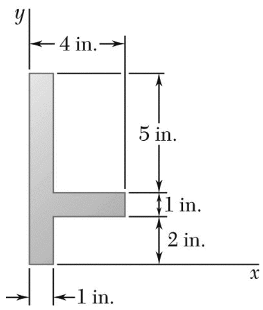
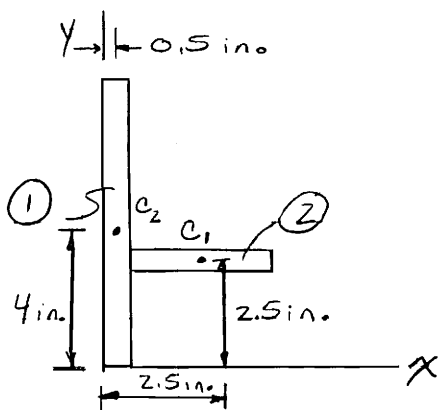
CHAPTER 5



PROBLEM 5.1

Locate the centroid of the plane area shown.

SOLUTION



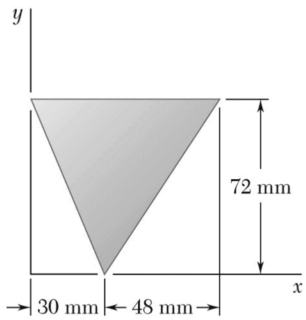
	A, in^2	$\bar{x}$, in	$\bar{y}$, in	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	8	0.5	4	4	32
2	3	2.5	2.5	7.5	7.5
Σ	11			11.5	39.5

$$\bar{X}\Sigma A = \bar{x}A$$

$$\bar{X}(11\text{ in}^2) = 11.5\text{ in}^3 \qquad \bar{X} = 1.045\text{ in.} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

$$\bar{Y}(11) = 39.5 \qquad \bar{Y} = 3.59\text{ in.} \blacktriangleleft$$



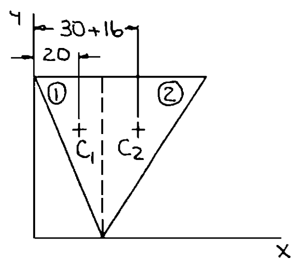
PROBLEM 5.2

Locate the centroid of the plane area shown.

SOLUTION

For the area as a whole, it can be concluded by observation that

$$\overline{Y} = \frac{2}{3}(72 \text{ mm}) \qquad \text{or } \overline{Y} = 48.0 \text{ mm} \blacktriangleleft$$

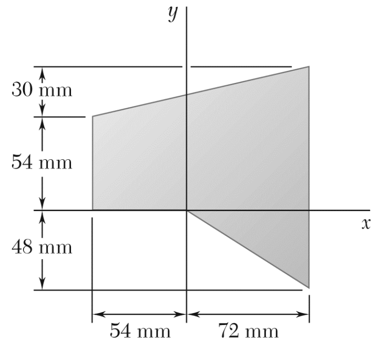


Dimensions in mm

	A, mm^2	$\bar{x}$, mm	$\bar{x}A$, mm^3
1	$\frac{1}{2}\times 30\times 72 = 1080$	20	21,600
2	$\frac{1}{2}\times 48\times 72 = 1728$	46	79,488
Σ	2808		101,088

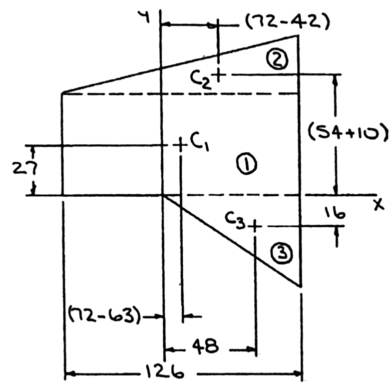
Then $\overline{X}A = \Sigma\bar{x}A \qquad \overline{X}(2808) = 101{,}088 \qquad$ or $\overline{X} = 36.0$ mm ◀

PROBLEM 5.3



Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$126\times 54=6804$	9	27	61,236	183,708
2	$\frac{1}{2}\times 126\times 30=1890$	30	64	56,700	120,960
3	$\frac{1}{2}\times 72\times 48=1728$	48	−16	82,944	−27,648
Σ	10,422			200,880	277,020

Then

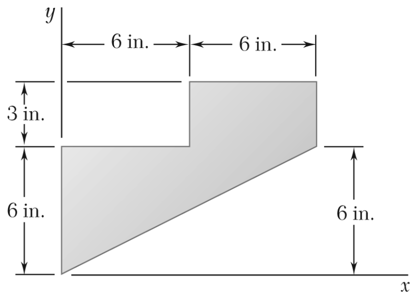
$$\bar{X}\Sigma A=\Sigma\bar{x}A$$

$$\bar{X}(10{,}422\text{ m}^2)=200{,}880\text{ mm}^2 \qquad \text{or } \bar{X}=19.27\text{ mm} \blacktriangleleft$$

and

$$\bar{Y}\Sigma A=\Sigma\bar{y}A$$

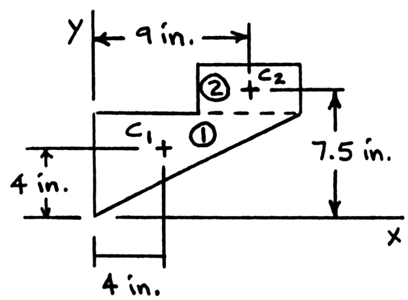
$$\bar{Y}(10{,}422\text{ m}^2)=270{,}020\text{ mm}^3 \qquad \text{or } \bar{Y}=26.6\text{ mm} \blacktriangleleft$$



PROBLEM 5.4

Locate the centroid of the plane area shown.

SOLUTION



	A, in^2	$\bar{x}$, in	$\bar{y}$, in	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{1}{2}(12)(6) = 36$	4	4	144	144
2	$(6)(3) = 18$	9	7.5	162	135
Σ	54			306	279

Then

$$\bar{X}A = \Sigma \bar{x}A$$

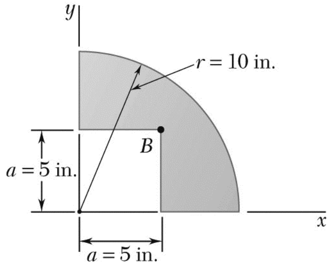
$$\bar{X}(54) = 306$$

$\bar{X} = 5.67$ in. ◀

$$\bar{Y}A = \Sigma \bar{y}A$$

$$\bar{Y}(54) = 279$$

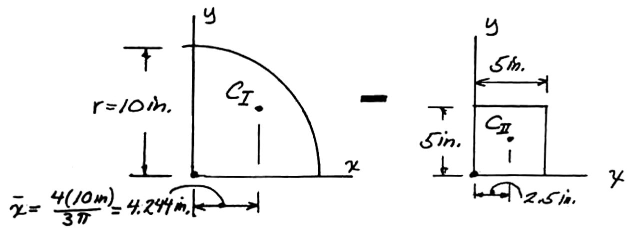
$\bar{Y} = 5.17$ in. ◀



PROBLEM 5.5

Locate the centroid of the plane area shown.

SOLUTION



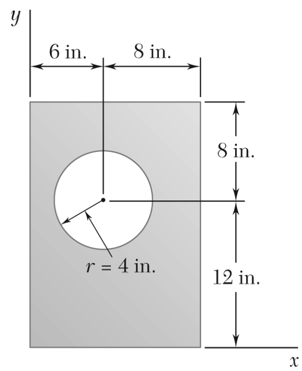
By symmetry, $\bar{X} = \bar{Y}$

	Component	A, in^2	$\bar{x}$, in.	$\bar{x}A$, in^3
I	Quarter circle	$\frac{\pi}{4}(10)^2 = 78.54$	4.2441	333.33
II	Square	$-(5)^2 = -25$	2.5	-62.5
Σ		53.54		270.83

$$\bar{X}\Sigma A = \Sigma \bar{x} A: \quad \bar{X}(53.54\text{ in}^2) = 270.83\text{ in}^3$$

$$\bar{X} = 5.0585\text{ in.}$$

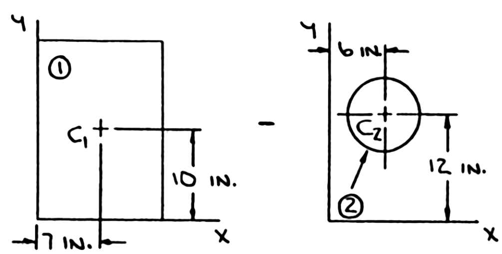
$\bar{X} = \bar{Y} = 5.06$ in. ◀



PROBLEM 5.6

Locate the centroid of the plane area shown.

SOLUTION



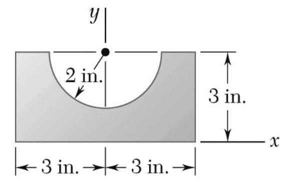
	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$14 \times 20 = 280$	7	10	1960	2800
2	$-\pi(4)^2 = -16\pi$	6	12	–301.59	–603.19
Σ	229.73			1658.41	2196.8

Then

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{1658.41}{229.73}$$ $\bar{X} = 7.22$ in. ◀

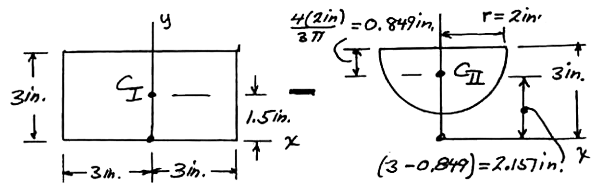
$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{2196.8}{229.73}$$ $\bar{Y} = 9.56$ in. ◀

PROBLEM 5.7



Locate the centroid of the plane area shown.

SOLUTION



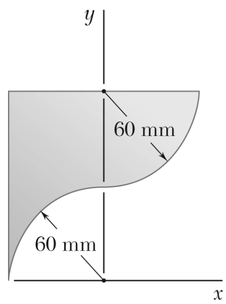
By symmetry, $\overline{X} = 0$

	Component	A, in^2	$\overline{y}$, in.	$\overline{y}A$, in^3
I	Rectangle	$(3)(6) = 18$	1.5	27.0
II	Semicircle	$-\dfrac{\pi}{2}(2)^2 = -6.28$	2.151	−13.51
Σ		11.72		13.49

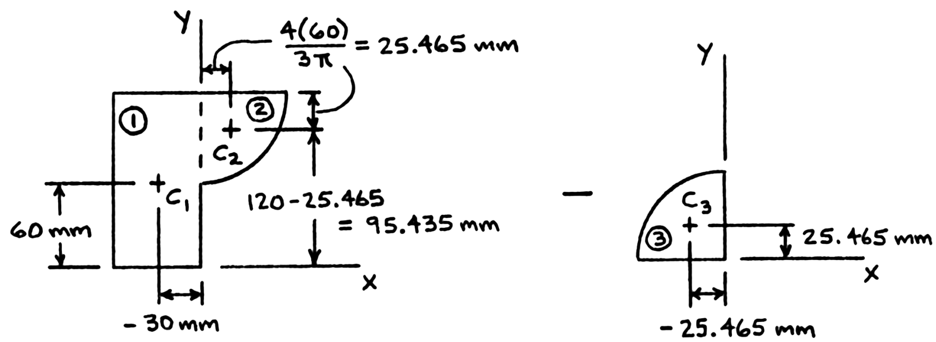
$$\overline{Y}\Sigma A = \Sigma \overline{y} A$$
$$\overline{Y}(11.72\text{ in.}^2) = 13.49\text{ in}^3$$
$$\overline{Y} = 1.151\text{ in.}$$

$\overline{X} = 0$

$\overline{Y} = 1.151\text{ in.}$ ◀

	**PROBLEM 5.8** Locate the centroid of the plane area shown.

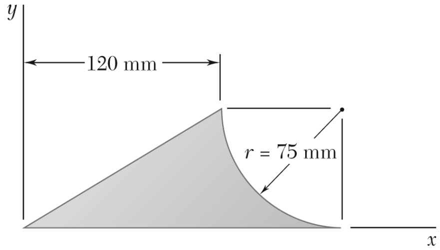
SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(60)(120) = 7200$	–30	60	-216×10^3	432×10^3
2	$\frac{\pi}{4}(60)^2 = 2827.4$	25.465	95.435	72.000×10^3	269.83×10^3
3	$-\frac{\pi}{4}(60)^2 = -2827.4$	–25.465	25.465	72.000×10^3	-72.000×10^3
Σ	7200			-72.000×10^3	629.83×10^3

Then $\quad \bar{X}A = \Sigma\bar{x}A \quad \bar{X}(7200) = -72.000\times10^3 \quad$ $\bar{X} = -10.00$ mm ◀

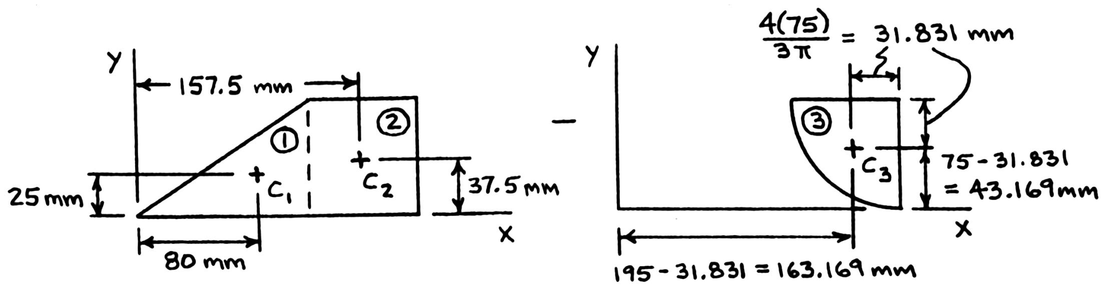
$\bar{Y}A = \Sigma\bar{y}A \quad \bar{Y}(7200) = 629.83\times10^3 \quad$ $\bar{Y} = 87.5$ mm ◀



PROBLEM 5.9

Locate the centroid of the plane area shown.

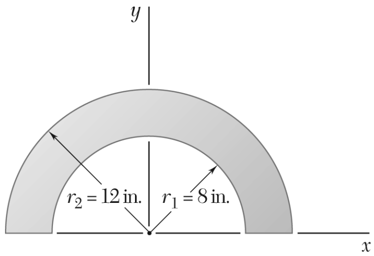
SOLUTION



	A, mm^2	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{x}A$, mm^3	$\overline{y}A$, mm^3
1	$\frac{1}{2}(120)(75) = 4500$	80	25	360×10^3	112.5×10^3
2	$(75)(75) = 5625$	157.5	37.5	885.94×10^3	210.94×10^3
3	$-\frac{\pi}{4}(75)^2 = -4417.9$	163.169	43.169	-720.86×10^3	-190.716×10^3
Σ	5707.1			525.08×10^3	132.724×10^3

Then $\overline{X}A = \Sigma\overline{x}A \qquad \overline{X}(5707.1) = 525.08\times10^3 \qquad \overline{X} = 92.0 \text{ mm}$ ◀

$\overline{Y}A = \Sigma\overline{y}A \qquad \overline{Y}(5707.1) = 132.724\times10^3 \qquad \overline{Y} = 23.3 \text{ mm}$ ◀

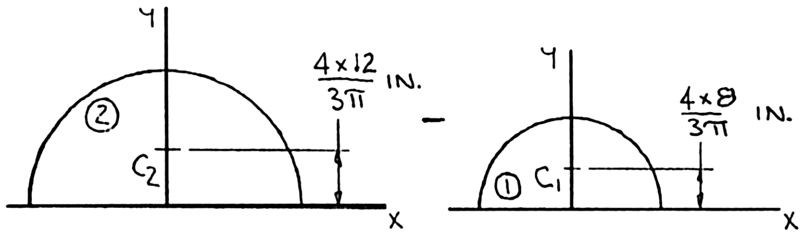


PROBLEM 5.10

Locate the centroid of the plane area shown.

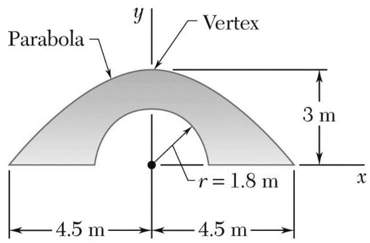
SOLUTION

First note that symmetry implies $\bar{X} = 0$ ◀



	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$-\dfrac{\pi(8)^2}{2} = -100.531$	3.3953	−341.33
2	$\dfrac{\pi(12)^2}{2} = 226.19$	5.0930	1151.99
Σ	125.659		810.66

Then $$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{810.66 \text{ in}^3}{125.66 \text{ in}^2}$$ or $\bar{Y} = 6.45$ in. ◀

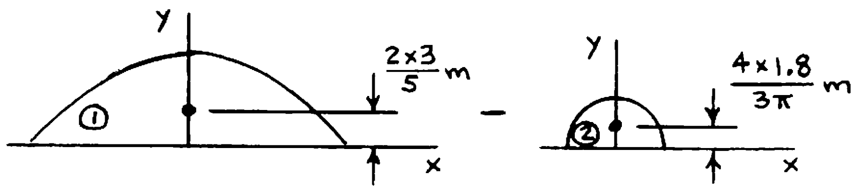


PROBLEM 5.11

Locate the centroid of the plane area shown.

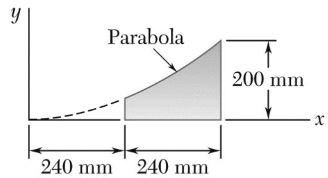
SOLUTION

First note that symmetry implies $\bar{X} = 0$ ◀



	A, m^2	$\bar{y}$, m	$\bar{y}A$, m^3
1	$\frac{4}{3}\times4.5\times3=18$	1.2	21.6
2	$-\frac{\pi}{2}(1.8)^2=-5.0894$	0.76394	−3.8880
Σ	12.9106		17.7120

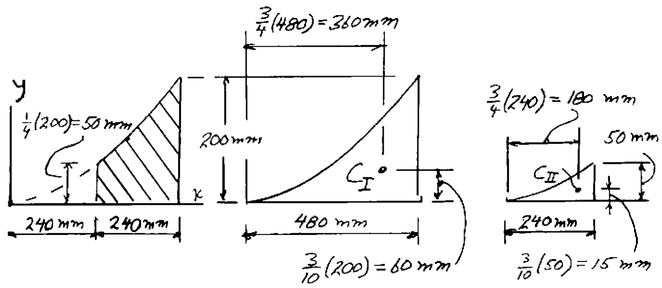
Then $$\bar{Y}=\frac{\Sigma\bar{y}A}{\Sigma A}=\frac{17.7120\text{ m}^3}{12.9106\text{ m}^2}$$ or $\bar{Y}=1.372$ m ◀



PROBLEM 5.12

Locate the centroid of the plane area shown.

SOLUTION



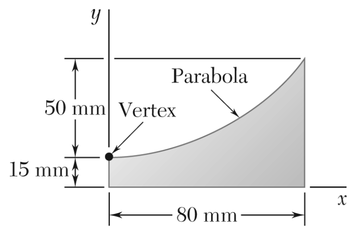
	Area mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{3}(200)(480) = 32\times10^3$	360	60	11.52×10^6	1.92×10^6
2	$-\frac{1}{3}(50)(240) = 4\times10^3$	180	15	-0.72×10^6	-0.06×10^6
Σ	28×10^3			10.80×10^6	1.86×10^6

$\bar{X}\Sigma A = \Sigma \bar{x} A$: $\quad \bar{X}(28\times10^3 \text{ mm}^2) = 10.80\times10^6 \text{ mm}^3$

$\bar{X} = 385.7$ mm $\qquad \bar{X} = 386$ mm ◀

$\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\quad \bar{Y}(28\times10^3 \text{ mm}^2) = 1.86\times10^6 \text{ mm}^3$

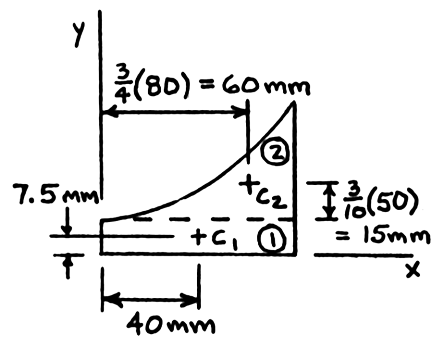
$\bar{Y} = 66.43$ mm $\qquad \bar{Y} = 66.4$ mm ◀



PROBLEM 5.13

Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(15)(80) = 1200$	40	7.5	48×10^3	9×10^3
2	$\frac{1}{3}(50)(80) = 1333.33$	60	30	80×10^3	40×10^3
Σ	2533.3			128×10^3	49×10^3

Then

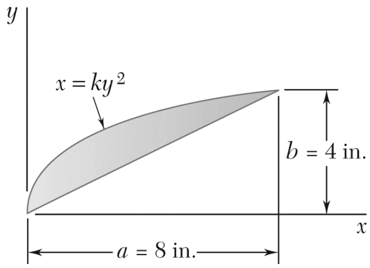
$$\bar{X}A = \Sigma\bar{x}A$$

$$\bar{X}(2533.3) = 128\times10^3 \qquad \bar{X} = 50.5 \text{ mm} \blacktriangleleft$$

$$\bar{Y}A = \Sigma\bar{y}A$$

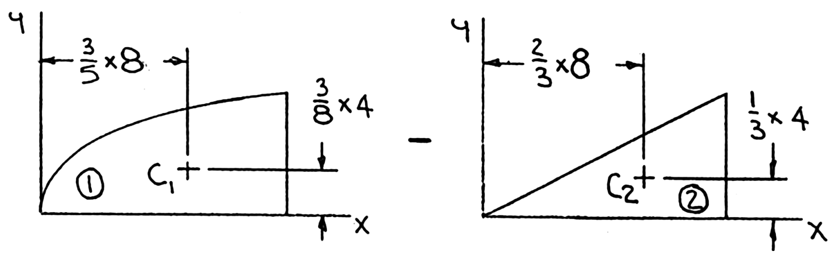
$$\bar{Y}(2533.3) = 49\times10^3 \qquad \bar{Y} = 19.34 \text{ mm} \blacktriangleleft$$

PROBLEM 5.14



Locate the centroid of the plane area shown.

