Let $u = x+y$ --- (x), $v = y-2x$

$\Rightarrow x = u - y$, $y = v + 2x$

$\Rightarrow x = u - (v + 2x) = u - v - 2x$

$\Rightarrow 3x = u - v$

then $x = \frac{u}{3} - \frac{v}{3}$ --- (1)

Now: $y = v + 2x = v + 2\left(\frac{u}{3} - \frac{v}{3}\right) =$

$\Rightarrow y = \frac{2u}{3} + \frac{v}{3}$ --- (2).

$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{3}{9} = \boxed{\frac{1}{3}}$

when $y = 0 \Rightarrow v = -2u$ from (2)

$y = 1-x \Rightarrow u = 1$ from (*)

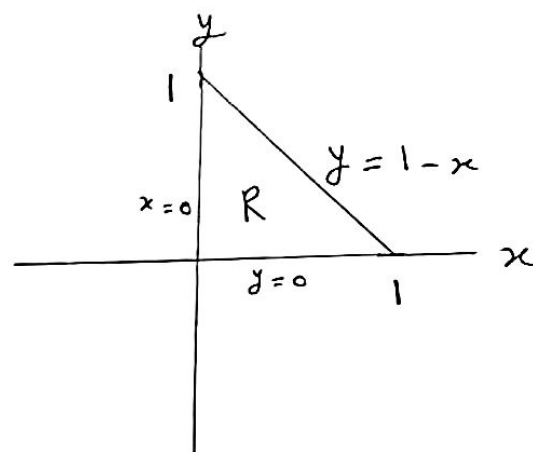
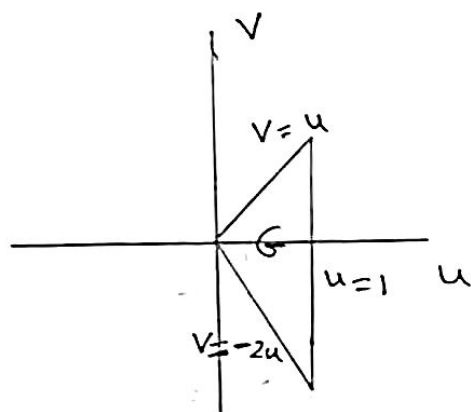
when $x = 0 \Rightarrow u = v$ from (1)

$x = 1 \Rightarrow 3 = u - v \Rightarrow v = u - 3$ from (1)

NOT Boundary

$$\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx$$

$$= \int_0^1 \int_{-2u}^u \sqrt{u} v^2 \left(\frac{1}{3}\right) d\check{v} du \quad \text{Vertical cross section} = \dots$$



$$= \frac{1}{3} \int_0^1 \sqrt{u} \left. \frac{v^3}{3} \right|_{-2u}^u du = \frac{1}{9} \int_0^1 \sqrt{u} (u^3 + 8u^3) du$$

$$= \frac{1}{9} \int_0^1 \sqrt{u} (9u^3) du = \int_0^1 u^{7/2} du$$

$$= \left. \frac{u^{9/2}}{9/2} \right|_0^1 = \boxed{\frac{2}{9}}$$

(We choose the Boundary of R to Identify the Boundary of G).

$(\frac{1}{2})(142)$

Example: Evaluate $\int_1^2 \int_{\frac{1}{y}}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$.

Sol: Let $u = \sqrt{xy}$, $v = \sqrt{\frac{y}{x}}$.

Squaring the equations:

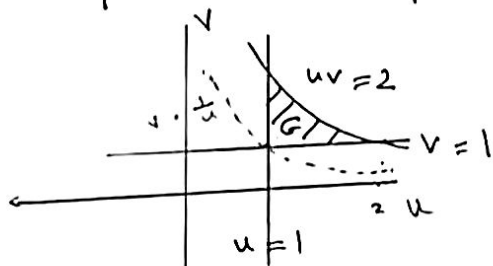
$$u^2 = xy \quad \dots (1) \quad \& \quad v^2 = \frac{y}{x} \quad \dots (2)$$

