Exercises of chapter 17 :

Q4: suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in Z[x]$. if r is rational and $x-r$ divides $f(x)$, show that r is an integer.

$(x-r) \mid f(x) \rightsquigarrow f(r) = 0$, $r \in \mathbb{Q}$, $r = \frac{p}{q}$, $g.c.d(\underbrace{p, q}_{\in \mathbb{Z}}) = 1$

$$0 = f(r) = f\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_1\left(\frac{p}{q}\right) + a_0$$

multiply by q^n $\quad 0 = p^n + a_{n-1}p^{n-1}q + \dots + a_1 p q^{n-1} + a_0 q^n$

$$p^n = -q\left[a_{n-1}p^{n-1} + \dots + a_1 p q^{n-2} + a_0 q^{n-1}\right]$$

So $q \mid p^n$ But $g.c.d(p, q) = 1 \Rightarrow q = \pm 1$

so $r = p$ or $-p \in \mathbb{Z}$.

Q8: suppose that $f(x) \in \mathbb{Z}_p[x]$ and $f(x)$ is irreducible over $\mathbb{Z}_p$, where p is a prime if $\deg f(x) = n$ prove that $\mathbb{Z}_p[x] / \langle f(x) \rangle$ is a field with p^n elements.

$\mathbb{Z}_p$ is a field and $\langle f(x) \rangle$ is irreducible $\Rightarrow \mathbb{Z}_p / \langle f(x) \rangle$ is a field

since $\deg f(x) = n \rightsquigarrow$ each element in F can be represented uniquely

are $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, $a \in \mathbb{Z}_p = F$

So $\exists\, n$ coefficent and each of them has p choices.

$\Rightarrow \mathbb{Z}_p[x] / \langle f(x) \rangle$ is a field with p^n elements.

Q9 + Q10 : easy

Q11: show that $x^3 + x^2 + x + 1$ is reducible over $\mathbb{Q}$, ~~does this~~

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$$x^3 + x^2 + x + 1 = (x^2+1)(x+1)$$

since it splits into factors , then it is reducible over $\mathbb{Q}$

corollary : for any prime p : $p(x) = \frac{x^p - 1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

in our case $x^{p-1} = x^3 \Rightarrow p = 4$ is Not prime

so our result does Not contradict the corollary .

Q12 : Determine which of the polynomials below is (are) irreducible over $\mathbb{Q}$.

1. $x^5 + 9x^4 + 12x^2 + 6$. using Eisten- - .

let $p = 3$

$$3 \nmid 1 \ , \ 3 \mid 9 \ , \ 3 \mid 12 \ , \ 3^2 \nmid 6$$

so f is irreducible over $\mathbb{Q}$.

2. $x^4 + x + 1$

$$\begin{array}{r|l} & x^2 + x \\ \hline x^2 + x + 1 & x^4 + x + 1 \\ & -x^4 + x^3 + x^2 \\ \hline & 1 + x - x^3 - x^2 \equiv x^3 + x^2 + x + 1 \pmod 2 \\ & x^3 + x^2 + x \\ \hline & 1 \end{array}$$

$x^2 + x \nmid x^4 + x + 1$

f is irreducible

3. $x^4 + 3x^2 + 3$.

let $p = 3$

$3 \nmid 1$, $3 \mid 3$, $3^2 \nmid 3$

so its irreducible over $\mathbb{Q}$.

4. $x^5 + 5x^2 + 1 \equiv x^5 + x^2 + 1 \pmod 2$

$x^5 + x^2 + 1 \nmid x^5 + 5x^2 + 1$.

so its irreducible over $\mathbb{Q}$

$$x^3 - x^2 + 2$$

$$x^5 + x^2 + 1 \,\overline{)\, x^5 + 5x^2 + 1}$$

$$\underline{x^5 + x^4 + x^3}$$

$$-x^4 - x^3 + x^2 + 1$$

$$\underline{-x^4 - x^3 - x^2}$$

$$2x^2 + 1$$

$$\underline{2x^2 + 2x + 2}$$

$$-2x - 1$$

5. $\left(\frac{5}{2}\right)x^5 + \left(\frac{9}{2}\right)x^4 + 15x^3 + \left(\frac{3}{7}\right)x^2 + 6x + \frac{3}{14}$

l.c.m of المقام $= 14$

$\rightarrow$ $14\, f(x) = 35x^5 + 63x^4 + 210x^3 + 6x^2 + 84x + 3$

By Einesteines criterion.

let $p = 3$

$3 \nmid 35$, $3 \mid 63$, $3 \mid 210$, $3 \mid 6$, $3 \mid 84$, $3^2 \nmid 3$.

so its irreducible over $\mathbb{Q}$.

Q13: Show that x^4+1 is irreducible over $\mathbb{Q}$ But reducible over $\mathbb{R}$.

on sol. book.

Q14: Show that ~~B~~ x^2+x+4 is irreducible over $\mathbb{Z}_{11}$.

$f(0) = 4$

$f(1) = 6$

$f(2) = 10$

$f(3) = 5$ so its irreducible

$f(4) = 2$

$f(5) = 1$

$f(6) = 2$

$f(7) = 5$

$f(8) = 10$

$f(9) = 6$

$f(10) = 4$