SOLUTION



Dimensions in in.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{2}{3}(4)(8) = 21.333$	4.8	1.5	102.398	32.000
2	$-\frac{1}{2}(4)(8) = -16.0000$	5.3333	1.33333	85.333	−21.333
Σ	5.3333			17.0650	10.6670

Then

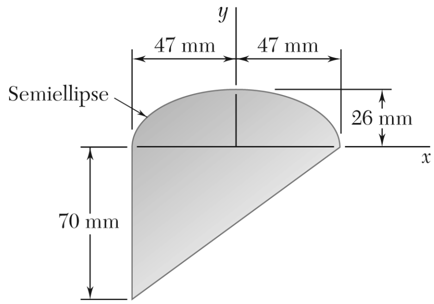
$$\bar{X}\Sigma A = \Sigma \bar{x}A$$

$$\bar{X}(5.3333 \text{ in}^2) = 17.0650 \text{ in}^3 \qquad \text{or} \quad \bar{X} = 3.20 \text{ in.} \blacktriangleleft$$

and

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(5.3333 \text{ in}^2) = 10.6670 \text{ in}^3 \qquad \text{or} \quad \bar{Y} = 2.00 \text{ in.} \blacktriangleleft$$

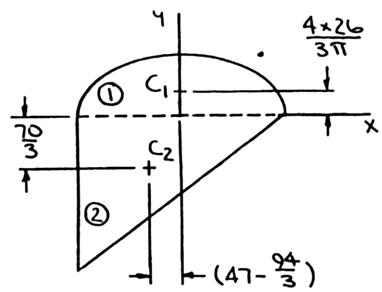


PROBLEM 5.15

Locate the centroid of the plane area shown.

SOLUTION

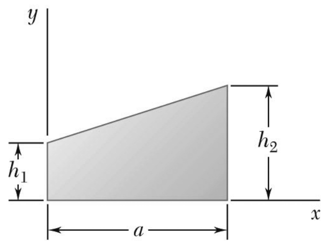
Dimensions in mm



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{\pi}{2}\times 47\times 26 = 1919.51$	0	11.0347	0	21,181
2	$\frac{1}{2}\times 94\times 70 = 3290$	−15.6667	−23.333	−51,543	−76,766
Σ	5209.5			−51,543	−55,584

Then
$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{-51,543}{5209.5}$$
$\bar{X} = -9.89$ mm ◀

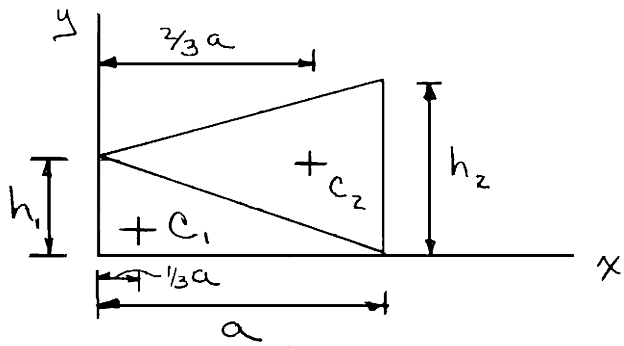
$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{-55,584}{5209.5}$$
$\bar{Y} = -10.67$ mm ◀



PROBLEM 5.16

Determine the x coordinate of the centroid of the trapezoid shown in terms of h_1, h_2, and a.

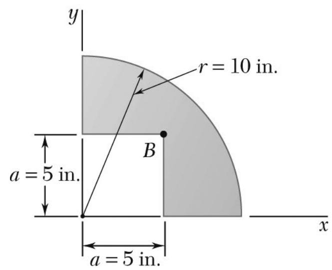
SOLUTION



	A	$\bar{x}$	$\bar{x}A$
1	$\frac{1}{2}h_1 a$	$\frac{1}{3}a$	$\frac{1}{6}h_1 a^2$
2	$\frac{1}{2}h_2 a$	$\frac{2}{3}a$	$\frac{2}{6}h_2 a^2$
Σ	$\frac{1}{2}a(h_1 + h_2)$		$\frac{1}{6}a^2(h_1 + 2h_2)$

$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{\frac{1}{6}a^2(h_1 + 2h_2)}{\frac{1}{2}a(h_1 + h_2)}$$

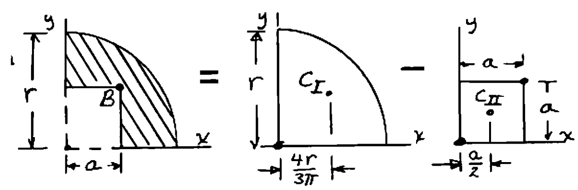
$$\bar{X} = \frac{1}{3}a\frac{h_1 + 2h_2}{h_1 + h_2} \blacktriangleleft$$



PROBLEM 5.17

For the plane area of Problem 5.5, determine the ratio a/r so that the centroid of the area is located at point B.

SOLUTION



By symmetry, $\bar{X} = \bar{Y}$. For centroid to be at B, $\bar{X} = a$.

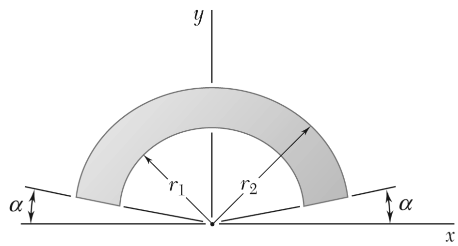
		Area	$\bar{x}$	$\bar{x}A$
I	Quarter circle	$\frac{1}{4}\pi r^2$	$\frac{4r}{3\pi}$	$\frac{1}{3}r^3$
II	Square	$-a^2$	$\frac{1}{2}a$	$-\frac{1}{2}a^3$
Σ		$\frac{\pi}{4}r^2 - a^2$		$\frac{1}{3}r^3 - \frac{1}{2}a^3$

$\bar{X}\Sigma A = \Sigma \bar{x} A$: $\quad \bar{X}\left(\frac{\pi}{4}r^2 - a^2\right) = \frac{1}{3}r^3 - \frac{1}{2}a^3$

Set $\bar{X} = a$: $\quad a\left(\frac{\pi}{4}r^2 - a^2\right) = \frac{1}{3}r^3 - \frac{1}{2}a^3$

$$\frac{1}{2}a^3 - \frac{\pi}{4}r^2 a + \frac{1}{3}r^3 = 0$$

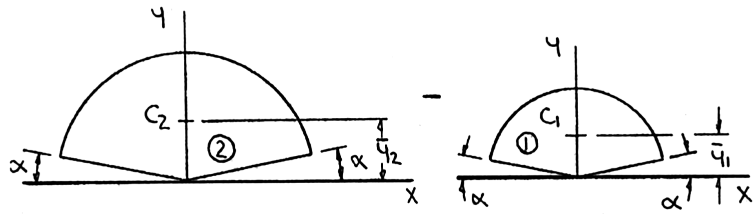
Divide by $\frac{1}{2}r^3$: $\quad \left(\frac{a}{r}\right)^3 - \frac{\pi}{2}\frac{a}{r} + \frac{2}{3} = 0 \qquad \frac{a}{r} = 0.508$ ◀



PROBLEM 5.18

Determine the y coordinate of the centroid of the shaded area in terms of r_1, r_2, and α.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A:

$$\bar{y}_2 = \frac{2}{3} r_2 \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_2 = \left(\frac{\pi}{2} - \alpha\right) r_2^2$$

$$= \frac{2}{3} r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)}$$

Similarly,

$$\bar{y}_1 = \frac{2}{3} r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_1 = \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

Then

$$\Sigma \bar{y} A = \frac{2}{3} r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_2^2\right] - \frac{2}{3} r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_1^2\right]$$

$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and

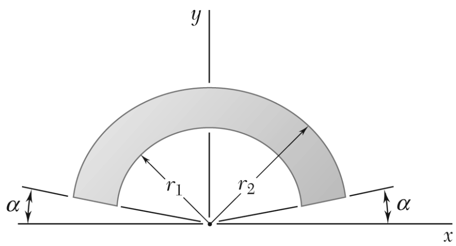
$$\Sigma A = \left(\frac{\pi}{2} - \alpha\right) r_2^2 - \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

$$= \left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)$$

Now

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

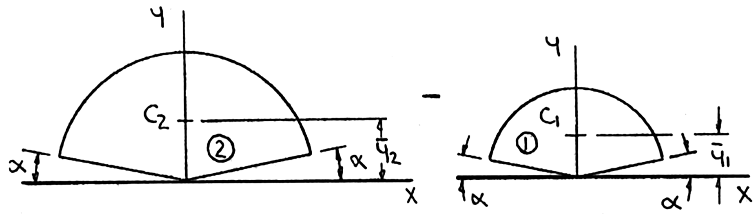
$$\bar{Y}\left[\left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha \qquad \bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2\cos\alpha}{\pi - 2\alpha}\right) \blacktriangleleft$$



PROBLEM 5.19

Show that as r_1 approaches r_2, the location of the centroid approaches that for an arc of circle of radius $(r_1 + r_2)/2$.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A:

$$\bar{y}_2 = \frac{2}{3} r_2 \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_2 = \left(\frac{\pi}{2} - \alpha\right) r_2^2$$

$$= \frac{2}{3} r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)}$$

Similarly,

$$\bar{y}_1 = \frac{2}{3} r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_1 = \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

Then

$$\Sigma \bar{y}A = \frac{2}{3} r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_2^2\right] - \frac{2}{3} r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_1^2\right]$$

$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and

$$\Sigma A = \left(\frac{\pi}{2} - \alpha\right) r_2^2 - \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

$$= \left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)$$

Now

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}\left[\left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

$$\bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2\cos\alpha}{\pi - 2\alpha}\right)$$

PROBLEM 5.19 (Continued)

Using Figure 5.8B, $\bar{Y}$ of an arc of radius $\frac{1}{2}(r_1 + r_2)$ is

$$\bar{Y} = \frac{1}{2}(r_1 + r_2)\frac{\sin(\frac{\pi}{2} - \alpha)}{(\frac{\pi}{2} - \alpha)}$$

$$= \frac{1}{2}(r_1 + r_2)\frac{\cos\alpha}{(\frac{\pi}{2} - \alpha)} \qquad (1)$$

Now

$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r_2 - r_1)\left(r_2^2 + r_1 r_2 + r_1^2\right)}{(r_2 - r_1)(r_2 + r_1)}$$

$$= \frac{r_2^2 + r_1 r_2 + r_1^2}{r_2 + r_1}$$

Let

$$r_2 = r + \Delta$$

$$r_1 = r - \Delta$$

Then

$$r = \frac{1}{2}(r_1 + r_2)$$

and

$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r + \Delta)^2 + (r + \Delta)(r - \Delta)(r - \Delta)^2}{(r + \Delta) + (r - \Delta)}$$

$$= \frac{3r^2 + \Delta^2}{2r}$$

In the limit as $\Delta \longrightarrow 0$ (i.e., $r_1 = r_2$), then

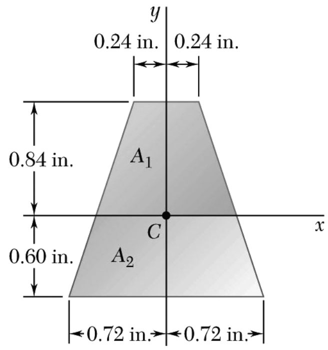
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{3}{2}r$$

$$= \frac{3}{2} \times \frac{1}{2}(r_1 + r_2)$$

So that

$$\bar{Y} = \frac{2}{3} \times \frac{3}{4}(r_1 + r_2)\frac{\cos\alpha}{\frac{\pi}{2} - \alpha} \qquad \text{or} \quad \bar{Y} = (r_1 + r_2)\frac{\cos\alpha}{\pi - 2\alpha} \blacktriangleleft$$

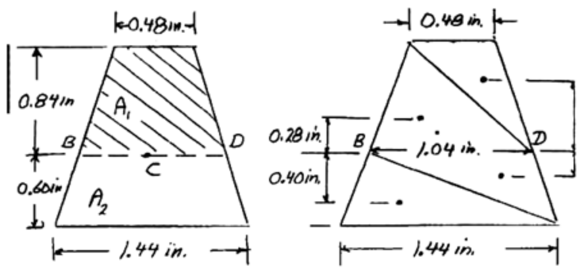
which agrees with Equation (1).



PROBLEM 5.20

The horizontal x-axis is drawn through the centroid C of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x-axis, and explain the results obtained.

SOLUTION



Length of BD:

$$BD = 0.48\text{ in.} + (1.44\text{ in.} - 0.48\text{ in.})\frac{0.84\text{in.}}{0.84\text{ in.} \times 0.60\text{ in.}} = 0.48 + 0.56 = 1.04\text{ in.}$$

Area above x-axis (consider two triangular areas):

$$Q_1 = \Sigma\,\overline{y}_A = (0.28\text{ in.})\left[\frac{1}{2}(0.84\text{ in.})(1.04\text{ in.})\right] + (0.56\text{ in.})\left[\frac{1}{2}(0.84\text{ in.})(0.48\text{ in.})\right]$$

$$= 0.122304\text{ in}^3 + 0.112896\text{ in}^3$$

$$Q_1 = 0.2352\text{ in}^3 \blacktriangleleft$$

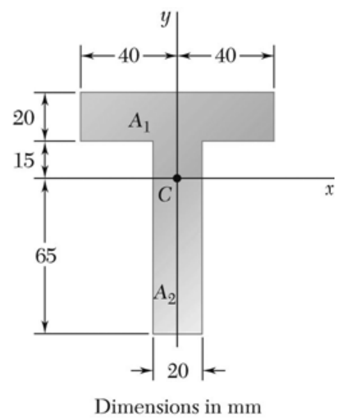
Area below x-axis:

$$Q_2 = \Sigma\,\overline{y}A = -(0.40\text{ in.})\left[\frac{1}{2}(0.60\text{ in.})(1.44\text{ in.})\right] - (0.20\text{ in.})\left[\frac{1}{2}(0.60\text{ in.})\right]$$

$$= -0.1728\text{ in}^3 - 0.0624\text{ in}^3$$

$$Q_2 = -0.2352\text{ in}^3 \blacktriangleleft$$

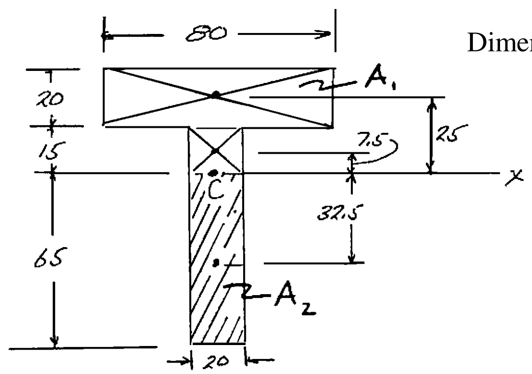
$$|Q| = |Q_2|, \text{ since } C \text{ is centroid and thus, } Q = \Sigma\,\overline{y}A = 0$$



PROBLEM 5.21

The horizontal x-axis is drawn through the centroid C of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x-axis, and explain the results obtained.

SOLUTION



Dimensions in mm

<u>Area above x-axis</u> (Area A_1):

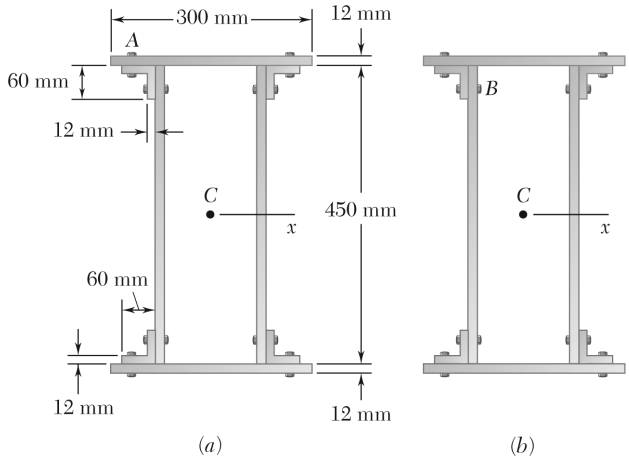
$$Q_1 = \Sigma \bar{y} A = (25)(20\times 80) + (7.5)(15\times 20)$$
$$= 40\times 10^3 + 2.25\times 10^3$$

$$Q_1 = 42.3\times 10^3 \text{ mm}^3 \blacktriangleleft$$

<u>Area below x-axis</u> (Area A_2):

$$Q_2 = \Sigma \bar{y} A = (-32.5)(65\times 20) \qquad Q_2 = -42.3\times 10^3 \text{ mm}^3 \blacktriangleleft$$

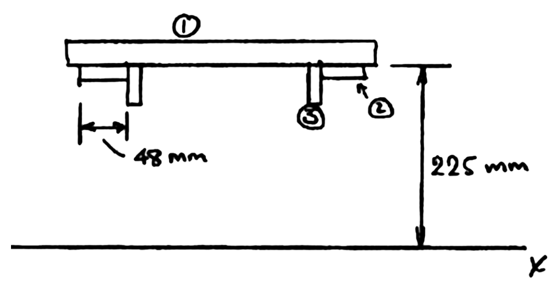
$|Q_1| = |Q_2|$, since C is centroid and thus, $Q = \Sigma \bar{y} A = 0$



PROBLEM 5.22

A composite beam is constructed by bolting four plates to four 60 × 60 × 12-mm angles as shown. The bolts are equally spaced along the beam, and the beam supports a vertical load. As proved in mechanics of materials, the shearing forces exerted on the bolts at *A* and *B* are proportional to the first moments with respect to the centroidal *x*-axis of the red-shaded areas shown, respectively, in parts *a* and *b* of the figure. Knowing that the force exerted on the bolt at *A* is 280 N, determine the force exerted on the bolt at *B*.

SOLUTION



From the problem statement, *F* is proportional to Q_x.

Therefore, $$\frac{F_A}{(Q_x)_A} = \frac{F_B}{(Q_x)_B}, \quad \text{or} \quad F_B = \frac{(Q_x)_B}{(Q_x)_A} F_A$$

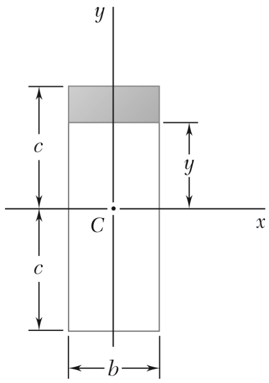
For the first moments, $$(Q_x)_A = \left(225 + \frac{12}{2}\right)(300 \times 12)$$

$$= 831{,}600 \text{ mm}^3$$

$$(Q_x)_B = (Q_x)A + 2\left(225 - \frac{12}{2}\right)(48 \times 12) + 2(225 - 30)(12 \times 60)$$

$$= 1{,}364{,}688 \text{ mm}^3$$

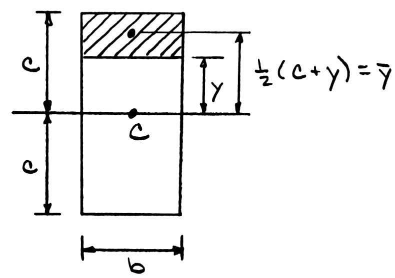
Then $$F_B = \frac{1{,}364{,}688}{831{,}600}(280 \text{ N})$$ or $F_B = 459$ N ◀



PROBLEM 5.23

The first moment of the shaded area with respect to the x-axis is denoted by Q_x. (a) Express Q_x in terms of b, c, and the distance y from the base of the shaded area to the x-axis. (b) For what value of y is Q_x maximum, and what is that maximum value?

SOLUTION



Shaded area:

$$A = b(c - y)$$

$$Q_x = \bar{y}A$$

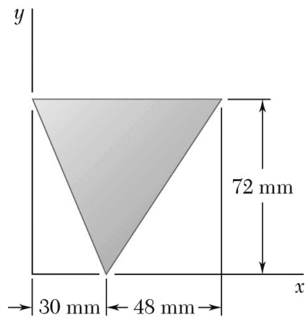
$$= \frac{1}{2}(c + y)[b(c - y)]$$

(a) $$Q_x = \frac{1}{2}b(c^2 - y^2) \quad ◀$$

(b) For $Q_{\max}$, $$\frac{dQ}{dy} = 0 \quad \text{or} \quad \frac{1}{2}b(-2y) = 0$$ $y = 0$ ◀

For $y = 0$, $$(Q_x) = \frac{1}{2}bc^2$$ $(Q_x) = \frac{1}{2}bc^2$ ◀

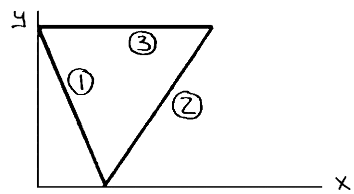
PROBLEM 5.24



A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



	L, mm	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{y}L$, mm^2	$\overline{y}L$, mm^2
1	$\sqrt{30^2+72^2}=78$	15	36	1170.0	2808.0
2	$\sqrt{48^2+72^2}=86.533$	54	36	4672.8	3115.2
3	78	39	72	3042.0	5616.0
Σ	242.53			8884.8	11,539.2

Then

$$\overline{X}\,\Sigma L = \Sigma \overline{x}\,L$$

$$\overline{X}(242.53) = 8884.8$$

or $\overline{X} = 36.6$ mm ◀

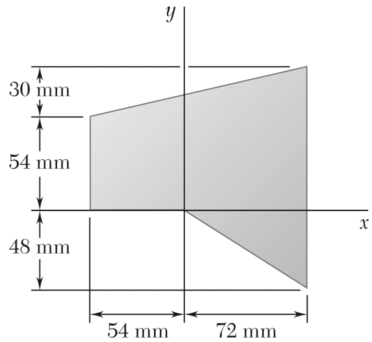
and

$$\overline{Y}\,\Sigma L = \Sigma \overline{y}\,L$$

$$\overline{Y}(242.53) = 11,539.2$$

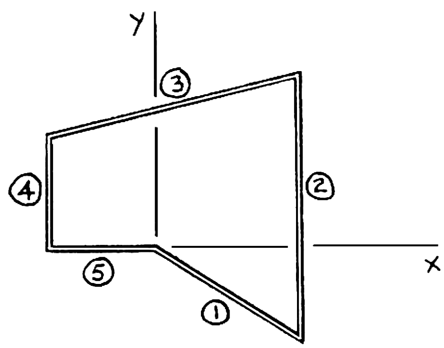
or $\overline{Y} = 47.6$ mm ◀

PROBLEM 5.25



A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION



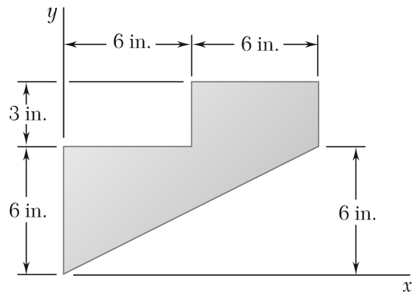
	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
1	$\sqrt{72^2+48^2}=86.533$	36	−24	3115.2	−2076.8
2	132	72	18	9504.0	2376.0
3	$\sqrt{126^2+30^2}=129.522$	9	69	1165.70	8937.0
4	54	−54	27	−2916.0	1458.0
5	54	−27	0	−1458.0	0
Σ	456.06			9410.9	10,694.2

Then $$\bar{X}\Sigma L=\Sigma\bar{x}L$$

$$\bar{X}(456.06)=9410.9$$ or $\bar{X}=20.6$ mm ◀

$$\bar{Y}\Sigma L=\Sigma\bar{y}L$$

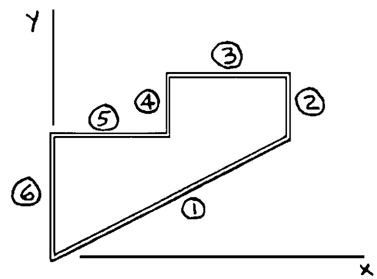
$$\bar{Y}(456.06)=10,694.2$$ or $\bar{Y}=23.4$ mm ◀



PROBLEM 5.26

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

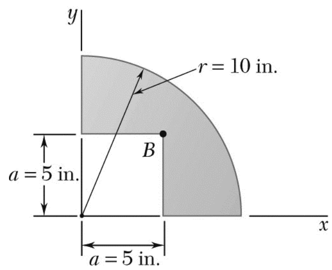
SOLUTION



	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
1	$\sqrt{12^2+6^2}=13.4164$	6	3	80.498	40.249
2	3	12	7.5	36	22.5
3	6	9	9	54	54.0
4	3	6	7.5	18	22.5
5	6	3	6	18	36.0
6	6	0	3	0	18.0
Σ	37.416			206.50	193.249

Then $\quad \bar{X}\,\Sigma L = \Sigma \bar{x} L \qquad \bar{X}(37.416) = 206.50 \qquad \bar{X} = 5.52$ in. ◀

$\bar{Y}\,\Sigma L = \Sigma \bar{y} L \qquad \bar{Y}(37.416) = 193.249 \qquad \bar{Y} = 5.16$ in. ◀

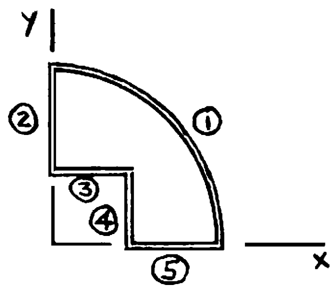


PROBLEM 5.27

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

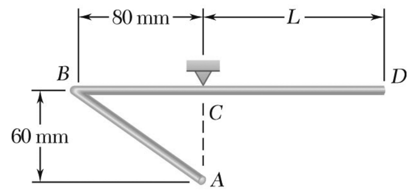
By symmetry, $\bar{X} = \bar{Y}$.