(2) $\Rightarrow y = v^2 x$. Now substitute y in (1), we have

$$u^2 = x(v^2 x) \Rightarrow \frac{u^2}{v^2} = x^2 \Rightarrow \boxed{x = \frac{u}{v}}$$

$$\text{Now from (2) : } y = v^2 x = v^2 \left(\frac{u}{v}\right) = \boxed{uv = y} \quad (*)$$

$$J(u, v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$



when $x = \frac{1}{y}$ Boundary of R
 $xy = 1 \Rightarrow u = 1$

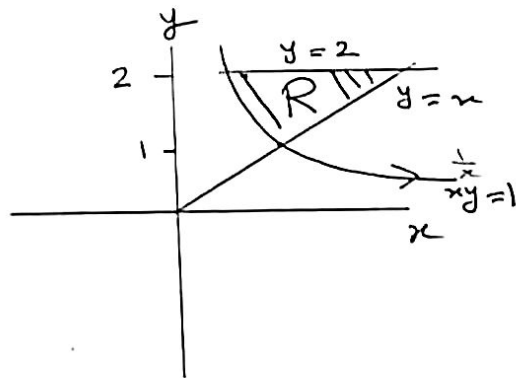
$y = x \Rightarrow v = 1$

when $y = 2 \Rightarrow v = \frac{2}{u}$

from (1)

from (2)

from (*)



$$\boxed{y=1 \Rightarrow uv=1}$$

$$v = \frac{1}{u}$$

G = region in

$$\rightarrow \int_1^2 \int_{\frac{1}{y}}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \int_1^2 \int_1^{\frac{2}{u}} v e^u \left(\frac{2u}{v}\right) dv du =$$

$$2e(e-2)$$

$$\left(\frac{2}{2}\right)(142)$$

2) Substitution in Triple Integrals:

- Suppose a region G in uvw -space is transformed (1-1) to a region R in xyz -space through the equations

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w).$$

- If $f(x, y, z)$ is defined on R , then f is defined on G

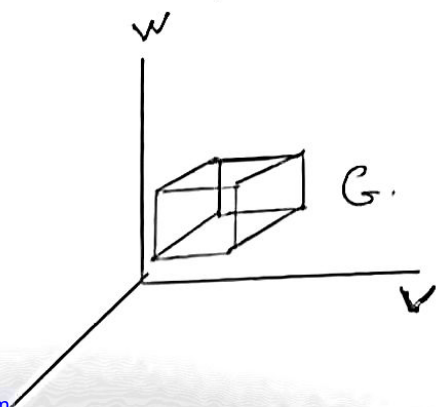
$$\begin{aligned} f(x, y, z) &= f(g(u, v, w), h(u, v, w), k(u, v, w)) \\ &= H(u, v, w). \end{aligned}$$

- If g, h, k & f have continuous first partial derivatives then:

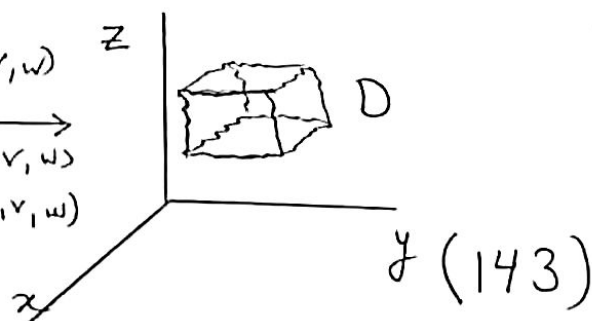
$$\iiint_R f(x, y, z) \, dx \, dy \, dz = \iiint_G H(u, v, w) |J(u, v, w)| \, du \, dv \, dw$$

where

$$J(u, v, w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$



$$\begin{aligned} x &= g(u, v, w) \\ y &= h(u, v, w) \\ z &= k(u, v, w) \end{aligned}$$



Example: (24) Let D be the region in xyz -plane defined by the inequalities:

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1$$

Evaluate
$$\iiint_D (x^2y + 3xyz) dx dy dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z.$$

$$\Rightarrow x = u, \quad y = \frac{v}{x} = \frac{v}{u}, \quad z = \frac{w}{3}$$

$$J(u, v, w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}.$$

$$\text{when } 1 \leq x \leq 2 \Leftrightarrow 1 \leq u \leq 2$$

$$0 \leq xy \leq 2 \Leftrightarrow 0 \leq v \leq 2.$$

$$0 \leq z \leq 1 \Leftrightarrow 0 \leq w \leq 3$$

$$\iiint_D (x^2y + 3xyz) dx dy dz = \int_1^2 \int_0^2 \int_0^3 (uv + vw) \frac{1}{3u} dw dv du$$

$$= 2 + 3 \ln 2.$$

Note: For Cylindrical Coordinates: r, θ & z

take the place of u, v, w . The transformation

from $r\theta z$ -space to xyz -space is given by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

$$J(r, \theta, z) = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = \boxed{r}$$

$$\Rightarrow \iiint_D f(x, y, z) dx dy dz = \iiint_G H(r, \theta, z) \underline{|r|} dr d\theta dz.$$