	L, in.	$\bar{x}$, in.	$\bar{y}L$, in^2
1	$\frac{1}{2}\pi(10) = 15.7080$	$\frac{2(10)}{\pi} = 6.3662$	100
2	5	0	0
3	5	2.5	12.5
4	5	5	25
5	5	7.5	37.5
Σ	35.708		175

Then $\bar{X}\Sigma L = \Sigma \bar{x} L \qquad \bar{X}(35.708) = 175$

$\bar{X} = \bar{Y} = 4.90$ in. ◀



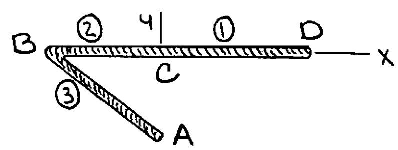
PROBLEM 5.28

The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* for which portion *BCD* of the wire is horizontal.

SOLUTION

First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *C*. Further, because the wire is homogeneous, the center of gravity of the wire will coincide with the centroid of the corresponding line. Thus,

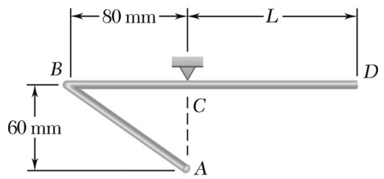
$$\overline{X} = 0 \text{ so that } \Sigma \overline{X} L = 0$$



Then $$\frac{L}{2} + (-40 \text{ mm})(80 \text{ mm}) + (-40 \text{ mm})(100 \text{ mm}) = 0$$

$$L^2 = 14,400 \text{ mm}^2$$

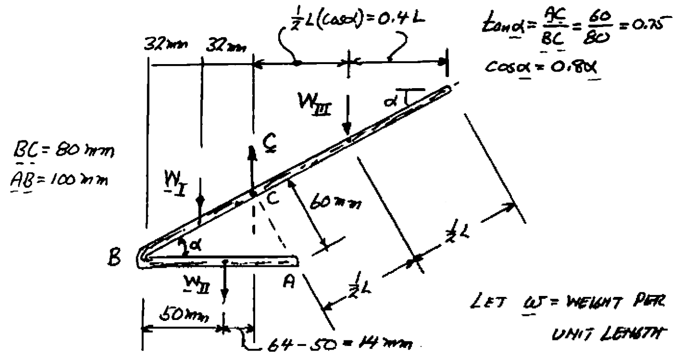
$L = 120.0$ mm ◀



PROBLEM 5.29

The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* for which portion *AB* of the wire is horizontal.

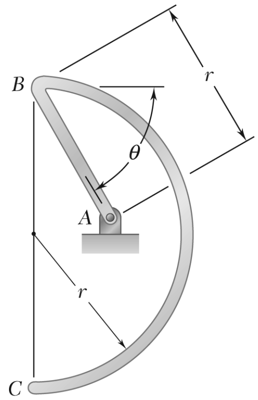
SOLUTION



$$W_{\mathrm{I}} = 80w \qquad W_{\mathrm{II}} = 100w \qquad W_{\mathrm{III}} = Lw$$

$$+\circlearrowleft \Sigma M_C = 0: \quad (80w)(32) + (100w)(14) - (Lw)(0.4L) = 0$$

$$L^2 = 9900 \qquad\qquad L = 99.5 \text{ mm} \blacktriangleleft$$

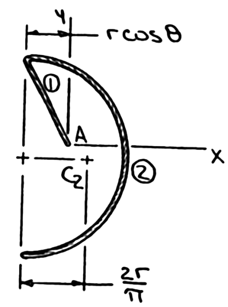


PROBLEM 5.30

The homogeneous wire *ABC* is bent into a semicircular arc and a straight section as shown and is attached to a hinge at *A*. Determine the value of θ for which the wire is in equilibrium for the indicated position.

SOLUTION

First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *A*. Further, because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line. Thus,



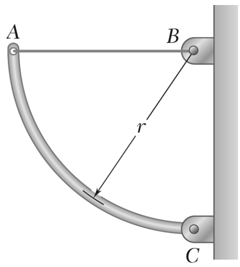
$$\bar{X} = 0$$

so that $$\Sigma \bar{x} L = 0$$

Then $$\left(-\frac{1}{2} r\cos\theta\right)(r) + \left(\frac{2r}{\pi} - r\cos\theta\right)(\pi r) = 0$$

or $$\cos\theta = \frac{4}{1+2\pi}$$
$$= 0.54921$$

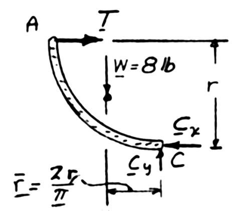
or $\theta = 56.7°$ ◀



PROBLEM 5.31

A uniform circular rod of weight 8 lb and radius 10 in. is attached to a pin at C and to the cable AB. Determine (a) the tension in the cable, (b) the reaction at C.

SOLUTION



For quarter circle, $\bar{r} = \dfrac{2r}{\pi}$

(a) $+\circlearrowleft \Sigma M_C = 0\text{:}\quad W\left(\dfrac{2r}{\pi}\right) - Tr = 0$

$$T = W\left(\frac{2}{\pi}\right) = (8\text{ lb})\left(\frac{2}{\pi}\right)$$

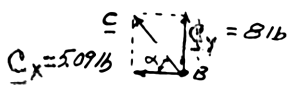
$T = 5.09\text{ lb}$ ◀

(b) $\xrightarrow{+} \Sigma F_x = 0\text{:}\quad T - C_x = 0 \quad 5.09\text{ lb} - C_x = 0$

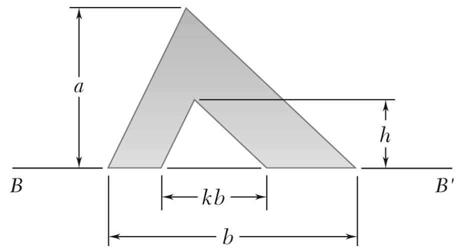
$\mathbf{C}_x = 5.09\text{ lb} \longleftarrow$

$+\uparrow \Sigma F_y = 0\text{:}\quad C_y - W = 0 \qquad C_y - 8\text{ lb} = 0$

$\mathbf{C}_y = 8\text{ lb}\uparrow$



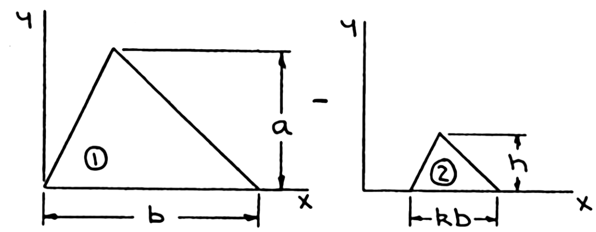
$\mathbf{C} = 9.48\text{ lb} \measuredangle 57.5°$ ◀



PROBLEM 5.32

Determine the distance h for which the centroid of the shaded area is as far above line BB' as possible when (*a*) $k = 0.10$, (*b*) $k = 0.80$.

SOLUTION



	A	$\bar{y}$	$\bar{y}A$
1	$\frac{1}{2}ba$	$\frac{1}{3}a$	$\frac{1}{6}a^2b$
2	$-\frac{1}{2}(kb)h$	$\frac{1}{3}h$	$-\frac{1}{6}kbh^2$
Σ	$\frac{b}{2}(a-kh)$		$\frac{b}{6}(a^2-kh^2)$

Then
$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$
$$\bar{Y}\left[\frac{b}{2}(a-kh)\right] = \frac{b}{6}(a^2-kh^2)$$

or
$$\bar{Y} = \frac{a^2-kh^2}{3(a-kh)} \qquad (1)$$

and
$$\frac{d\bar{Y}}{dh} = \frac{1}{3}\frac{-2kh(a-kh)-(a^2-kh^2)(-k)}{(a-kh)^2} = 0$$

or
$$2h(a-kh)-a^2+kh^2 = 0 \qquad (2)$$

Simplifying Eq. (2) yields
$$kh^2 - 2ah + a^2 = 0$$

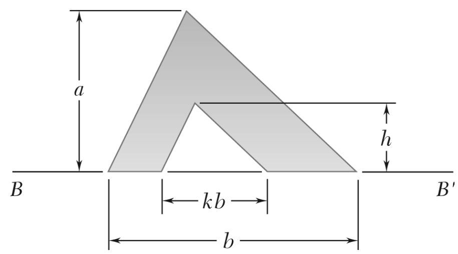
PROBLEM 5.32 (Continued)

Then
$$h = \frac{2a \pm \sqrt{(-2a)^2 - 4(k)(a^2)}}{2k}$$
$$= \frac{a}{k}\left[1 \pm \sqrt{1-k}\right]$$

Note that only the negative root is acceptable since $h < a$. Then

(*a*) $$k = 0.10$$
$$h = \frac{a}{0.10}\left[1 - \sqrt{1-0.10}\right]$$
or $h = 0.513a$ ◀

(*b*) $$k = 0.80$$
$$h = \frac{a}{0.80}\left[1 - \sqrt{1-0.80}\right]$$
or $h = 0.691a$ ◀



PROBLEM 5.33

Knowing that the distance h has been selected to maximize the distance $\bar{y}$ from line BB' to the centroid of the shaded area, show that $\bar{y} = 2h/3$.

SOLUTION

See solution to Problem 5.32 for analysis leading to the following equations:

$$\bar{Y} = \frac{a^2 - kh^2}{3(a - kh)} \tag{1}$$

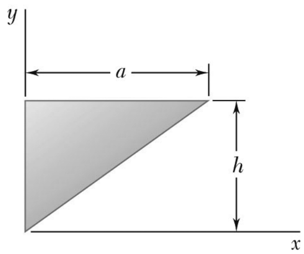
$$2h(a - kh) - a^2 + kh^2 = 0 \tag{2}$$

Rearranging Eq. (2) (which defines the value of h which maximizes $\bar{Y}$) yields

$$a^2 - kh^2 = 2h(a - kh)$$

Then substituting into Eq. (1) (which defines $\bar{Y}$),

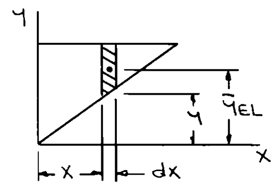
$$\bar{Y} = \frac{1}{3(a - kh)} \times 2h(a - kh) \qquad \text{or} \quad \bar{Y} = \frac{2}{3}h \blacktriangleleft$$



PROBLEM 5.34

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

SOLUTION



We have $$y = \frac{h}{a}x$$

and
$$dA = (h - y)dx$$
$$= h\left(1 - \frac{x}{a}\right)dx$$
$$\bar{x}_{EL} = x$$
$$\bar{y}_{EL} = \frac{1}{2}(h + y)$$
$$= \frac{h}{2}\left(1 + \frac{x}{a}\right)$$

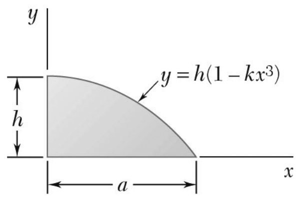
Then $$A = \int dA = \int_0^a h\left(1 - \frac{x}{a}\right)dx = h\left[x - \frac{x^2}{2a}\right]_0^a = \frac{1}{2}ah$$

and $$\int \bar{x}_{EL}\,dA = \int_0^a x\left[h\left(1 - \frac{x}{a}\right)dx\right] = h\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a = \frac{1}{6}a^2h$$

$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{h}{2}\left(1 + \frac{x}{a}\right)\left[h\left(1 - \frac{x}{a}\right)dx\right]$$
$$= \frac{h^2}{2}\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx = \frac{h^2}{2}\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{1}{3}ah^2$$

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{1}{2}ah\right) = \frac{1}{6}a^2h \qquad \bar{x} = \frac{2}{3}a \blacktriangleleft$$

$$\bar{y}A = \int y_{EL}\,dA: \quad \bar{y}\left(\frac{1}{2}ah\right) = \frac{1}{3}ah^2 \qquad \bar{y} = \frac{2}{3}h \blacktriangleleft$$



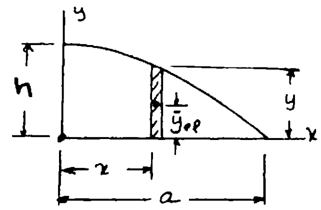
PROBLEM 5.35

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

SOLUTION

$$y = h(1 - kx^3)$$

For $x = a,\ y = 0.$



$$0 = h(1 - k\,a^3)$$

$$\therefore\quad k = \frac{1}{a^3}$$

$$y = h\left(1 - \frac{x^3}{a^3}\right)$$

$$\overline{x}_{EL} = x,\quad \overline{y}_{EL} = \frac{1}{2}y \quad dA = ydx$$

$$A = \int dA = \int_0^a ydx = \int_0^a h\left(1 - \frac{x^3}{a^3}\right)dx = h\left[x - \frac{x^4}{4a^3}\right]_0^a = \frac{3}{4}ah$$

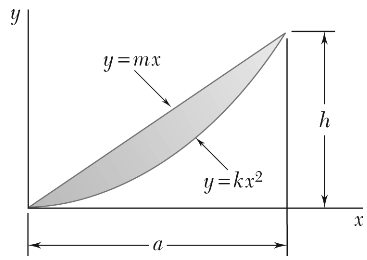
$$\int \overline{x}_{EL}dA = \int_0^a xydx = \int_0^a h\left(x - \frac{x^4}{a^3}\right)dx = h\left[\frac{x^2}{2} - \frac{x^5}{5a^3}\right]_0^a = \frac{3}{10}a^2h$$

$$\int \overline{y}_{EL}dA = \int_0^a \left(\frac{1}{2}y\right)ydx = \frac{1}{2}\int_0^a h^2\left(1 - \frac{x^3}{a^3}\right)dx = \frac{h^2}{2}\int_0^a \left(1 - \frac{2x^3}{a^3} + \frac{x^6}{a^6}\right)dx$$

$$= \frac{h^2}{2}\left[x - \frac{x^4}{2a^3} + \frac{x^7}{7a^6}\right]_0^a = \frac{9}{28}ah^2$$

$$\overline{x}A = \int \overline{x}_{EL}dA:\quad \overline{x}\left(\frac{3}{4}ah\right) = \frac{3}{10}a^2h \qquad \overline{x} = \frac{2}{5}a \ ◀$$

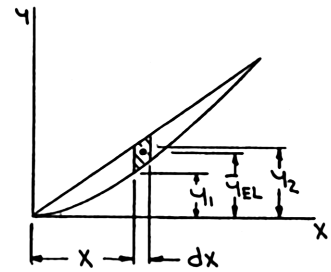
$$\overline{y}A = \int \overline{y}_{EL}dA:\quad \overline{y}\left(\frac{3}{4}ah\right) = \frac{9}{28}ah^2 \qquad \overline{y} = \frac{3}{7}h \ ◀$$



PROBLEM 5.36

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

SOLUTION



At (a, h), $\quad y_1: \quad h = ka^2$

or $$k = \frac{h}{a^2}$$

$$y_2: \quad h = ma$$

or $$m = \frac{h}{a}$$

Now $$\overline{x}_{EL} = x$$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

and $$dA = (y_2 - y_1)dx = \left[\frac{h}{a}x - \frac{h}{a^2}x^2\right]dx$$

$$= \frac{h}{a^2}(ax - x^2)dx$$

Then $$A = \int dA = \int_0^a \frac{h}{a^2}(ax - x^2)dx = \frac{h}{a^2}\left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_0^a = \frac{1}{6}ah$$

and $$\int \overline{x}_{EL}dA = \int_0^a x\left[\frac{h}{a^2}(ax - x^2)dx = \frac{h}{a^2}\left[\frac{a}{3}x^3 - \frac{1}{4}x^4\right]_0^a = \frac{1}{12}a^2h\right.$$

$$\int \overline{y}_{EL}dA = \int \frac{1}{2}(y_1 + y_2)[(y_2 - y_1)dx] = \int \frac{1}{2}\left(y_2^2 - y_1^2\right)dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{h^2}{a^2}x^2 - \frac{h^2}{a^4}x^4\right)dx$$

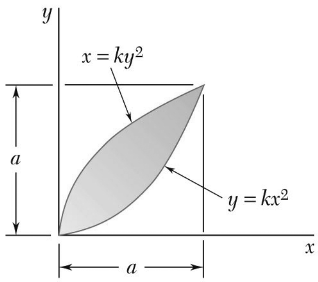
$$= \frac{1}{2}\frac{h^2}{a^4}\left[\frac{a^2}{3}x^3 - \frac{1}{5}x^5\right]_0^a$$

$$= \frac{1}{15}ah^2$$

PROBLEM 5.36 (Continued)

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{1}{6}ah\right) = \frac{1}{12}a^2h \qquad \bar{x} = \frac{1}{2}a \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\frac{1}{6}ah\right) = \frac{1}{15}ah^2 \qquad \bar{y} = \frac{2}{5}h \blacktriangleleft$$

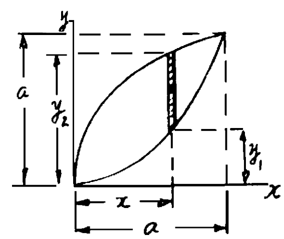


PROBLEM 5.37

Determine by direct integration the centroid of the area shown.

SOLUTION

$$y_2 = \sqrt{\frac{x}{k}}, \quad y_1 = kx^2$$



But $$a = ka^2, \text{ thus, } k = \frac{1}{a^2}$$

$$y_2 = \sqrt{ax}, \quad y_1 = \frac{x^2}{a}$$

$$\overline{x}_{EL} = x$$

$$dA = (y_2 - y_1)dx = \left(\sqrt{ax} - \frac{x^2}{a}\right)dx$$

$$A = \int dA = \int_0^a \left(\sqrt{ax} - \frac{x^2}{a}\right)dx$$

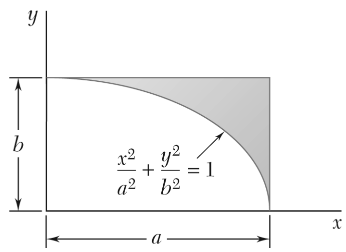
$$= \left[\frac{2}{3}\sqrt{a}x^{3/2} - \frac{x^3}{3a}\right]_0^a = \frac{1}{3}a^2$$

$$\int \overline{x}_{EL}dA = \int_0^a x\left(\sqrt{ax} - \frac{x^2}{a}\right)dx = \int_0^a \left(\sqrt{a}x^{3/2} - \frac{x^3}{a}\right)dx = \left[\frac{2}{5}\sqrt{a}x^{5/2} - \frac{x^4}{4a}\right]_0^a = \frac{3}{20}a^3$$

$$\overline{x}A = \int x_{EL}dA: \quad \overline{x}\left(\frac{1}{3}a^2\right) = \frac{3}{20}a^3 \qquad \overline{x} = \frac{9a}{20}$$

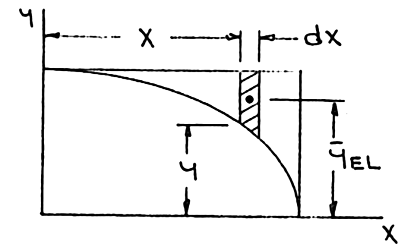
By symmetry, $$\overline{y} = \overline{x} = \frac{9a}{20} \blacktriangleleft$$

PROBLEM 5.38



Determine by direct integration the centroid of the area shown.

SOLUTION



For the element (EL) shown,
$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

and
$$dA=(b-y)dx$$
$$=\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx$$
$$\bar{x}_{EL}=x$$
$$\bar{y}_{EL}=\frac{1}{2}(y+b)$$
$$=\frac{b}{2a}\left(a+\sqrt{a^2-x^2}\right)$$

Then
$$A=\int dA=\int_0^a\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx$$

To integrate, let
$$x=a\sin\theta:\quad \sqrt{a^2-x^2}=a\cos\theta,\ dx=a\cos\theta\, d\theta$$

Then
$$A=\int_0^{\pi/2}\frac{b}{a}(a-a\cos\theta)(a\cos\theta d\theta)$$
$$=\frac{b}{a}\left[a^2\sin\theta-a^2\left(\frac{\theta}{2}+\sin\frac{2\theta}{4}\right)\right]_0^{\pi/2}$$
$$=ab\left(1-\frac{\pi}{4}\right)$$

and
$$\int\bar{x}_{EL}dA=\int_0^a x\left[\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx\right]$$
$$=\frac{b}{a}\left[\left(\frac{a}{2}x^2+\frac{1}{3}(a^2-x^2)^{3/2}\right)\right]_0^{\pi/2}$$
$$=\frac{1}{6}a^3b$$

PROBLEM 5.38 (Continued)

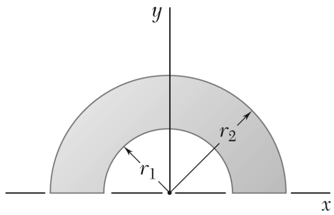
$$\int \overline{y}_{EL} dA = \int_0^a \frac{b}{2a}\left(a + \sqrt{a^2 - x^2}\right)\left[\frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx\right]$$

$$= \frac{b^2}{2a^2}\int_0^a (x^2)\ dx = \frac{b^2}{2a^2}\left(\frac{x^3}{3}\right)\Bigg|_0^a$$

$$= \frac{1}{6}ab^2$$

$$\overline{x}A = \int \overline{x}_{EL} dA: \quad \overline{x}\left[ab\left(1 - \frac{\pi}{4}\right)\right] = \frac{1}{6}a^2 b \qquad \text{or } \overline{x} = \frac{2a}{3(4-\pi)} \blacktriangleleft$$

$$\overline{y}A = \int \overline{y}_{EL} dA: \quad \overline{y}\left[ab\left(1 - \frac{\pi}{4}\right)\right] = \frac{1}{6}ab^2 \qquad \text{or } \overline{y} = \frac{2b}{3(4-\pi)} \blacktriangleleft$$



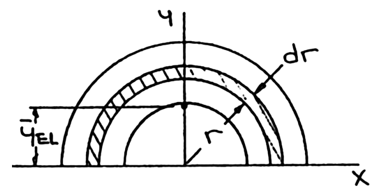
PROBLEM 5.39

Determine by direct integration the centroid of the area shown.

SOLUTION

First note that symmetry implies $\qquad \bar{x} = 0$ ◀

For the element (EL) shown,



$$\bar{y}_{EL} = \frac{2r}{\pi} \quad \text{(Figure 5.8B)}$$

$$dA = \pi r dr$$

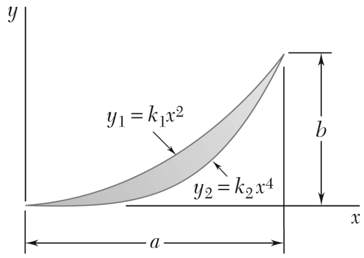
Then $$A = \int dA = \int_{r_1}^{r_2} \pi r dr = \pi \left(\frac{r^2}{2} \right) \Bigg|_{r_1}^{r_2} = \frac{\pi}{2}\left(r_2^2 - r_1^2\right)$$

and $$\int \bar{y}_{EL} dA = \int_{r_1}^{r_2} \frac{2r}{\pi}(\pi r dr) = 2\left(\frac{1}{3}r^3\right)\Bigg|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

So $$\bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

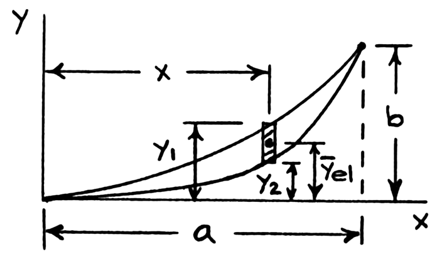
or $\bar{y} = \dfrac{4}{3\pi}\dfrac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$ ◀

PROBLEM 5.40



Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



$$y_1 = k_1x^2 \quad \text{but} \quad b = k_1a^2 \qquad y_1 = \frac{b}{a^2}x^2$$

$$y_2 = k_2x^4 \quad \text{but} \quad b = k_2a^4 \qquad y_2 = \frac{b}{a^4}x^4$$

$$dA = (y_2 - y_1)dx = \frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

$$= \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)$$

$$A = \int dA = \frac{b}{a^2}\int_0^a \left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^a$$

$$= \frac{2}{15}ba$$

$$\int \bar{x}_{EL}dA = \int_0^a x\frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\int_0^a \left(x^3 - \frac{x^5}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^4}{4} - \frac{x^6}{6a^2}\right]_0^a$$

$$= \frac{1}{12}a^2b$$

PROBLEM 5.40 (Continued)

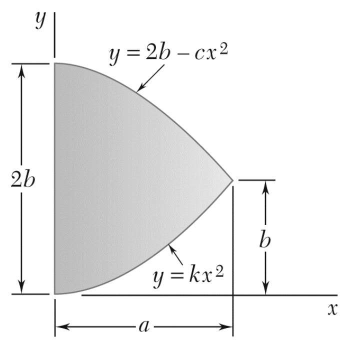
$$\int \bar{y}_{EL}\, dA = \int_0^a \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)\frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b^2}{2a^4}\int_0^a \left(x^4 - \frac{x^8}{a^4}\right)dx$$

$$= \frac{b^2}{2a^4}\left[\frac{x^5}{5} - \frac{x^9}{9a^4}\right]_0^a = \frac{2}{45}ab^2$$

$$\bar{x}A = \int \bar{x}_{EL}\, dA: \quad \bar{x}\left(\frac{2}{15}ba\right) = \frac{1}{12}a^2 b \qquad \bar{x} = \frac{5}{8}a \ \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}\, dA: \quad \bar{y}\left(\frac{2}{15}ba\right) = \frac{2}{45}ab^2 \qquad \bar{y} = \frac{1}{3}b \ \blacktriangleleft$$

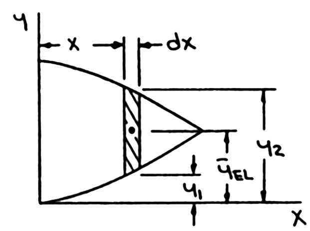


PROBLEM 5.41

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION

First note that symmetry implies $\bar{y} = b$ ◀



At $$x = a, \quad y = b$$

$$y_1: \quad b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

Then $$y_1 = \frac{b}{a^2}x^2$$

$$y_2: \quad b = 2b - ca^2$$

or $$c = \frac{b}{a^2}$$

Then $$y_2 = b\left(2 - \frac{x^2}{a^2}\right)$$

Now $$dA = (y_2 - y_1)dx_2 = \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx$$

$$= 2b\left(1 - \frac{x^2}{a^2}\right)dx$$

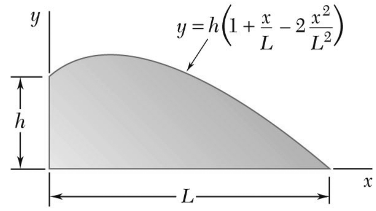
and $$\bar{x}_{EL} = x$$

Then $$A = \int dA \int_0^a 2b\left(1 - \frac{x^2}{a^2}\right)dx = 2b\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4}{3}ab$$

and $$\int \bar{x}_{EL}dA = \int_0^a x\left[2b\left(1 - \frac{x^2}{a^2}\right)dx\right] = 2b\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a = \frac{1}{2}a^2b$$

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{4}{3}ab\right) = \frac{1}{2}a^2b$$

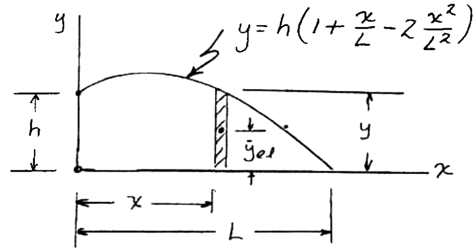
$\bar{x} = \frac{3}{8}a$ ◀



PROBLEM 5.42

Determine by direct integration the centroid of the area shown.

SOLUTION



$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y \quad dA = y\,dx$$

$$A = \int dA = \int_0^L h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\left[x + \frac{x^2}{2L} - \frac{2}{3}\frac{x^3}{L^2}\right]_0^L = \frac{5}{6}hL$$

$$\int x_{EL}dA = \int_0^L xh\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\int_0^L\left(x + \frac{x^2}{L} - 2\frac{x^3}{L^2}\right)dx$$

$$= h\left[\frac{x^2}{2} + \frac{1}{3}\frac{x^3}{L} - \frac{2}{4}\frac{x^4}{L^2}\right]_0^L = \frac{1}{3}hL^2$$

$$\bar{x}A = \int x_{EL}dA: \quad \bar{x}\left(\frac{5}{6}hL\right) = \frac{1}{3}hL^2 \qquad \bar{x} = \frac{2}{5}L \blacktriangleleft$$

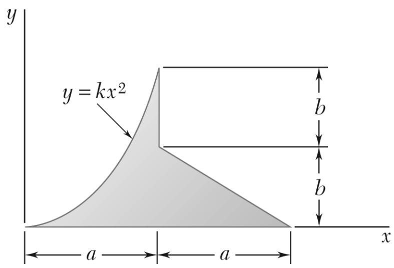
$$A = \frac{5}{6}hL \quad \bar{y}_{EL} = \frac{1}{2}y \quad y = h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)$$

$$\int \bar{y}_{EL}dA = \frac{1}{2}\int y^2dx = \frac{h^2}{2}\int_0^L\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{h^2}{2}\int_0^L\left(1 + \frac{x^2}{L^2} + 4\frac{x^4}{L^4} + 2\frac{x}{L} - 4\frac{x^2}{L^2} - 4\frac{x^3}{L^3}\right)dx$$

$$= \frac{h^2}{2}\left[x + \frac{x^3}{3L^2} + \frac{4x^5}{5L^4} + \frac{x^2}{L} - \frac{4x^3}{3L^2} - \frac{x^4}{L^3}\right]_0^L = \frac{4}{10}h^2L$$

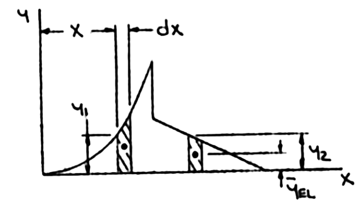
$$\bar{y}A = \int y_{EL}dA: \quad \bar{y}\left(\frac{5}{6}hL\right) = \frac{4}{10}h^2L \qquad \bar{y} = \frac{12}{25}h \blacktriangleleft$$



PROBLEM 5.43

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



For y_1 at $x = a$, $\quad y = 2b, \quad 2b = ka^2, \quad \text{or} \quad k = \dfrac{2b}{a^2}$

Then $\quad y_1 = \dfrac{2b}{a^2}x^2$

By observation, $\quad y_2 = -\dfrac{b}{a}(x + 2b) = b\left(2 - \dfrac{x}{a}\right)$

Now $\quad \bar{x}_{EL} = x$

and for $0 \le x \le a$, $\quad \bar{y}_{EL} = \dfrac{1}{2}y_1 = \dfrac{b}{a^2}x^2 \quad \text{and} \quad dA = y_1\,dx = \dfrac{2b}{a^2}x^2dx$

For $a \le x \le 2a$, $\quad \bar{y}_{EL} = \dfrac{1}{2}y_2 = \dfrac{b}{2}\left(2 - \dfrac{x}{a}\right) \quad \text{and} \quad dA = y_2\,dx = b\left(2 - \dfrac{x}{a}\right)dx$

Then

$$A = \int dA = \int_0^a \frac{2b}{a^2}x^2dx + \int_a^{2a} b\left(2 - \frac{x}{a}\right)dx$$

$$= \frac{2b}{a^2}\left[\frac{x^3}{3}\right]_0^a + b\left[-\frac{a}{2}\left(2 - \frac{x}{a}\right)^2\right]_0^{2a} = \frac{7}{6}ab$$

and

$$\int \bar{x}_{EL}dA = \int_0^a x\left(\frac{2b}{a^2}x^2dx\right) + \int_a^{2a} x\left[b\left(2 - \frac{x}{a}\right)dx\right]$$

$$= \frac{2b}{a^2}\left[\frac{x^4}{4}\right]_0^a + b\left[x^2 - \frac{x^3}{3a}\right]_0^{2a}$$

$$= \frac{1}{2}a^2b + b\left\{\left[(2a)^2 - (a)^2\right] + \frac{1}{3a}\left[(2a^2) - (a)^3\right]\right\}$$

$$= \frac{7}{6}a^2b$$

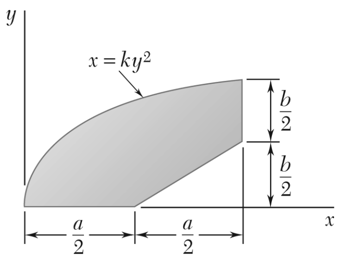
$$\int \bar{y}_{EL}\, dA = \int_0^a \frac{b}{a^2}x^2\left[\frac{2b}{a^2}x^2 dx\right] + \int_0^{2a} \frac{b}{2}\left(2-\frac{x}{a}\right)\left[b\left(2-\frac{x}{a}\right)dx\right]$$

$$= \frac{2b^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + \frac{b^2}{2}\left[-\frac{a}{3}\left(2-\frac{x}{a}\right)^3\right]_a^{2a}$$

$$= \frac{17}{30}ab^2$$

Hence, $\quad \bar{x}A = \int \bar{x}_{EL}\, dA: \quad \bar{x}\left(\frac{7}{6}ab\right) = \frac{7}{6}a^2 b \qquad \bar{x} = a$ ◀

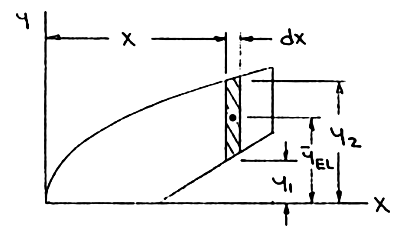
$\bar{y}A = \int \bar{y}_{EL}\, dA: \quad \bar{y}\left(\frac{7}{6}ab\right) = \frac{17}{30}ab^2 \qquad \bar{y} = \frac{17}{35}b$ ◀



PROBLEM 5.44

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



For y_2 at $x = a$, $\quad y = b, \quad a = kb^2, \quad$ or $\quad k = \dfrac{a}{b^2}$

Then $\quad y_2 = \dfrac{b}{\sqrt{a}} x^{1/2}$

Now $\quad \overline{x}_{EL} = x$

and for $0 \le x \le \dfrac{a}{2}$, $\quad \overline{y}_{EL} = \dfrac{y_2}{2} = \dfrac{b}{2}\dfrac{x^{1/2}}{\sqrt{a}}$

$$dA = y_2\, dx = b\frac{x^{1/2}}{\sqrt{a}}\, dx$$

For $\dfrac{a}{2} \le x \le a$, $\quad \overline{y}_{EL} = \dfrac{1}{2}(y_1 + y_2) = \dfrac{b}{2}\left(\dfrac{x}{a} - \dfrac{1}{2} + \dfrac{x^{1/2}}{\sqrt{a}}\right)$

$$dA = (y_2 - y_1)dx = b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$$

Then $\quad A = \displaystyle\int dA = \int_0^{a/2} b\frac{x^{1/2}}{\sqrt{a}}\,dx + \int_{a/2}^{a} b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{3}x^{3/2}\right]_0^{a/2} + b\left[\frac{2}{3}\frac{x^{3/2}}{\sqrt{a}} - \frac{x^2}{2a} + \frac{1}{2}x\right]_{a/2}^{a}$$

$$= \frac{2}{3}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{3/2} + (a)^{3/2} - \left(\frac{a}{2}\right)^{3/2}\right]$$

$$+ b\left\{-\frac{1}{2a}\left[(a^2) - \left(\frac{a}{2}\right)^2\right] + \frac{1}{2}\left[(a) - \left(\frac{a}{2}\right)\right]\right\}$$

$$= \frac{13}{24}ab$$

and
$$\int \bar{x}_{EL}\,dA = \int_0^{a/2} x\left(b\frac{x^{1/2}}{\sqrt{a}}dx\right) + \int_{a/2}^{a} x\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)\right]dx$$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{5}x^{5/2}\right]_0^{a/2} + b\left[\frac{2}{5}\frac{x^{5/2}}{\sqrt{a}} - \frac{x^3}{3a} + \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \frac{2}{5}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{5/2} + (a)^{5/2} - \left(\frac{a}{2}\right)^{5/2}\right]$$

$$+ b\left\{-\frac{1}{3a}\left[(a)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{1}{4}\left[(a)^2 - \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{71}{240}a^2b$$

$$\int \bar{y}_{EL}\,dA = \int_0^{a/2} \frac{b}{2}\frac{x^{1/2}}{\sqrt{a}}\left[b\frac{x^{1/2}}{\sqrt{a}}dx\right]$$

$$+ \int_{a/2}^{a} \frac{b}{2}\left(\frac{x}{a} - \frac{1}{2} + \frac{x^{1/2}}{\sqrt{a}}\right)\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx\right]$$

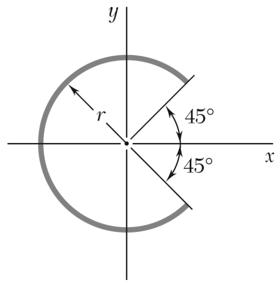
$$= \frac{b^2}{2a}\left[\frac{1}{2}x^2\right]_0^{a/2} + \frac{b^2}{2}\left[\left(\frac{x^2}{2a} - \frac{1}{3a}\left(\frac{x}{a} - \frac{1}{2}\right)^3\right)\right]_{a/2}^{a}$$

$$= \frac{b}{4a}\left[\left(\frac{a}{2}\right)^2 + (a)^2 - \left(\frac{a}{2}\right)^2\right] - \frac{b^2}{6a}\left(\frac{a}{2} - \frac{1}{2}\right)^3$$

$$= \frac{11}{48}ab^2$$

Hence,
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{13}{24}ab\right) = \frac{71}{240}a^2b \qquad \bar{x} = \frac{17}{130}a = 0.546a \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{13}{24}ab\right) = \frac{11}{48}ab^2 \qquad \bar{y} = \frac{11}{26}b = 0.423b \blacktriangleleft$$

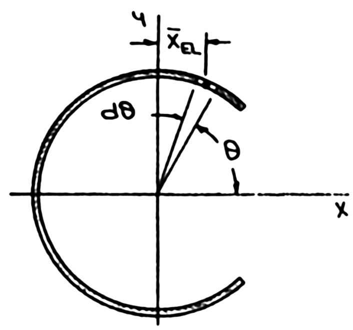


PROBLEM 5.45

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line.



Now $\quad \bar{x}_{EL} = r\cos\theta \quad$ and $\quad dL = r\,d\theta$

Then $$L = \int dL = \int_{\pi/4}^{7\pi/4} r\,d\theta = r[\theta]_{\pi/4}^{7\pi/4} = \frac{3}{2}\pi r$$

and $$\int \bar{x}_{EL}\,dL = \int_{\pi/4}^{7\pi/4} r\cos\theta(r\,d\theta)$$

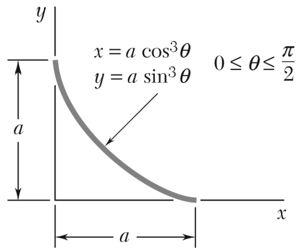
$$= r^2[\sin\theta]_{\pi/4}^{7\pi/4}$$

$$= r^2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= -r^2\sqrt{2}$$

Thus $$\bar{x}L = \int \bar{x}\,dL: \quad \bar{x}\left(\frac{3}{2}\pi r\right) = -r^2\sqrt{2}$$

$$\bar{x} = -\frac{2\sqrt{2}}{3\pi}r \quad ◀$$

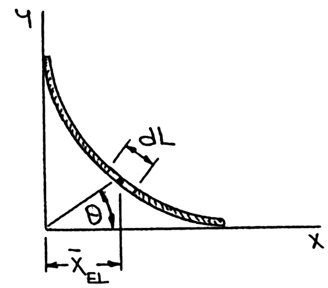


PROBLEM 5.46

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line.



Now $\quad \bar{x}_{EL} = a\cos^3\theta \quad$ and $\quad dL = \sqrt{dx^2 + dy^2}$

where $\quad x = a\cos^3\theta: \quad dx = -3a\cos^2\theta\sin\theta d\theta$

$$y = a\sin^3\theta: \quad dy = 3a\sin^2\theta\cos\theta d\theta$$

Then

$$dL = [(-3a\cos^2\theta\sin\theta d\theta)^2 + (3a\sin^2\theta\cos\theta d\theta)^2]^{1/2}$$
$$= 3a\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta)^{1/2}d\theta$$
$$= 3a\cos\theta\sin\theta d\theta$$

$$L = \int dL = \int_0^{\pi/2} 3a\cos\theta\sin\theta d\theta = 3a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$
$$= \frac{3}{2}a$$

and

$$\int \bar{x}_{EL}dL = \int_0^{\pi/2} a\cos^3\theta(3a\cos\theta\sin\theta d\theta)$$
$$= 3a^2\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = \frac{3}{5}a^2$$

Hence, $\quad \bar{x}L = \int \bar{x}_{EL}dL: \quad \bar{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2 \qquad \bar{x} = \frac{2}{5}a$ ◀

Alternative Solution:

$$x = a\cos^3\theta \Rightarrow \cos^2\theta = \left(\frac{x}{a}\right)^{2/3}$$

$$y = a\sin^3\theta \Rightarrow \sin^2\theta = \left(\frac{y}{a}\right)^{2/3}$$

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1 \quad \text{or} \quad y = (a^{2/3} - x^{2/3})^{3/2}$$

PROBLEM 5.46 (Continued)

Then $$\frac{dy}{dx} = (a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})$$

Now $$\overline{x}_{EL} = x$$

and $$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

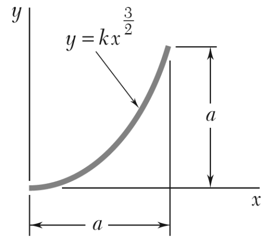
$$dx = \left\{1 + \left[(a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})\right]^2\right\}^{1/2} dx$$

Then $$L = \int dL = \int_0^a \frac{a^{1/3}}{x^{1/3}} dx = a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = \frac{3}{2}a$$

and $$\int \overline{x}_{EL} dL = \int_0^a x\left(\frac{a^{1/3}}{x^{1/3}} dx\right) = a^{1/3}\left[\frac{3}{5}x^{5/3}\right]_0^a = \frac{3}{5}a^2$$

Hence $$\overline{x}L = \int \overline{x}_{EL} dL: \quad \overline{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2$$

$$\overline{x} = \frac{2}{5}a \blacktriangleleft$$

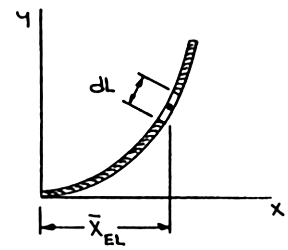


PROBLEM 5.47*

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid. Express your answer in terms of a.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



We have at $x = a$, $\quad y = a, \quad a = ka^{3/2}, \quad$ or $\quad k = \dfrac{1}{\sqrt{a}}$

Then $$y = \frac{1}{\sqrt{a}} x^{3/2}$$

and $$\frac{dy}{dx} = \frac{3}{2\sqrt{a}} x^{1/2}$$

Now $$\bar{x}_{EL} = x$$

and
$$\begin{aligned} dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \left[1 + \left(\frac{3}{2\sqrt{a}} x^{1/2}\right)^2\right]^{1/2} dx \\ &= \frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\, dx \end{aligned}$$

Then
$$\begin{aligned} L = \int dL &= \int_0^a \frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\, dx \\ &= \frac{1}{2\sqrt{a}} \left[\frac{2}{3} \times \frac{1}{9} (4a + 9x)^{3/2}\right]_0^a \\ &= \frac{a}{27}[(13)^{3/2} - 8] \\ &= 1.43971a \end{aligned}$$

and $$\int \bar{x}_{EL}\, dL = \int_0^a x \left[\frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\, dx\right]$$

PROBLEM 5.47* (Continued)

Use integration by parts with

$$u = x \qquad dv = \sqrt{4a + 9x}\, dx$$

$$du = dx \qquad v = \frac{2}{27}(4a + 9x)^{3/2}$$

Then

$$\int \overline{x}_{EL} dL = \frac{1}{2\sqrt{a}}\left\{\left[x \times \frac{2}{27}(4a + 9x)^{3/2}\right]_0^a - \int_0^a \frac{2}{27}(4a + 9x)^{3/2} dx\right\}$$

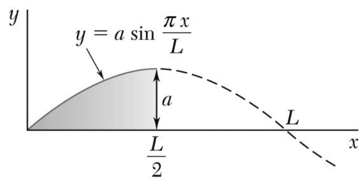
$$= \frac{(13)^{3/2}}{27}a^2 - \frac{1}{27\sqrt{a}}\left[\frac{2}{45}(4a + 9x)^{5/2}\right]_0^a$$

$$= \frac{a^2}{27}\left\{(13)^{3/2} - \frac{2}{45}[(13)^{5/2} - 32]\right\}$$

$$= 0.78566a^2$$

$$\overline{x}L = \int x_{EL} dL: \quad \overline{x}(1.43971a) = 0.78566a^2$$

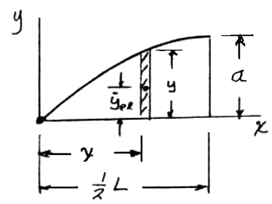
or $\overline{x} = 0.546a$ ◀



PROBLEM 5.48

Determine by direct integration the centroid of the area shown.