Note: For Spherical Coordinates: ρ, ϕ and θ take

the place of u, v, w . The transformation from

$\rho\phi\theta$ -space to xyz -space is given by:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$J(\rho, \phi, \theta) = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \rho^2 \sin \phi.$$

$$\Rightarrow \iiint_D f(x, y, z) dx dy dz = \iiint_G H(\rho, \phi, \theta) \underline{|\rho^2 \sin \phi|} d\rho d\phi d\theta.$$

(145)

⑨ Let R be the region in the first quadrant of xy -plane bounded by $xy = 1$, $xy = 9$ & $y = x$ & $y = 4x$

Use the transformation

$$x = \frac{u}{v}, \quad y = uv \quad \text{with } u > 0, \quad v > 0 \quad \text{to rewrite}$$

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy \quad \text{as an Integral in } uv\text{-plane}$$

$$\begin{aligned} \bullet \quad x = \frac{u}{v} \quad \& \quad y = uv \Rightarrow \frac{y}{x} &= \frac{uv}{(u/v)} = v^2 \\ &\Rightarrow xy = u^2. \end{aligned}$$

$$J(u, v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

$$\text{when } y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v^2 = 1 \Rightarrow v = 1$$

$$y = 4x \Rightarrow uv = \frac{4u}{v} \Rightarrow v^2 = 4 \Rightarrow v = 2$$

$$\text{when } xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1$$

$$xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3$$

$$\Rightarrow \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy = \int_1^3 \int_1^2 (v + u) \left(\frac{2u}{v} \right) dv du$$

$$= 8 + \frac{52}{3} (\ln 2).$$

$$\left(\frac{1}{3} \right) (146)$$

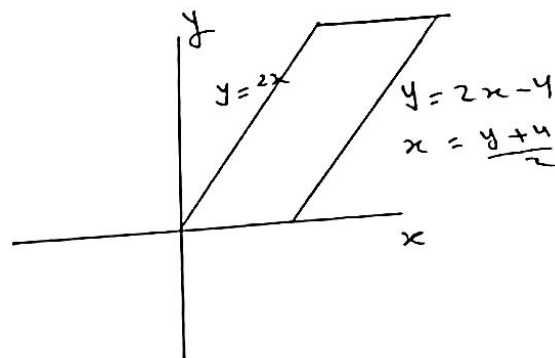
⑭ Use the transformation $x = u + \frac{1}{2}v$, $y = v$

to evaluate :

$$\int_0^2 \int_{y/2}^{\frac{y+4}{2}} y^3 (2x-y) e^{(2x-y)^2} dx dy.$$

• $x = u + \frac{1}{2}v$, $y = v$

$$J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$$



Now : $x = \frac{y}{2} \Rightarrow y = 2x \Rightarrow v = 2x \Rightarrow v = 2(u + \frac{1}{2}v)$

$$\Rightarrow 2u = 0 \Rightarrow \boxed{u = 0}$$

$$x = \frac{y+4}{2} \Rightarrow u + \frac{1}{2}v = \frac{v+4}{2} \Rightarrow$$

$$u + \frac{1}{2}v = \frac{1}{2}v + 2 \Rightarrow \boxed{u = 2}$$

$$y = 0 \Rightarrow v = 0$$

$$y = 2 \Rightarrow v = 2$$

$$\int_0^2 \int_{y/2}^{\frac{y+4}{2}} y^3 (2x-y) e^{(2x-y)^2} dx dy = \int_0^2 \int_0^2 v^3 (2u) e^{4u^2} du dv$$

$$= \boxed{16e - 1}$$

(2) (146)

(22) Find the Volume of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Hint : Let $x = au$, $y = bv$, and $z = cw$)

$$J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow u^2 + v^2 + w^2 = 1$$

$$V = \iiint_R dx dy dz = \iiint_G abc \, du dv dw$$

$$= \frac{4\pi abc}{3}$$

(spherical Region) $V = \frac{4}{3}\pi r^3 = \left(\frac{4}{3}\pi\right)$

$$\left(\frac{4}{3}\right)(146)$$