SOLUTION



$$y = a \sin \frac{\pi x}{L}$$

$$\bar{x}_{EL} = x, \quad \bar{y}_{EL} = \frac{1}{2}y, \quad dA = y\,dx$$

$$A = \int_0^{L/2} y\,dx = a\int_0^{L/2} \sin\frac{\pi x}{L}dx$$

$$A = a\left[\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)\right]_0^{L/2} = \frac{aL}{\pi}$$

$$\int \bar{x}_{EL}dA = \int_0^{L/2} xydx = \int_0^{L/2} xa\sin\frac{\pi x}{L}dx$$

Setting $u = \frac{\pi x}{L}$, we have $x = \frac{L}{\pi}u, \quad dx = \frac{L}{\pi}du,$

$$\int \bar{x}_{EL}dA = \int_0^{\pi/2}\left(\frac{L}{\pi}u\right)a\sin u\left(\frac{L}{\pi}du\right) = a\left(\frac{L}{\pi}\right)^2\int_0^{\pi/2} u\sin x\,du$$

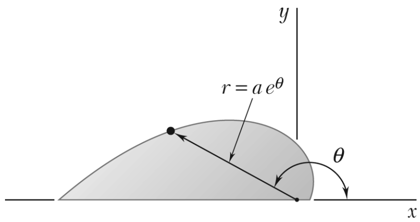
Integrating by parts, $$\int \bar{x}_{EL}dA = a\left(\frac{L}{\pi}\right)^2\left\{[-u\cos u]_0^{\pi/2} + \int_0^{\pi/2}\cos u\,du\right\} = \frac{aL^2}{\pi^2}$$

$$\int \bar{y}_{EL}dA = \int_0^{L/2}\frac{1}{2}y^2dx = \frac{1}{2}a^2\int_0^{L/2}\sin^2\frac{\pi x}{L}dx = \frac{a^2L}{2\pi}\int_0^{\pi/2}\sin^2 u\,du$$

$$= \frac{a^2L}{2\pi^2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2u)du = \frac{a^2L}{4\pi}\left[u - \frac{1}{2}\sin 2u\right]_0^{\pi/2} = \frac{1}{8}a^2L$$

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{aL}{\pi}\right) = \frac{aL^2}{\pi^2} \qquad \bar{x} = \frac{L}{\pi} \blacktriangleleft$$

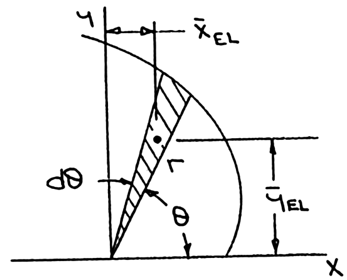
$$\bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\frac{aL}{\pi}\right) = \frac{1}{8}a^2L \qquad \bar{y} = \frac{\pi}{8}a \blacktriangleleft$$



PROBLEM 5.49*

Determine by direct integration the centroid of the area shown.

SOLUTION



We have
$$\bar{x}_{EL} = \frac{2}{3}r\cos\theta = \frac{2}{3}ae^\theta\cos\theta$$
$$\bar{y}_{EL} = \frac{2}{3}r\sin\theta = \frac{2}{3}ae^\theta\sin\theta$$

and
$$dA = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}a^2e^{2\theta}d\theta$$

Then
$$A = \int dA = \int_0^\pi \frac{1}{2}a^2e^{2\theta}d\theta = \frac{1}{2}a^2\left[\frac{1}{2}e^{2\theta}\right]_0^\pi$$
$$= \frac{1}{4}a^2(e^{2\pi} - 1)$$
$$= 133.623a^2$$

and
$$\int \bar{x}_{EL}dA = \int_0^\pi \frac{2}{3}ae^\theta\cos\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$
$$= \frac{1}{3}a^3\int_0^\pi e^{3\theta}\cos\theta d\theta$$

To proceed, use integration by parts, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$
$$dv = \cos\theta d\theta \quad \text{and} \quad v = \sin\theta$$

Then
$$\int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - \int\sin\theta(3e^{3\theta}d\theta)$$

Now let
$$u = e^{3\theta} \quad \text{then} \quad du = 3e^{3\theta}d\theta$$
$$dv = \sin\theta d\theta, \quad \text{then} \quad v = -\cos\theta$$

Then
$$\int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - 3\left[-e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)\right]$$

PROBLEM 5.49* (Continued)

so that $$\int e^{3\theta}\cos\theta d\theta = \frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)$$

$$\int x_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(-3e^{3\pi} - 3) = -1239.26a^3$$

Also, $$\int \overline{y}_{EL}dA = \int_0^\pi \frac{2}{3}ae^\theta \sin\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$

$$= \frac{1}{3}a^3\int_0^\pi e^{3\theta}\sin\theta d\theta$$

Use integration by parts, as above, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \int \sin\theta d\theta \quad \text{and} \quad v = -\cos\theta$$

Then $$\int e^{3\theta}\sin\theta d\theta = -e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)$$

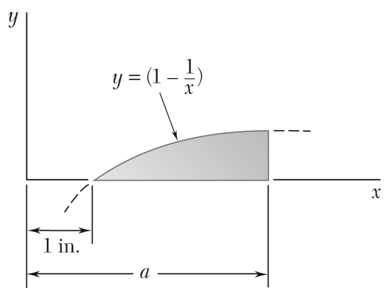
so that $$\int e^{3\theta}\sin\theta d\theta = \frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)$$

$$\int \overline{y}_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(e^{3\pi} + 1) = 413.09a^3$$

Hence, $$\overline{x}A = \int x_{EL}dA: \quad \overline{x}(133.623a^2) = -1239.26a^3$$ or $\overline{x} = -9.27a$ ◀

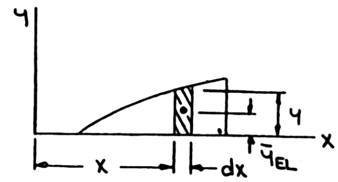
$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}(133.623a^2) = 413.09a^3$$ or $\overline{y} = 3.09a$ ◀



PROBLEM 5.50

Determine the centroid of the area shown when $a = 2$ in.

SOLUTION



We have $\quad \bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $\quad dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$

Then $\quad A = \int dA = \int_1^a \left(1 - \frac{1}{x}\right)\frac{dx}{2} = [x - \ln x]_1^a = (a - \ln a - 1)\text{ in}^2$

and $\quad \int \bar{x}_{EL}\,dA = \int_1^a x\left[\left(1 - \frac{1}{x}\right)dx\right] = \left[\frac{x^2}{2} - x\right]_1^a = \left(\frac{a^2}{2} - a + \frac{1}{2}\right)\text{in}^3$

$$\int \bar{y}_{EL}\,dA = \int_1^a \frac{1}{2}\left(1 - \frac{1}{x}\right)\left[\left(1 - \frac{1}{x}\right)dx\right] = \frac{1}{2}\int_1^a \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{2}\left[x - 2\ln x - \frac{1}{x}\right]_1^a = \frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)\text{in}^3$$

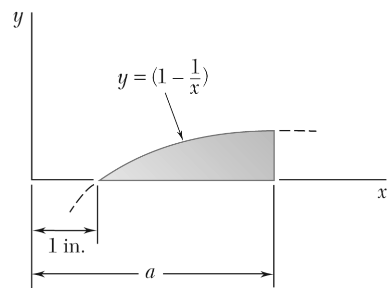
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x} = \frac{\frac{a^2}{2} - a + \frac{1}{2}}{a - \ln a - 1}\text{ in.}$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y} = \frac{a - 2\ln a - \frac{1}{a}}{2(a - \ln a - 1)}\text{ in.}$$

Find $\bar{x}$ and $\bar{y}$ when $a = 2$ in.

We have $\quad \bar{x} = \dfrac{\frac{1}{2}(2)^2 - 2 + \frac{1}{2}}{2 - \ln 2 - 1}$ $\quad$ or $\quad \bar{x} = 1.629$ in. ◀

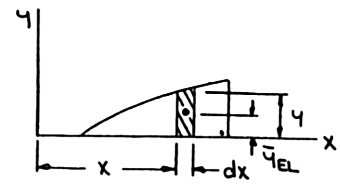
and $\quad \bar{y} = \dfrac{2 - 2\ln 2 - \frac{1}{2}}{2(2 - \ln 2 - 1)}$ $\quad$ or $\quad \bar{y} = 0.1853$ in. ◀



PROBLEM 5.51

Determine the value of a for which the ratio $\bar{x}/\bar{y}$ is 9.

SOLUTION



We have
$$\bar{x}_{EL} = x$$
$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and
$$dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$$

Then
$$A = \int dA = \int_1^a \left(1 - \frac{1}{x}\right)\frac{dx}{2} = [x - \ln x]_1^a$$
$$= (a - \ln a - 1)\text{ in}^2$$

and
$$\int \bar{x}_{EL}\,dA = \int_1^a x\left[\left(1 - \frac{1}{x}\right)dx\right] = \left[\frac{x^2}{2} - x\right]_1^a$$
$$= \left(\frac{a^2}{2} - a + \frac{1}{2}\right)\text{in}^3$$

$$\int \bar{y}_{EL}\,dA = \int_1^a \frac{1}{2}\left(1 - \frac{1}{x}\right)\left[\left(1 - \frac{1}{x}\right)dx\right] = \frac{1}{2}\int_1^a \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$
$$= \frac{1}{2}\left[x - 2\ln x - \frac{1}{x}\right]_1^a$$
$$= \frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)\text{in}^3$$

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x} = \frac{\frac{a^2}{2} - a + \frac{1}{2}}{a - \ln a - 1}\text{ in.}$$
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y} = \frac{a - 2\ln a - \frac{1}{a}}{2(a - \ln a - 1)}\text{ in.}$$

PROBLEM 5.51 (Continued)

Find a so that $\dfrac{\overline{x}}{\overline{y}} = 9$.

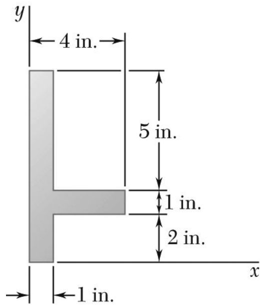
We have
$$\frac{\overline{x}}{\overline{y}} = \frac{\overline{x}A}{\overline{y}A} = \frac{\int \overline{x}_{EL}\,dA}{\int \overline{y}_{EL}\,dA}$$

Then
$$\frac{\frac{1}{2}a^2 - a + \frac{1}{2}}{\frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)} = 9$$

or
$$a^3 - 11a^2 + a + 18a\ln a + 9 = 0$$

Using trial and error or numerical methods, and ignoring the trivial solution $a = 1$ in., we find

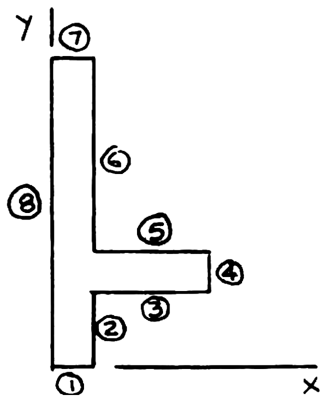
$$a = 1.901\text{ in.} \quad \text{and} \quad a = 3.74\text{ in.} \blacktriangleleft$$



PROBLEM 5.52

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.1 about (*a*) the *x*-axis, (*b*) the *y*-axis.

SOLUTION



From the solution of Problem 5.1, we have

$$A = 11\text{ in}^2$$

$$\Sigma \overline{x}A = 11.5\text{ in}^3$$

$$\Sigma \overline{y}A = 39.5\text{ in}^3$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi \overline{y}_{\text{area}} A = 2\pi \Sigma \overline{y} A$$

$$= 2\pi(39.5\text{ in}^3)$$

or $\text{Volume} = 248\text{ in}^3$ ◀

$$\text{Area} = 2\pi\, \overline{y}_{\text{line}} L = 2\pi \Sigma(\overline{y}_{\text{line}})L$$

$$= 2\pi(\overline{y}_2 L_2 + \overline{y}_3 L_3 + \overline{y}_4 L_4 + \overline{y}_5 L_5 + \overline{y}_6 L_6 + \overline{y}_7 L_7 + \overline{y}_8 L_8)$$

$$= 2\pi[(1)(2) + (2)(3) + (2.5)(1) + (3)(3) + (5.5)(5) + (8)(1) + (4)(8)]$$

or $\text{Area} = 547\text{ in}^2$ ◀

(*b*) Rotation about the *y*-axis:

$$\text{Volume} = 2\pi \overline{x}_{\text{area}} A = 2\pi \Sigma \overline{x} A$$

$$= 2\pi(11.5\text{ in}^3)$$

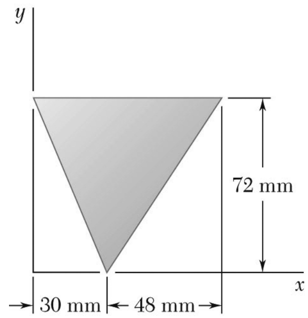
or $\text{Volume} = 72.3\text{ in}^3$ ◀

$$\text{Area} = 2\pi\, \overline{x}_{\text{line}} L = 2\pi \Sigma(\overline{x}_{\text{line}})L$$

$$= 2\pi(\overline{x}_1 L_1 + \overline{x}_2 L_2 + \overline{x}_3 L_3 + \overline{x}_4 L_4 + \overline{x}_5 L_5 + \overline{x}_6 L_6 + \overline{x}_7 L_7)$$

$$= 2\pi[(0.5)(1) + (1)(2) + (2.5)(3) + (4)(1) + (2.5)(3) + (1)(5) + (0.5)(1)]$$

or $\text{Area} = 169.6\text{ in}^2$ ◀

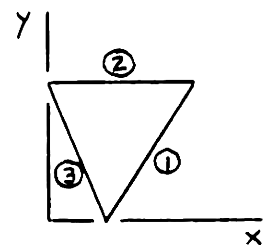


PROBLEM 5.53

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.2 about (*a*) the line $y = 72$ mm, (*b*) the *x*-axis.

SOLUTION

From the solution of Problem 5.2, we have



$$A = 2808 \text{ mm}^2$$
$$\overline{x} = 36 \text{ mm}$$
$$\overline{y} = 48 \text{ mm}$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the line $y = 72$ mm:

$$\text{Volume} = 2\pi(72 - \overline{y})A$$
$$= 2\pi(72 - 48)(2808)$$

$$\text{Volume} = 423 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi \overline{y}_{\text{line}} L$$
$$= 2\pi \Sigma(\overline{y}_{\text{line}})L$$
$$= 2\pi(\overline{y}_1 L_1 + \overline{y}_3 L_3)$$

where $\overline{y}_1$ and $\overline{y}_3$ are measured with respect to line $y = 72$ mm.

$$\text{Area} = 2\pi\left[(36)\left(\sqrt{48^2 + 72^2}\right) + (36)\left(\sqrt{30^2 + 72^2}\right)\right]$$

$$\text{Area} = 37.2 \times 10^3 \text{ mm}^2 \blacktriangleleft$$

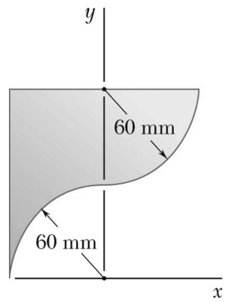
(*b*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi \overline{y}_{\text{area}} A$$
$$= 2\pi(48)(2808)$$

$$\text{Volume} = 847 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi \overline{y}_{\text{line}} L = 2\pi \Sigma(\overline{y}_{\text{line}})L$$
$$= 2\pi(\overline{y}_1 L_1 + \overline{y}_2 L_2 + \overline{y}_3 L_3)$$
$$= 2\pi\left[(36)\left(\sqrt{48^2 + 72^2}\right) + (72)(78) + (36)\left(\sqrt{30^2 + 72^2}\right)\right]$$

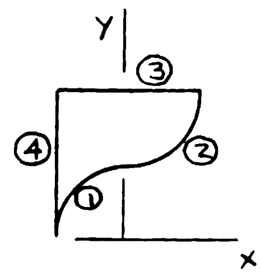
$$\text{Area} = 72.5 \times 10^3 \text{ mm}^2 \blacktriangleleft$$



PROBLEM 5.54

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.8 about (*a*) the line $x = -60$ mm, (*b*) the line $y = 120$ mm.

SOLUTION



From the solution of Problem 5.8, we have

$$A = 7200 \text{ mm}^2$$
$$\Sigma \bar{x} A = -72 \times 10^3 \text{ mm}^3$$
$$\Sigma \bar{y} A = 629.83 \times 10^3 \text{ mm}^3$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about line $x = -60$ mm:

$$\text{Volume} = 2\pi(\bar{x} + 60)A = 2\pi(\Sigma \bar{x}A + 60A)$$
$$= 2\pi[-72 \times 10^3 + 60(7200)]$$

$\text{Volume} = 2.26 \times 10^6 \text{ mm}^3$ ◀

$$\text{Area} = 2\pi \bar{x}_{\text{line}} L = 2\pi \Sigma(\bar{x}_{\text{line}})L$$
$$= 2\pi(\bar{x}_1 L_1 + \bar{x}_2 L_2 + \bar{x}_3 L_3)$$
$$= 2\pi\left[\left(60 - \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + \left(60 + \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + (60)(120)\right]$$

where $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are measured with respect to line $x = -60$ mm.

$\text{Area} = 116.3 \times 10^3 \text{ mm}^2$ ◀

(*b*) Rotation about line $y = 120$ mm:

$$\text{Volume} = 2\pi(120 - \bar{y})A = 2\pi(120A - \Sigma \bar{y}A)$$
$$= 2\pi[120(7200) - 629.83 \times 10^3]$$

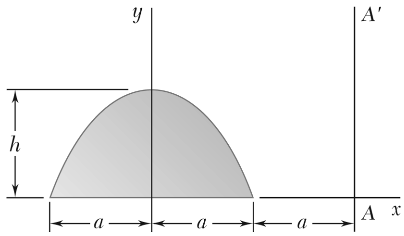
$\text{Volume} = 1.471 \times 10^6 \text{ mm}^3$ ◀

$$\text{Area} = 2\pi \bar{y}_{\text{line}} L = 2\pi \Sigma(\bar{y}_{\text{line}})L$$
$$= 2\pi(\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_4 L_4)$$

where $\bar{y}_1, \bar{y}_2, \bar{y}_4$ are measured with respect to line $y = 120$ mm.

$$\text{Area} = 2\pi\left[\left(120 - \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + \left(\frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + (60)(120)\right]$$

$\text{Area} = 116.3 \times 10^3 \text{ mm}^2$ ◀



PROBLEM 5.55

Determine the volume of the solid generated by rotating the parabolic area shown about (*a*) the *x*-axis, (*b*) the axis *AA*′.

SOLUTION

First, from Figure 5.8a, we have $A = \dfrac{4}{3}ah$

$$\bar{y} = \frac{2}{5}h$$

Applying the second theorem of Pappus-Guldinus, we have

(*a*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi\bar{y}A$$

$$= 2\pi\left(\frac{2}{5}h\right)\left(\frac{4}{3}ah\right)$$

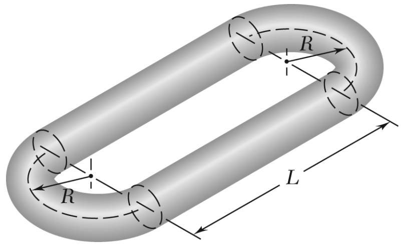
or $\text{Volume} = \dfrac{16}{15}\pi ah^2$ ◀

(*b*) Rotation about the line *AA*′:

$$\text{Volume} = 2\pi(2a)A$$

$$= 2\pi(2a)\left(\frac{4}{3}ah\right)$$

or $\text{Volume} = \dfrac{16}{3}\pi a^2 h$ ◀



PROBLEM 5.56

Determine the volume and the surface area of the chain link shown, which is made from a 6-mm-diameter bar, if $R = 10$ mm and $L = 30$ mm.

SOLUTION

The area A and circumference C of the cross section of the bar are

$$A = \frac{\pi}{4}d^2 \quad \text{and} \quad C = \pi d.$$

Also, the semicircular ends of the link can be obtained by rotating the cross section through a horizontal semicircular arc of radius R. Now, applying the theorems of Pappus-Guldinus, we have for the volume V,

$$\begin{aligned} V &= 2(V_{\text{side}}) + 2(V_{\text{end}}) \\ &= 2(AL) + 2(\pi RA) \\ &= 2(L + \pi R)A \end{aligned}$$

or

$$V = 2[30 \text{ mm} + \pi(10 \text{ mm})]\left[\frac{\pi}{4}(6 \text{ mm})^2\right]$$

$$= 3470 \text{ mm}^3 \qquad \text{or } V = 3470 \text{ mm}^3 \blacktriangleleft$$

For the area A,

$$\begin{aligned} A &= 2(A_{\text{side}}) + 2(A_{\text{end}}) \\ &= 2(CL) + 2(\pi RC) \\ &= 2(L + \pi R)C \end{aligned}$$

or

$$A = 2[30 \text{ mm} + \pi(10 \text{ mm})][\pi(6 \text{ mm})]$$

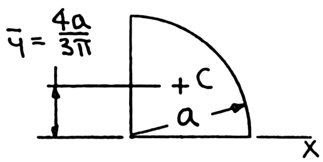
$$= 2320 \text{ mm}^2 \qquad \text{or } A = 2320 \text{ mm}^2 \blacktriangleleft$$

PROBLEM 5.57

Verify that the expressions for the volumes of the first four shapes in Figure 5.21 on Page 264 are correct.

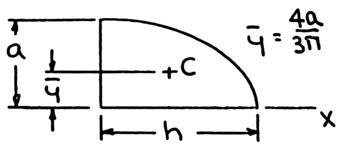
SOLUTION

Following the second theorem of Pappus-Guldinus, in each case, a specific generating area A will be rotated about the x-axis to produce the given shape. Values of $\overline{y}$ are from Figure 5.8a.



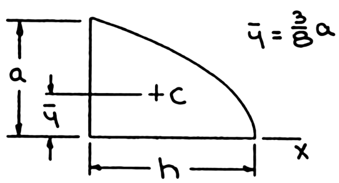
(1) Hemisphere: the generating area is a quarter circle.

We have $V = 2\pi \overline{y} A = 2\pi \left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}a^2\right)$ or $V = \frac{2}{3}\pi a^3$ ◀



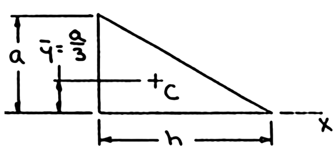
(2) Semiellipsoid of revolution: the generating area is a quarter ellipse.

We have $V = 2\pi \overline{y} A = 2\pi \left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}ha\right)$ or $V = \frac{2}{3}\pi a^2 h$ ◀



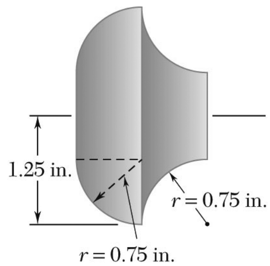
(3) Paraboloid of revolution: the generating area is a quarter parabola.

We have $V = 2\pi \overline{y} A = 2\pi \left(\frac{3}{8}a\right)\left(\frac{2}{3}ah\right)$ or $V = \frac{1}{2}\pi a^2 h$ ◀



(4) Cone: the generating area is a triangle.

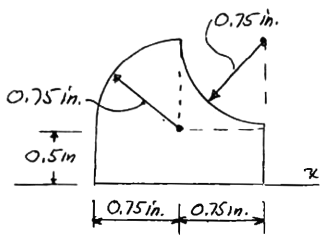
We have $V = 2\pi \overline{y} A = 2\pi \left(\frac{a}{3}\right)\left(\frac{1}{2}ha\right)$ or $V = \frac{1}{3}\pi a^2 h$ ◀



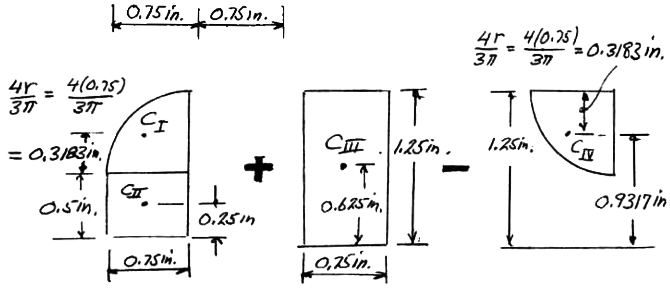
PROBLEM 5.58

Determine the volume and weight of the solid brass knob shown, knowing that the specific weight of brass is 0.306 lb/in^3.

SOLUTION



Volume of knob is obtained by rotating area at left about the x-axis. Consider area as made of components shown below.

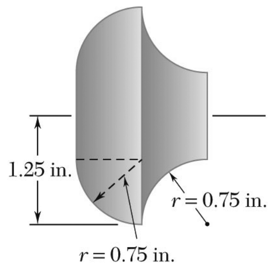


	Area, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$\frac{\pi}{4}(0.75)^2 = 0.4418$	0.8183	0.3615
2	$(0.5)(0.75) = 0.375$	0.25	0.0938
3	$(1.25)(0.75) = 0.9375$	0.625	0.5859
4	$\frac{-\pi}{4}(0.75)^2 = -0.4418$	0.9317	−0.4116
Σ			0.6296

$$V = 2\pi\Sigma\bar{y}A = 2\pi(0.6296 \text{ in}^3) = 3.9559 \text{ in}^3 \qquad V = 3.96 \text{ in}^3 \blacktriangleleft$$

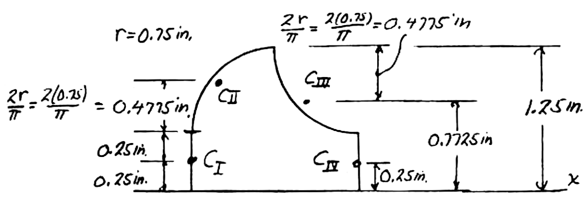
$$W = \gamma V = (0.306 \text{ lb/in}^3)(3.9559 \text{ in}^3) \qquad W = 1.211 \text{ lb} \blacktriangleleft$$

PROBLEM 5.59



Determine the total surface area of the solid brass knob shown.

SOLUTION



Area is obtained by rotating lines shown about the x-axis.

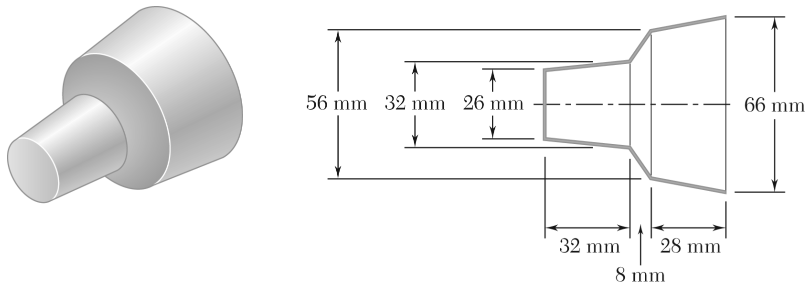
	L, in.	$\overline{y}$, in.	$\overline{y}L$, in^2
1	0.5	0.25	0.1250
2	$\frac{\pi}{2}(0.75) = 1.1781$	0.9775	1.1516
3	$\frac{\pi}{2}(0.75) = 1.1781$	0.7725	0.9101
4	0.5	0.25	0.1250
Σ			2.3117

$$A = 2\pi\Sigma\overline{y}L = 2\pi(2.3117 \text{ in}^2)$$

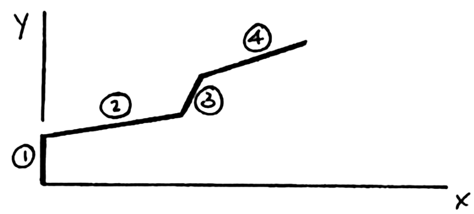
$A = 14.52 \text{ in}^2$ ◀

PROBLEM 5.60

The aluminum shade for the small high-intensity lamp shown has a uniform thickness of 1 mm. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass of the shade.



SOLUTION



The mass of the lamp shade is given by

$$m = \rho V = \rho A t$$

where A is the surface area and t is the thickness of the shade. The area can be generated by rotating the line shown about the x-axis. Applying the first theorem of Pappus Guldinus, we have

$$A = 2\pi \bar{y} L = 2\pi \Sigma \bar{y} L$$
$$= 2\pi(\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_3 L_3 + \bar{y}_4 L_4)$$

or

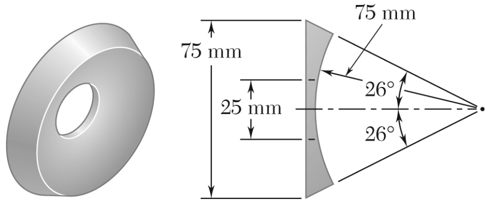
$$A = 2\pi \left[\frac{13 \text{ mm}}{2}(13 \text{ mm}) + \left(\frac{13+16}{2}\right) \text{mm} \times \sqrt{(32 \text{ mm})^2 + (3 \text{ mm})^2} \right.$$
$$+ \left(\frac{16+28}{2}\right) \text{mm} \times \sqrt{(8 \text{ mm})^2 + (12 \text{ mm})^2}$$
$$\left. + \left(\frac{28+33}{2}\right) \text{mm} \times \sqrt{(28 \text{ mm})^2 + (5 \text{ mm})^2} \right]$$
$$= 2\pi(84.5 + 466.03 + 317.29 + 867.51)$$
$$= 10{,}903.4 \text{ mm}^2$$

Then

$$m = \rho A t$$
$$= (2800 \text{ kg/m}^3)(10.9034 \times 10^{-3} \text{ m}^2)(0.001 \text{ m})$$

or

$$m = 0.0305 \text{ kg} \blacktriangleleft$$

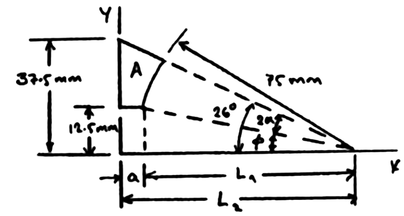


PROBLEM 5.61

The escutcheon (a decorative plate placed on a pipe where the pipe exits from a wall) shown is cast from brass. Knowing that the density of brass is 8470 kg/m^3, determine the mass of the escutcheon.

SOLUTION

The mass of the escutcheon is given by $m = (\text{density})V$, where V is the volume. V can be generated by rotating the area A about the x-axis.



From the figure:

$$L_1 = \sqrt{75^2 - 12.5^2} = 73.9510 \text{ m}$$

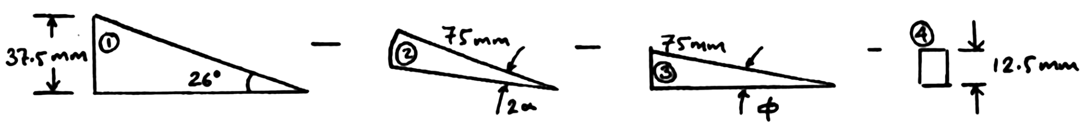
$$L_2 = \frac{37.5}{\tan 26°} = 76.8864 \text{ mm}$$

$$a = L_2 - L_1 = 2.9324 \text{ mm}$$

$$\phi = \sin^{-1}\frac{12.5}{75} = 9.5941°$$

$$\alpha = \frac{26° - 9.5941°}{2} = 8.2030° = 0.143168 \text{ rad}$$

Area A can be obtained by combining the following four areas:



Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

$$V = 2\pi\bar{y}A = 2\pi\Sigma\bar{y}A$$

Seg.	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{1}{2}(76.886)(37.5) = 1441.61$	$\frac{1}{3}(37.5) = 12.5$	18,020.1
2	$-\alpha(75)^2 = -805.32$	$\frac{2(75)\sin\alpha}{3\alpha}\sin(\alpha+\phi) = 15.2303$	−12,265.3
3	$-\frac{1}{2}(73.951)(12.5) = -462.19$	$\frac{1}{3}(12.5) = 4.1667$	−1925.81
4	$-(2.9354)(12.5) = -36.693$	$\frac{1}{2}(12.5) = 6.25$	−229.33
Σ			3599.7

PROBLEM 5.61 (Continued)

Then $V = 2\pi\Sigma\bar{y}A$

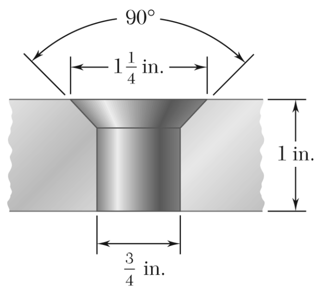
$= 2\pi(3599.7 \text{ mm}^3)$

$= 22{,}618 \text{ mm}^3$

$m = (\text{density})V$

$= (8470 \text{ kg/m}^3)(22.618\times10^{-6} \text{ m}^3)$

$= 0.191574 \text{ kg}$ or $m = 0.1916 \text{ kg}$ ◀

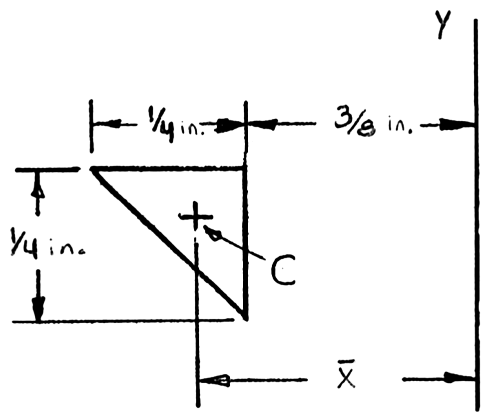


PROBLEM 5.62

A $\frac{3}{4}$- in.-diameter hole is drilled in a piece of 1-in.-thick steel; the hole is then countersunk as shown. Determine the volume of steel removed during the countersinking process.

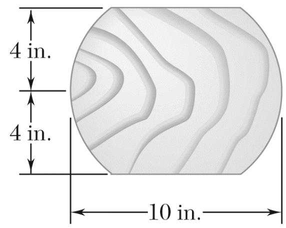
SOLUTION

The required volume can be generated by rotating the area shown about the *y*-axis. Applying the second theorem of Pappus-Guldinus, we have



$$V = 2\pi \bar{x} A$$

$$= 2\pi\left[\frac{3}{8}+\frac{1}{3}\left(\frac{1}{4}\right) \text{ in.}\right]\times\left[\frac{1}{2}\times\frac{1}{4} \text{ in.}\times\frac{1}{4} \text{ in.}\right] \qquad V = 0.0900 \text{ in}^3 \blacktriangleleft$$



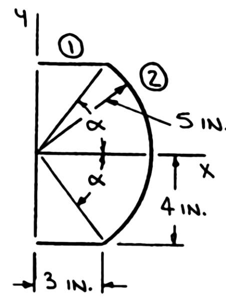
PROBLEM 5.63

Knowing that two equal caps have been removed from a 10-in.-diameter wooden sphere, determine the total surface area of the remaining portion.

SOLUTION

The surface area can be generated by rotating the line shown about the *y*-axis. Applying the first theorem of Pappus-Guldinus, we have

$$A = 2\pi \overline{X} L = 2\pi \Sigma \overline{x} L$$
$$= 2\pi(2\overline{x}_1 L_1 + \overline{x}_2 L_2)$$



Now $\tan\alpha = \dfrac{4}{3}$

or $\alpha = 53.130°$

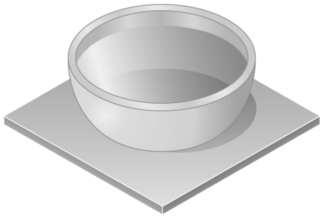
Then $$\overline{x}_2 = \frac{5\text{ in.} \times \sin 53.130°}{53.130° \times \frac{\pi}{180°}}$$
$$= 4.3136 \text{ in.}$$

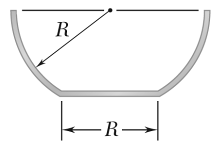
and $$L_2 = 2\left(53.130° \times \frac{\pi}{180°}\right)(5\text{ in.})$$
$$= 9.2729 \text{ in.}$$

$$A = 2\pi\left[2\left(\frac{3}{2}\text{ in.}\right)(3\text{ in.}) + (4.3136\text{ in.})(9.2729\text{ in.})\right]$$

or $A = 308 \text{ in}^2$ ◀

PROBLEM 5.64

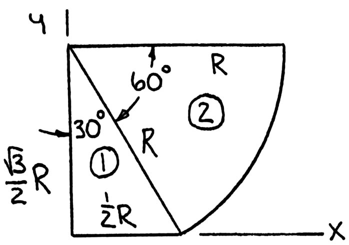




Determine the capacity, in liters, of the punch bowl shown if $R = 250$ mm.

SOLUTION

The volume can be generated by rotating the triangle and circular sector shown about the y-axis. Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

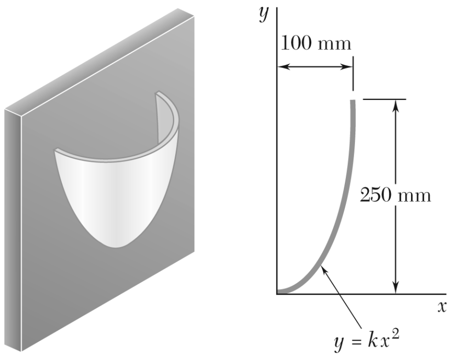


$$
\begin{aligned}
V &= 2\pi\bar{x}A = 2\pi\Sigma\bar{x}A \\
&= 2\pi(\bar{x}_1A_1 + \bar{x}_2A_2) \\
&= 2\pi\left[\left(\frac{1}{3}\times\frac{1}{2}R\right)\left(\frac{1}{2}\times\frac{1}{2}R\times\frac{\sqrt{3}}{2}R\right)+\left(\frac{2R\sin 30^\circ}{3\times\frac{\pi}{6}}\right)\left(\frac{\pi}{6}R^2\right)\right] \\
&= 2\pi\left(\frac{R^3}{16\sqrt{3}}+\frac{R^3}{2\sqrt{3}}\right) \\
&= \frac{3\sqrt{3}}{8}\pi R^3 \\
&= \frac{3\sqrt{3}}{8}\pi(0.25\text{ m})^3 \\
&= 0.031883\text{ m}^3
\end{aligned}
$$

Since $\qquad 10^3\text{ l} = 1\text{ m}^3$

$$V = 0.031883\text{ m}^3\times\frac{10^3\text{ l}}{1\text{ m}^3}$$

$V = 31.9\text{ l}$ ◀



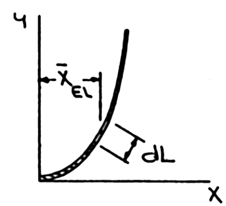
PROBLEM 5.65*

The shade for a wall-mounted light is formed from a thin sheet of translucent plastic. Determine the surface area of the outside of the shade, knowing that it has the parabolic cross section shown.

SOLUTION

First note that the required surface area A can be generated by rotating the parabolic cross section through π radians about the y-axis. Applying the first theorem of Pappus-Guldinus, we have

$$A = \pi \bar{x} L$$



Now at $\quad x = 100 \text{ mm}, \quad y = 250 \text{ mm}$

$$250 = k(100)^2 \quad \text{or} \quad k = 0.025 \text{ mm}^{-1}$$

and $\quad x_{EL} = x$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $\quad \dfrac{dy}{dx} = 2kx$

Then $\quad dL = \sqrt{1 + 4k^2x^2}\, dx$

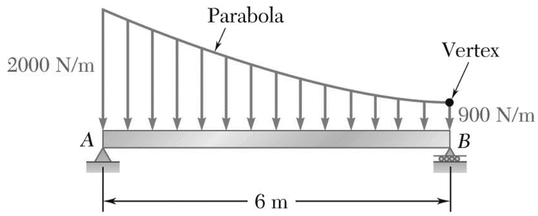
We have $\quad \bar{x}L = \int x_{EL}\, dL = \int_0^{100} x\left(\sqrt{1 + 4k^2x^2 dx}\right)$

$$\bar{x}L = \left[\frac{1}{3}\frac{1}{4k^2}(1 + 4k^2x^2)^{3/2}\right]_0^{100}$$

$$= \frac{1}{12}\frac{1}{(0.025)^2}\left\{[1 + 4(0.025)^2(100)^2]^{3/2} - (1)^{3/2}\right\}$$

$$= 17{,}543.3 \text{ mm}^2$$

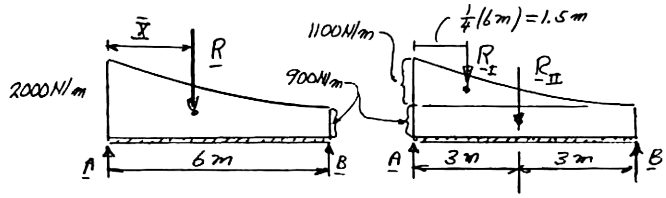
Finally, $\quad A = \pi(17{,}543.3 \text{ mm}^2) \qquad$ or $\quad A = 55.1 \times 10^3 \text{ mm}^2$ ◀



PROBLEM 5.66

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION



(*a*) $$R_{\mathrm{I}} = \frac{1}{3}(1100\text{ N/m})(6\text{ m}) = 2200\text{ N}$$

$$R_{\mathrm{II}} = (900\text{ N/m})(6\text{ m}) = 5400\text{ N}$$

$$R = R_{\mathrm{I}} + R_{\mathrm{II}} = 2200 + 5400 = 7600\text{ N}$$

$$XR = \Sigma xR: \quad X(7600) = (2200)(1.5) + (5400)(3)$$

$$X = 2.5658\text{ m}$$

$\mathbf{R} = 7.60\text{ kN}\downarrow, \quad X = 2.57\text{ m}$ ◀

(*b*) $$+\circlearrowleft \Sigma M_A = 0: \quad B(6\text{ m}) - (7600\text{ N})(2.5658\text{ m}) = 0$$

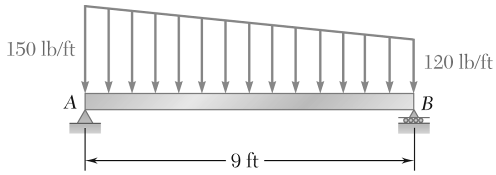
$$B = 3250.0\text{ N}$$

$\mathbf{B} = 3.25\text{ kN}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad A + 3250.0\text{ N} - 7600\text{ N} = 0$$

$$A = 4350.0\text{ N}$$

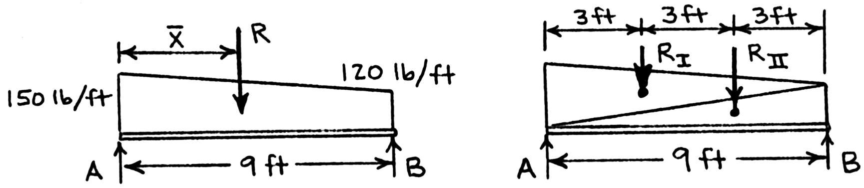
$\mathbf{A} = 4.35\text{ kN}\uparrow$ ◀



PROBLEM 5.67

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION



$$R_{\text{I}} = \frac{1}{2}(150\,\text{lb/ft})(9\text{ ft}) = 675\text{ lb}$$

$$R_{\text{II}} = \frac{1}{2}(120\text{ lb/ft})(9\,\text{ft}) = 540\,\text{lb}$$

$$R = R_{\text{I}} + R_{\text{II}} = 675 + 540 = 1215\text{ lb}$$

$$\bar{X}R = \Sigma\bar{x}R: \quad \bar{X}(1215) = (3)(675) + (6)(540) \quad \bar{X} = 4.3333\text{ ft}$$

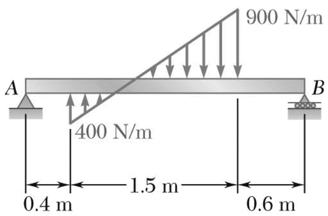
(*a*) $\mathbf{R} = 1215\text{ lb}\downarrow \quad \bar{X} = 4.33\text{ ft}$ ◀

(*b*) Reactions: $+\circlearrowleft\Sigma M_A = 0\!: \; B(9\text{ ft}) - (1215\text{ lb})(4.3333\text{ ft}) = 0$

$B = 585.00\text{ lb}$ $\quad \mathbf{B} = 585\text{ lb}\uparrow$ ◀

$+\uparrow\Sigma F_y = 0\!: \; A + 585.00\text{ lb} - 1215\text{ lb} = 0$

$A = 630.00\text{ lb}$ $\quad \mathbf{A} = 630\text{ lb}\uparrow$ ◀

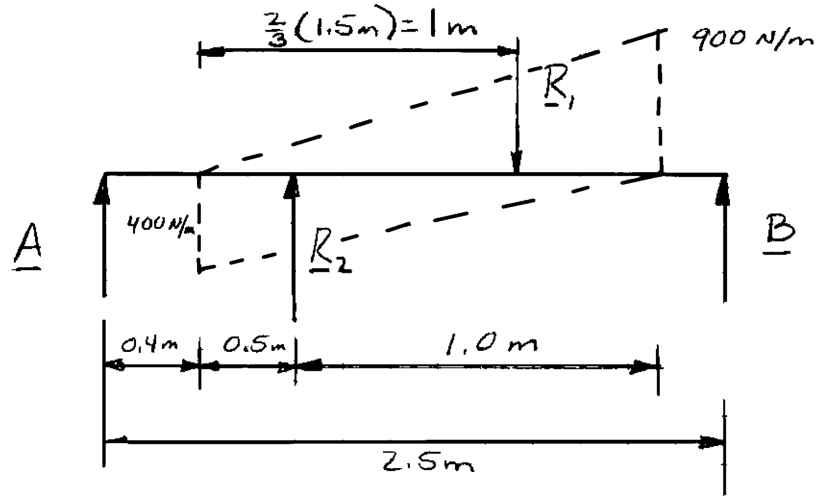


PROBLEM 5.68

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading by the loadings shown below. Both loading are equivalent since they are both defined by a linear relation between load and distance and have the same values at the end points.



$$R_1 = \frac{1}{2}(900\text{ N/m})(1.5\text{ m}) = 675\text{ N}$$

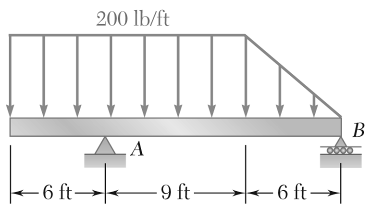
$$R_2 = \frac{1}{2}(400\text{ N/m})(1.5\text{ m}) = 300\text{ N}$$

$+\circlearrowleft \Sigma M_A = 0\colon\quad -(675\text{ N})(1.4\text{ m}) + (300\text{ N})(0.9\text{ m}) + B(2.5\text{ m}) = C$

$B = 270\text{ N}$ $\qquad$ $\mathbf{B} = 270\text{ N}\uparrow$ ◀

$+\uparrow \Sigma F_y = 0\colon\quad A - 675\text{ N} + 300\text{ N} + 270\text{ N} = 0$

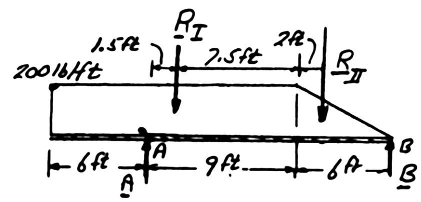
$A = 105.0\text{ N}$ $\qquad$ $\mathbf{A} = 105.0\text{ N}\uparrow$ ◀



PROBLEM 5.69

Determine the reactions at the beam supports for the given loading.

SOLUTION



$$R_{\mathrm{I}} = (200 \text{ lb/ft})(15 \text{ ft})$$
$$R_{\mathrm{I}} = 3000 \text{ lb}$$
$$R_{\mathrm{II}} = \frac{1}{2}(200 \text{ lb/ft})(6 \text{ ft})$$
$$R_{\mathrm{II}} = 600 \text{ lb}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -(3000 \text{ lb})(1.5 \text{ ft}) - (600 \text{ lb})(9 \text{ ft} + 2 \text{ ft}) + B(15 \text{ ft}) = 0$$

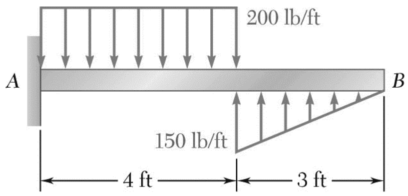
$$B = 740 \text{ lb}$$

$\mathbf{B} = 740 \text{ lb}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad A + 740 \text{ lb} - 3000 \text{ lb} - 600 \text{ lb} = 0$$

$$A = 2860 \text{ lb}$$

$\mathbf{A} = 2860 \text{ lb}\uparrow$ ◀



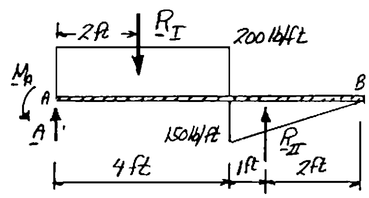
PROBLEM 5.70

Determine the reactions at the beam supports for the given loading.

SOLUTION

$$R_{\mathrm{I}} = (200 \text{ lb/ft})(4 \text{ ft}) = 800 \text{ lb}$$

$$R_{\mathrm{II}} = \frac{1}{2}(150 \text{ lb/ft})(3 \text{ ft}) = 225 \text{ lb}$$

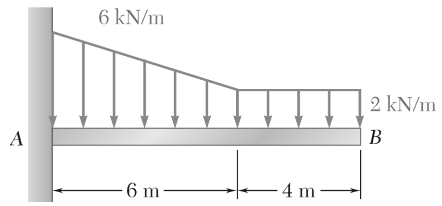


$$+\uparrow \Sigma F_y = 0: \quad A - 800 \text{ lb} + 225 \text{ lb} = 0$$

$$\mathbf{A} = 575 \text{ lb} \uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (800 \text{ lb})(2 \text{ ft}) + (225 \text{ lb})(5 \text{ ft}) = 0$$

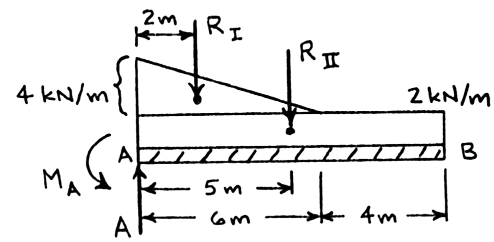
$$\mathbf{M}_A = 475 \text{ lb} \cdot \text{ft} \circlearrowleft \blacktriangleleft$$



PROBLEM 5.71

Determine the reactions at the beam supports for the given loading.

SOLUTION



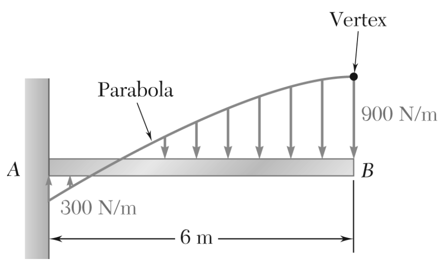
$$R_{\mathrm{I}} = \frac{1}{2}(4\text{ kN/m})(6\text{ m})$$
$$= 12\text{ kN}$$
$$R_{\mathrm{II}} = (2\text{ kN/m})(10\text{ m})$$
$$= 20\text{ kN}$$

$$+\uparrow \Sigma F_y = 0: \quad A - 12\text{ kN} - 20\text{ kN} = 0$$

$$\mathbf{A} = 32.0\text{ kN}\uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (12\text{ kN})(2\text{ m}) - (20\text{ kN})(5\text{ m}) = 0$$

$$\mathbf{M}_A = 124.0\text{ kN}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

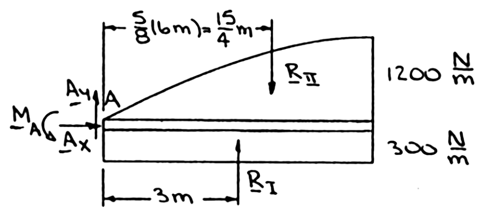


PROBLEM 5.72

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading with the loading shown below. The two loadings are equivalent because both are defined by a parabolic relation between load and distance and the values at the end points are the same.



We have $$R_{\mathrm{I}} = (6\text{ m})(300\text{ N/m}) = 1800\text{ N}$$

$$R_{\mathrm{II}} = \frac{2}{3}(6\text{ m})(1200\text{ N/m}) = 4800\text{ N}$$

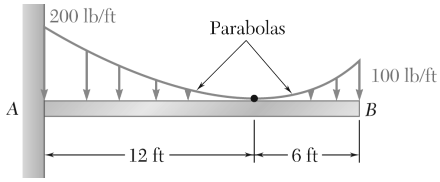
Then $$\xrightarrow{+}\Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 1800\text{ N} - 4800\text{ N} = 0$$

or $$A_y = 3000\text{ N}$$ $\mathbf{A} = 3.00\text{ kN}\uparrow$ ◀

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A + (3\text{ m})(1800\text{ N}) - \left(\frac{15}{4}\text{ m}\right)(4800\text{ N}) = 0$$

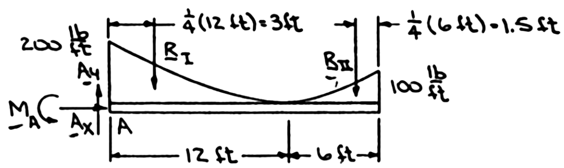
or $$M_A = 12.6\text{ kN}\cdot\text{m}$$ $\mathbf{M}_A = 12.60\text{ kN}\cdot\text{m}\circlearrowleft$ ◀



PROBLEM 5.73

Determine the reactions at the beam supports for the given loading.

SOLUTION



We have $$R_{\mathrm{I}} = \frac{1}{3}(12\text{ ft})(200\text{ lb/ft}) = 800\text{ lb}$$

$$R_{\mathrm{II}} = \frac{1}{3}(6\text{ ft})(100\text{ lb/ft}) = 200\text{ lb}$$

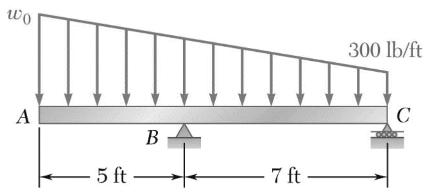
Then $$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 800\text{ lb} - 200\text{ lb} = 0$$

or $$A_y = 1000\text{ lb} \qquad \mathbf{A} = 1000\text{ lb}\uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (3\text{ ft})(800\text{ lb}) - (16.5\text{ ft})(200\text{ lb}) = 0$$

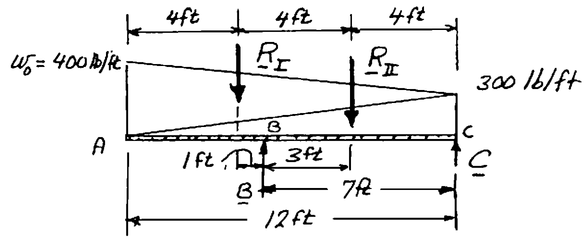
or $$M_A = 5700\text{ lb}\cdot\text{ft} \qquad \mathbf{M}_A = 5700\text{ lb}\cdot\text{ft} \circlearrowleft \blacktriangleleft$$



PROBLEM 5.74

Determine the reactions at the beam supports for the given loading when $w_O = 400$ lb/ft.

SOLUTION



$$R_{\mathrm{I}} = \frac{1}{2} w_O (12 \text{ ft}) = \frac{1}{2}(400 \text{ lb/ft})(12 \text{ ft}) = 2400 \text{ lb}$$

$$R_{\mathrm{II}} = \frac{1}{2}(300 \text{ lb/ft})(12 \text{ ft}) = 1800 \text{ lb}$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (2400 \text{ lb})(1 \text{ ft}) - (1800 \text{ lb})(3 \text{ ft}) + C(7 \text{ ft}) = 0$$

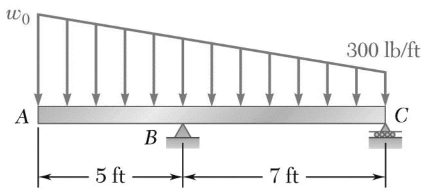
$$C = 428.57 \text{ lb}$$

$\mathbf{C} = 429 \text{ lb} \uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad B + 428.57 \text{ lb} - 2400 \text{ lb} - 1800 \text{ lb} = 0$$

$$B = 3771 \text{ lb}$$

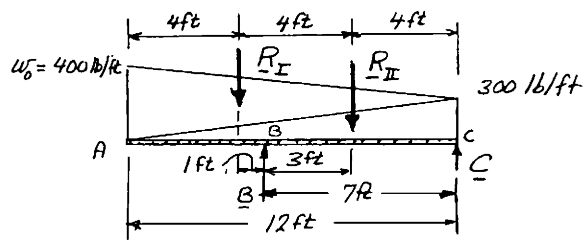
$\mathbf{B} = 3770 \text{ lb} \uparrow$ ◀



PROBLEM 5.75

Determine (*a*) the distributed load w_O at the end *A* of the beam *ABC* for which the reaction at *C* is zero, (*b*) the corresponding reaction at *B*.

SOLUTION



For w_O, $\qquad R_{\text{I}} = \frac{1}{2} w_O (12 \text{ ft}) = 6\, w_O$

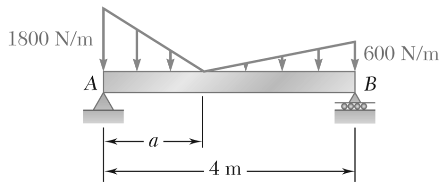
$$R_{\text{II}} = \frac{1}{2}(300 \text{ lb/ft})(12 \text{ ft}) = 1800 \text{ lb}$$

(*a*) For $C = 0$, $\qquad +\circlearrowleft \Sigma M_B = 0\text{:}\quad (6\, w_O)(1 \text{ ft}) - (1800 \text{ lb})(3 \text{ ft}) = 0$ $\qquad w_O = 900 \text{ lb/ft}$ ◀

(*b*) Corresponding value of R_{I}:

$$R_{\text{I}} = 6(900) = 5400 \text{ lb}$$

$+\!\uparrow \Sigma F_y = 0\text{:}\quad B - 5400 \text{ lb} - 1800 \text{ lb} = 0$ $\qquad \mathbf{B} = 7200 \text{ lb} \uparrow$ ◀

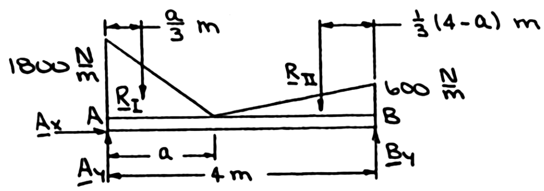


PROBLEM 5.76

Determine (*a*) the distance *a* so that the vertical reactions at supports *A* and *B* are equal, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)



We have $$R_{\mathrm{I}} = \frac{1}{2}(a \text{ m})(1800 \text{ N/m}) = 900a \text{ N}$$

$$R_{\mathrm{II}} = \frac{1}{2}[(4-a) \text{ m}](600 \text{ N/m}) = 300(4-a) \text{ N}$$

Then $$+\uparrow \Sigma F_y = 0: \quad A_y - 900a - 300(4-a) + B_y = 0$$

or $$A_y + B_y = 1200 + 600a$$

Now $$A_y = B_y \Rightarrow A_y = B_y = 600 + 300a \text{ (N)} \qquad (1)$$

Also, $$+\circlearrowleft \Sigma M_B = 0: \quad -(4 \text{ m})A_y + \left[\left(4 - \frac{a}{3}\right) \text{m}\right][(900a) \text{ N}]$$

$$+\left[\frac{1}{3}(4-a) \text{ m}\right][300(4-a) \text{ N}] = 0$$

or $$A_y = 400 + 700a - 50a^2 \qquad (2)$$

Equating Eqs. (1) and (2), $$600 + 300a = 400 + 700a - 50a^2$$

or $$a^2 - 8a + 4 = 0$$

Then $$a = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(4)}}{2}$$

or $a = 0.53590$ m $\qquad a = 7.4641$ m

Now $a \le 4 \text{ m} \Rightarrow$ $\qquad a = 0.536$ m ◀

PROBLEM 5.76 (Continued)

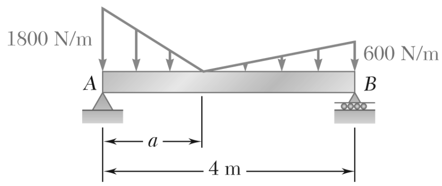
(*b*) We have $\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

From Eq. (1): $A_y = B_y$

$$= 600 + 300(0.53590)$$

$$= 761 \text{ N}$$

$\mathbf{A} = \mathbf{B} = 761 \text{ N} \uparrow$ ◀

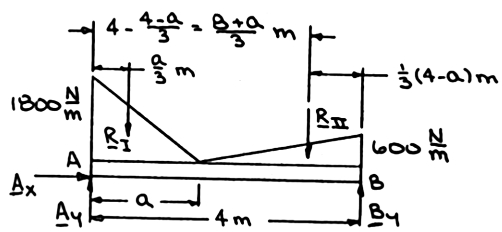


PROBLEM 5.77

Determine (*a*) the distance *a* so that the reaction at support *B* is minimum, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)



We have $$R_{\mathrm{I}} = \frac{1}{2}(a\text{ m})(1800\text{ N/m}) = 900a\text{ N}$$

$$R_{\mathrm{II}} = \frac{1}{2}[(4-a)\text{m}](600\text{ N/m}) = 300(4-a)\text{ N}$$

Then $$+\circlearrowleft \Sigma M_A = 0: \quad -\left(\frac{a}{3}\text{m}\right)(900a\text{ N}) - \left(\frac{8+a}{3}\text{m}\right)[300(4-a)\text{N}] + (4\text{ m})B_y = 0$$

or $$B_y = 50a^2 - 100a + 800 \qquad (1)$$

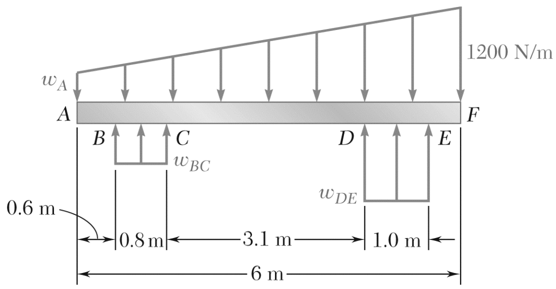
Then $$\frac{dB_y}{da} = 100a - 100 = 0$$ or $a = 1.000$ m ◀

(*b*) From Eq. (1): $$B_y = 50(1)^2 - 100(1) + 800 = 750\text{ N}$$ $\mathbf{B} = 750\text{ N}\uparrow$ ◀

and $$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 900(1)\text{ N} - 300(4-1)\text{ N} + 750\text{ N} = 0$$

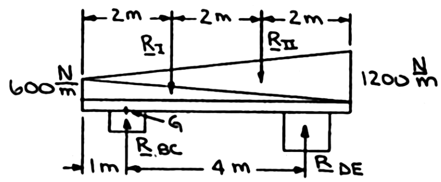
or $$A_y = 1050\text{ N}$$ $\mathbf{A} = 1050\text{ N}\uparrow$ ◀



PROBLEM 5.78

A beam is subjected to a linearly distributed downward load and rests on two wide supports BC and DE, which exert uniformly distributed upward loads as shown. Determine the values of w_{BC} and w_{DE} corresponding to equilibrium when $w_A = 600$ N/m.

SOLUTION



We have $$R_{\text{I}} = \frac{1}{2}(6 \text{ m})(600 \text{ N/m}) = 1800 \text{ N}$$

$$R_{\text{II}} = \frac{1}{2}(6 \text{ m})(1200 \text{ N/m}) = 3600 \text{ N}$$

$$R_{BC} = (0.8 \text{ m})(w_{BC} \text{ N/m}) = (0.8w_{BC}) \text{ N}$$

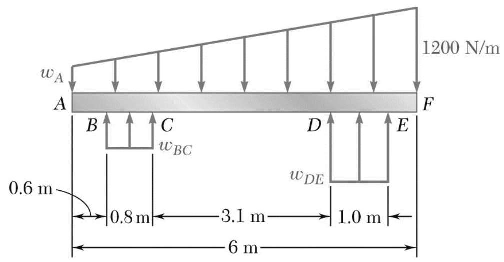
$$R_{DE} = (1.0 \text{ m})(w_{DE} \text{ N/m}) = (w_{DE}) \text{ N}$$

Then $$+\circlearrowleft \Sigma M_G = 0: \quad -(1 \text{ m})(1800 \text{ N}) - (3 \text{ m})(3600 \text{ N}) + (4 \text{ m})(w_{DE} \text{ N}) = 0$$

or $$w_{DE} = 3150 \text{ N/m} \blacktriangleleft$$

and $$+\uparrow \Sigma F_y = 0: \quad (0.8w_{BC}) \text{ N} - 1800 \text{ N} - 3600 \text{ N} + 3150 \text{ N} = 0$$

or $$w_{BC} = 2812.5 \text{ N/m} \qquad w_{BC} = 2810 \text{ N/m} \blacktriangleleft$$

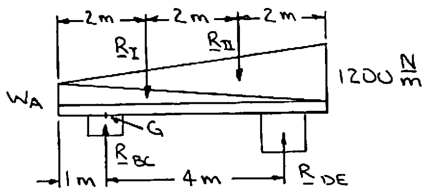


PROBLEM 5.79

A beam is subjected to a linearly distributed downward load and rests on two wide supports *BC* and *DE*, which exert uniformly distributed upward loads as shown. Determine (*a*) the value of w_A so that $w_{BC} = w_{DE}$, (*b*) the corresponding values of w_{BC} and w_{DE}.

SOLUTION

(*a*)



We have
$$R_{\text{I}} = \frac{1}{2}(6\text{ m})(w_A\text{ N/m}) \cdot (3\,w_A)\text{ N}$$

$$R_{\text{II}} = \frac{1}{2}(6\text{ m})(1200\text{ N/m}) = 3600\text{ N}$$

$$R_{BC} = (0.8\text{ m})(w_{BC}\text{ N/m}) = (0.8w_{BC})\text{ N}$$

$$R_{DE} = (1\text{ m})(w_{DE}\text{ N/m}) = (w_{DE})\text{ N}$$

Then
$$+\uparrow\ \Sigma F_y = 0: \quad (0.8w_{BC})\text{ N} - (3w_A)\text{ N} - 3600\text{ N} + (w_{DE})\text{ N} = 0$$

or
$$0.8w_{BC} + w_{DE} = 3600 + 3w_A$$

Now
$$w_{BC} = w_{DE} \Rightarrow w_{BC} = w_{DE} = 2000 + \frac{5}{3}w_A \qquad (1)$$

Also,
$$+\circlearrowleft \Sigma M_G = 0: \quad -(1\text{ m})(3w_A\text{ N}) - (3\text{ m})(3600\text{ N}) + (4\text{ m})(w_{DE}\text{ N}) = 0$$

or
$$w_{DE} = 2700 + \frac{3}{4}w_A \qquad (2)$$

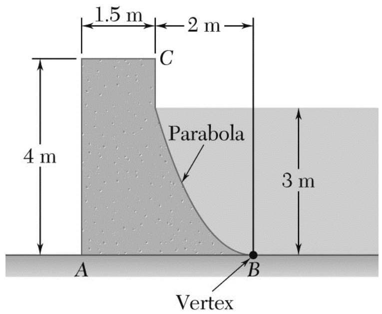
Equating Eqs. (1) and (2),
$$2000 + \frac{5}{3}w_A = 2700 + \frac{3}{4}w_A$$

or
$$w_A = \frac{8400}{11}\text{ N/m}$$

$w_A = 764$ N/m ◀

(*b*) Eq. (1)⇒
$$w_{BC} = w_{DE} = 2000 + \frac{5}{3}\left(\frac{8400}{11}\right)$$

or $w_{BC} = w_{DE} = 3270$ N/m ◀

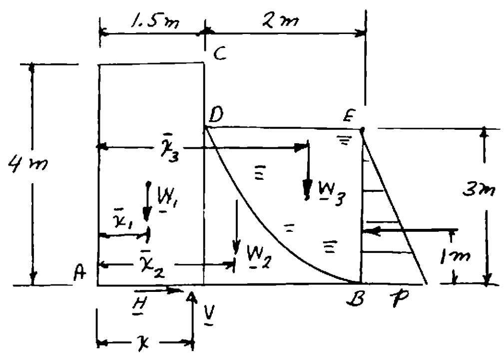


PROBLEM 5.80

The cross section of a concrete dam is as shown. For a 1-m-wide dam section, determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

(*a*) Consider free body made of dam and section *BDE* of water. (Thickness = 1 m)



$$p = (3\text{ m})(10\text{ kg/m}^3)(9.81\text{ m/s}^2)$$

$$W_1 = (1.5\text{ m})(4\text{ m})(1\text{ m})(2.4\times 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 144.26\text{ kN}$$

$$W_2 = \frac{1}{3}(2\text{ m})(3\text{ m})(1\text{ m})(2.4\times 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 47.09\text{ kN}$$

$$W_3 = \frac{2}{3}(2\text{ m})(3\text{ m})(1\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 39.24\text{ kN}$$

$$P = \frac{1}{2}Ap = \frac{1}{2}(3\text{ m})(1\text{ m})(3\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 44.145\text{ kN}$$

$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad H - 44.145\text{ kN} = 0$

$$H = 44.145\text{ kN} \qquad \mathbf{H} = 44.1\text{ kN} \longrightarrow \blacktriangleleft$$

$+\uparrow \Sigma F_y = 0\text{:}\quad V - 141.26 - 47.09 - 39.24 = 0$

$$V = 227.6\text{ kN} \qquad \mathbf{V} = 228\text{ kN} \uparrow \blacktriangleleft$$

PROBLEM 5.80 (Continued)

$$\bar{x}_1 = \frac{1}{2}(1.5\text{ m}) = 0.75\text{ m}$$

$$\bar{x}_2 = 1.5\text{ m} + \frac{1}{4}(2\text{ m}) = 2\text{ m}$$

$$\bar{x}_3 = 1.5\text{ m} + \frac{5}{8}(2\text{ m}) = 2.75\text{ m}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad xV - \Sigma\bar{x}W + P(1\text{ m}) = 0$$

$$x(227.6\text{ kN}) - (141.26\text{ kN})(0.75\text{ m}) - (47.09\text{ kN})(2\text{ m})$$
$$- (39.24\text{ kN})(2.75\text{ m}) + (44.145\text{ kN})(1\text{ m}) = 0$$

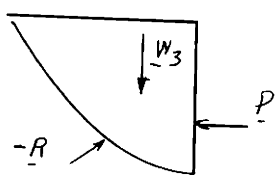
$$x(227.6\text{ kN}) - 105.9 - 94.2 - 107.9 + 44.145 = 0$$

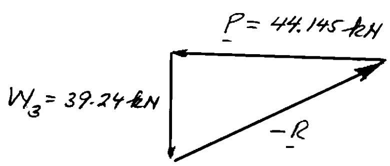
$$x(227.6) - 263.9 = 0$$

$x = 1.159$ m (to right of A) ◀

(*b*) Resultant of face *BC*:

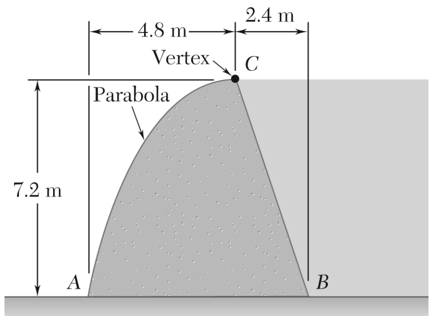
Consider free body of section *BDE* of water.





$-\mathbf{R} = 59.1$ kN ⦣ 41.6°

$\mathbf{R} = 59.1$ kN ⦣ 41.6° ◀

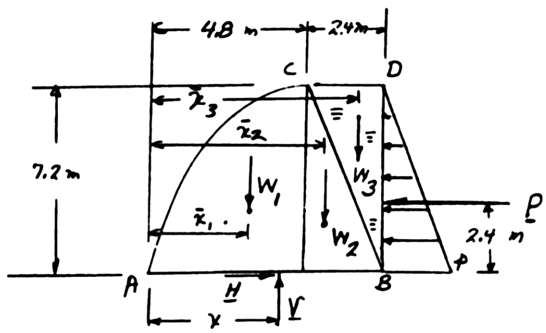


PROBLEM 5.81

The cross section of a concrete dam is as shown. For a 1-m-wide dam section, determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

(*a*) Consider free body made of dam and triangular section of water shown. (Thickness = 1 m.)



$$p = (7.2\,\text{m})(10^3\,\text{kg/m}^3)(9.81\,\text{m/s}^2)$$

$$W_1 = \frac{2}{3}(4.8\text{ m})(7.2\text{ m})(1\text{ m})(2.4\times10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 542.5\text{ kN}$$

$$W_2 = \frac{1}{2}(2.4\text{ m})(7.2\text{ m})(1\text{ m})(2.4\times10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 203.4\text{ kN}$$

$$W_3 = \frac{1}{2}(2.4\text{ m})(7.2\text{ m})(1\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 84.8\text{ kN}$$

$$P = \frac{1}{2}Ap = \frac{1}{2}(7.2\text{ m})(1\text{ m})(7.2\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 254.3\text{ kN}$$

$\xrightarrow{+} \Sigma F_x = 0{:}\quad H - 254.3\text{ kN} = 0$ $\qquad$ $\mathbf{H} = 254\text{ kN} \longrightarrow$ ◀

$+\!\uparrow \Sigma F_y = 0{:}\quad V - 542.5 - 203.4 - 84.8 = 0$

$V = 830.7\text{ kN}$ $\qquad$ $\mathbf{V} = 831\text{ kN}\uparrow$ ◀

PROBLEM 5.81 (Continued)

(*b*)

$$\bar{x}_1 = \frac{5}{8}(4.8 \text{ m}) = 3 \text{ m}$$

$$\bar{x}_2 = 4.8 + \frac{1}{3}(2.4) = 5.6 \text{ m}$$

$$\bar{x}_3 = 4.8 + \frac{2}{3}(2.4) = 6.4 \text{ m}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad xV - \Sigma \bar{x}W + P(2.4 \text{ m}) = 0$$

$$x(830.7 \text{ kN}) - (3 \text{ m})(542.5 \text{ kN}) - (5.6 \text{ m})(203.4 \text{ kN})$$
$$- (6.4 \text{ m})(84.8 \text{ kN}) + (2.4 \text{ m})(254.3 \text{ kN}) = 0$$

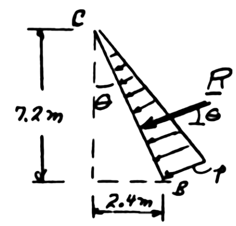
$$x(830.7) - 1627.5 - 1139.0 - 542.7 + 610.3 = 0$$

$$x(830.7) - 2698.9 = 0$$

$x = 3.25$ m (to right of A) ◀

(*c*) Resultant on face BC:

Direct computation:



$$P = \rho g h = (10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(7.2 \text{ m})$$

$$P = 70.63 \text{ kN/m}^2$$

$$BC = \sqrt{(2.4)^2 + (7.2)^2}$$
$$= 7.589 \text{ m}$$

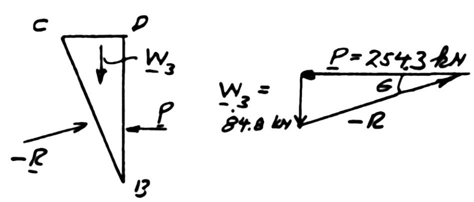
$$\theta = 18.43°$$

$$R = \frac{1}{2} PA$$

$$= \frac{1}{2}(70.63 \text{ kN/m}^2)(7.589 \text{ m})(1 \text{ m})$$

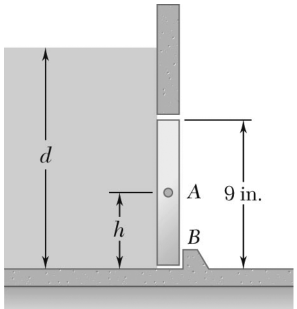
R = 268 kN ⦨ 18.43° ◀

Alternate computation: Use free body of water section BCD.



−**R** = 268 kN ⦨ 18.43°

R = 268 kN ⦨ 18.43° ◀



PROBLEM 5.82

An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through A located at a distance h = 3.6 in. above the lower edge. Determine the depth of water d for which the valve will open.

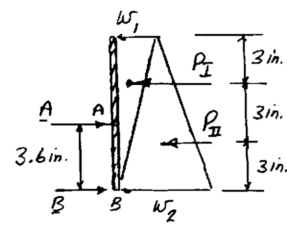
SOLUTION

Since valve is 9 in. wide, $w = 9p = 9\gamma h$, where all dimensions are in inches.

$$w_1 = 9\gamma(d-9), \quad w_2 = 9\gamma d$$

$$P_{\text{I}} = \frac{1}{2}(9\text{ in.})w_1 = \frac{1}{2}(9)(9\gamma)(d-9)$$

$$P_{\text{II}} = \frac{1}{2}(9\text{ in.})w_2 = \frac{1}{2}(9)(9\gamma d)$$



Valve opens when $B = 0$.

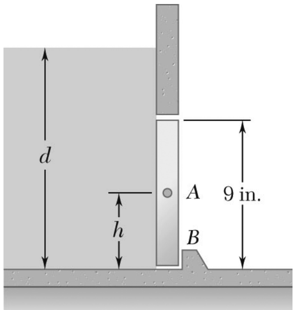
$$+\circlearrowleft \Sigma M_A = 0: \quad P_{\text{I}}(6\text{ in.} - 3.6\text{ in.}) - P_{\text{II}}(3.6\text{ in.} - 3\text{ in.}) = 0$$

$$\left[\frac{1}{2}(9)(9\gamma)(d-9)\right](2.4) - \left[\frac{1}{2}(9)(9\gamma d)\right](0.6) = 0$$

$$(d-9)(2.4) - d(0.6) = 0$$

$$1.8d - 21.6 = 0$$

$d = 12.00$ in. ◀



PROBLEM 5.83

An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through A. If the valve is to open when the depth of water is d = 18 in., determine the distance h from the bottom of the valve to the pivot A.

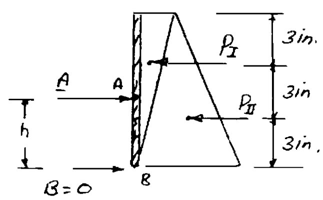
SOLUTION

Since valve is 9 in. wide, $w = 9p = 9\gamma h$, where all dimensions are in inches.

$$w_1 = 9\gamma(d-9)$$
$$w_2 = 9\gamma d$$

For $d = 18$ in.,

$$w_1 = 9\gamma(18-9) = 81\gamma$$
$$w_2 = 9\gamma(18) = 162\gamma$$
$$P_1 = \frac{1}{2}(9)(9\gamma)(18-9) = \frac{1}{2}(729\gamma)$$
$$P_{\text{II}} = \frac{1}{2}(9)(9\gamma)(18) = 729\gamma$$



Valve opens when $B = 0$.

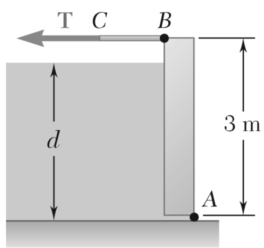
$$+\circlearrowleft \Sigma M_A = 0: \quad P_1(6-h) - P_{\text{II}}(h-3) = 0$$

$$\frac{1}{2}729\gamma(6-h) - 729(h-3) = 0$$

$$3 - \frac{1}{2}h - h + 3 = 0$$

$$6 - 1.5h = 0$$

$h = 4.00$ in. ◀

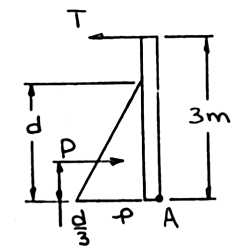


PROBLEM 5.84

The 3×4-m side AB of a tank is hinged at its bottom A and is held in place by a thin rod BC. The maximum tensile force the rod can withstand without breaking is 200 kN, and the design specifications require the force in the rod not to exceed 20 percent of this value. If the tank is slowly filled with water, determine the maximum allowable depth of water d in the tank.

SOLUTION

Consider the free-body diagram of the side.



We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$$

Now $$+\circlearrowleft \Sigma M_A = 0: \quad hT - \frac{d}{3}P = 0$$

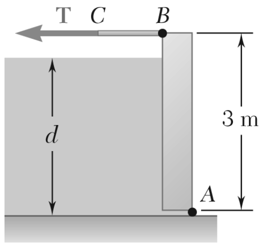
where $$h = 3 \text{ m}$$

Then for d_{max},

$$(3 \text{ m})(0.2 \times 200 \times 10^3 \text{N}) - \frac{d_{max}}{3}\left[\frac{1}{2}(4 \text{ m} \times d_{max}) \times (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2 \times d_{max})\right] = 0$$

or $$120 \text{ N}\cdot\text{m} - 6.54 d_{max}^3 \text{N/m}^2 = 0$$

or $$d_{max} = 2.64 \text{ m} \blacktriangleleft$$

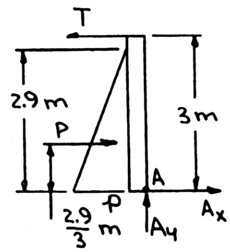


PROBLEM 5.85

The 3 × 4-m side of an open tank is hinged at its bottom A and is held in place by a thin rod BC. The tank is to be filled with glycerine, whose density is 1263 kg/m^3. Determine the force **T** in the rod and the reactions at the hinge after the tank is filled to a depth of 2.9 m.

SOLUTION

Consider the free-body diagram of the side.



We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$$

$$= \frac{1}{2}[(2.9\text{ m})(4\text{ m})]\,[(1263\text{ kg/m}^3)(9.81\text{ m/s}^2)(2.9\text{ m})]$$

$$= 208.40\text{ kN}$$

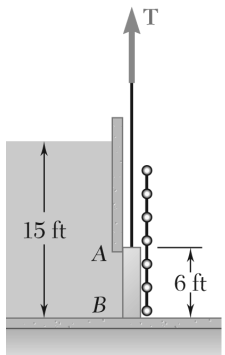
Then $$+\uparrow \Sigma F_y = 0: \quad A_y = 0$$

$$+\circlearrowleft \Sigma M_A = 0: \quad (3\text{ m})T - \left(\frac{2.9}{3}\text{ m}\right)(208.4\text{ kN}) = 0$$

or $$T = 67.151\text{ kN}$$ $\mathbf{T} = 67.2\text{ kN} \longleftarrow$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 208.40\text{ kN} - 67.151\text{ kN} = 0$$

or $$A_x = -141.249\text{ kN}$$ $\mathbf{A} = 141.2\text{ kN} \longleftarrow$ ◀

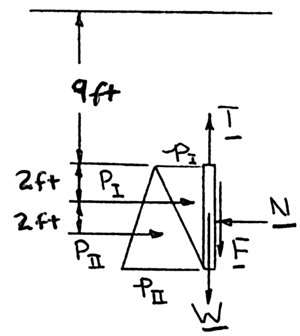


PROBLEM 5.86

The friction force between a 6 × 6-ft square sluice gate *AB* and its guides is equal to 10 percent of the resultant of the pressure forces exerted by the water on the face of the gate. Determine the initial force needed to lift the gate if it weighs 1000 lb.

SOLUTION

Consider the free-body diagram of the gate.



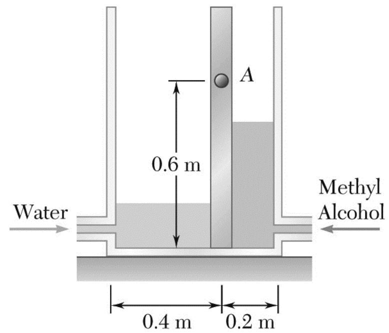
Now $$P_I = \frac{1}{2}Ap_I = \frac{1}{2}[(6\times 6)\text{ ft}^2][(62.4\text{ lb/ft}^3)(9\text{ ft})]$$
$$= 10{,}108.8\text{ lb}$$

$$P_{II} = \frac{1}{2}Ap_{II} = \frac{1}{2}[(6\times 6)\text{ ft}^2][(62.4\text{ lb/ft}^3)(15\text{ ft})]$$
$$= 16{,}848\text{ lb}$$

Then $$F = 0.1P = 0.1(P_I + P_{II})$$
$$= 0.1(10{,}108.8 + 16{,}848)\text{ lb}$$
$$= 2695.7\text{ lb}$$

Finally $+\uparrow \Sigma F_y = 0{:}\quad T - 2695.7\text{ lb} - 1000\text{ lb} = 0$ or $\mathbf{T} = 3.70\text{ kips}\uparrow$ ◀

PROBLEM 5.87

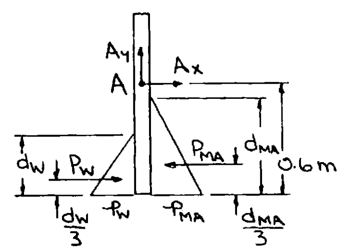


A tank is divided into two sections by a 1×1-m square gate that is hinged at A. A couple of magnitude 490 N · m is required for the gate to rotate. If one side of the tank is filled with water at the rate of 0.1 m^3/min and the other side is filled simultaneously with methyl alcohol (density $\rho_{ma} = 789$ kg/m^3) at the rate of 0.2 m^3/min, determine at what time and in which direction the gate will rotate.

SOLUTION

Consider the free-body diagram of the gate.

First note $V = A_{\text{base}}d$ and $V = rt$.



Then
$$d_W = \frac{0.1 \text{ m}^3/\text{min} \times t(\text{min})}{(0.4 \text{ m})(1 \text{ m})} = 0.25t(\text{m})$$

$$d_{MA} = \frac{0.2 \text{ m}^3/\text{min} \times t(\text{min})}{(0.2 \text{ m})(1 \text{ m})} = t(\text{m})$$

Now
$$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho gh) \text{ so that}$$

$$P_W = \frac{1}{2}[(0.25t) \text{ m} \times (1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.25t) \text{ m}]$$
$$= 306.56t^2 \text{ N}$$

$$P_{MA} = \frac{1}{2}[(t) \text{ m} \times (1 \text{ m})][(789 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(t) \text{ m}]$$
$$= 3870t^2 \text{ N}$$

Now assume that the gate will rotate clockwise and when $d_{MA} \le 0.6$ m. When rotation of the gate is impending, we require

$$\Sigma M_A: \quad M_R = \left(0.6 \text{ m} - \frac{1}{3}d_{MA}\right)P_{MA} - \left(0.6 \text{ m} - \frac{1}{3}d_W\right)P_W$$

Substituting

$$490 \text{ N} \cdot \text{m} = \left(0.6 - \frac{1}{3}t\right) \text{m} \times (3870t^2) \text{ N} - \left(0.6 - \frac{1}{3} \times 0.25t\right) \text{m} \times (306.56t^2) \text{ N}$$

PROBLEM 5.87 (Continued)

Simplifying $$1264.45t^3 - 2138.1t^2 + 490 = 0$$

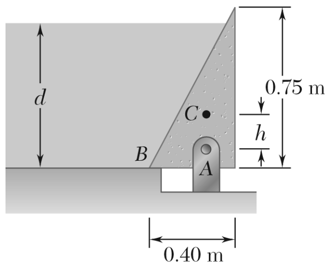
Solving (positive roots only)

$$t = 0.59451 \text{ min} \quad \text{and} \quad t = 1.52411 \text{ min}$$

Now check assumption using the smaller root. We have

$$d_{MA} = (t) \text{ m} = 0.59451 \text{ m} < 0.6 \text{ m} \qquad \therefore \quad t = 0.59451 \text{min} = 35.7 \text{ s} \blacktriangleleft$$

and the gate rotates clockwise. ◀



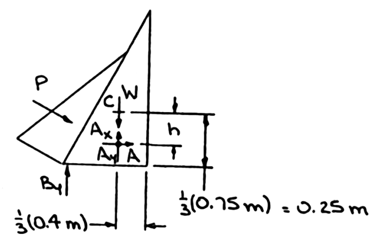
PROBLEM 5.88

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. The pin is located at a distance $h = 0.10$ m below the center of gravity C of the gate. Determine the depth of water d for which the gate will open.

SOLUTION

First note that when the gate is about to open (clockwise rotation is impending), $B_y \longrightarrow 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at A. In addition, if it is assumed that the gate is homogeneous, then its center of gravity C coincides with the centroid of the triangular area. Then

$$a = \frac{d}{3} - (0.25 - h)$$

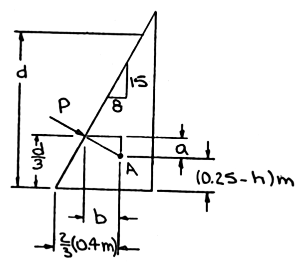


and

$$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now

$$\frac{a}{b} = \frac{8}{15}$$



so that

$$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \tag{1}$$

Alternative solution:

Consider a free body consisting of a 1-m thick section of the gate and the triangular section BDE of water above the gate.

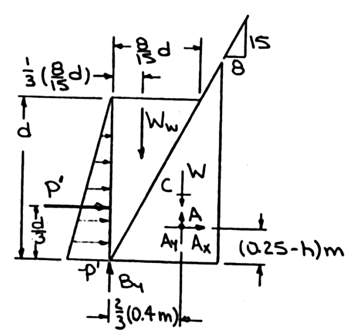
Now

$$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$

$$= \frac{1}{2}\rho g d^2 \text{ (N)}$$

$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$

$$= \frac{4}{15}\rho g d^2 \text{ (N)}$$



PROBLEM 5.88 (Continued)

Then with $B_y = 0$ (as explained above), we have

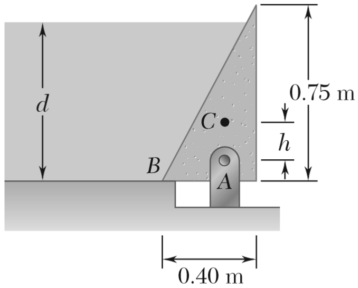
$$+\circlearrowleft \Sigma M_A = 0: \quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $$\frac{289}{45}d + 15h = \frac{70.6}{12}$$

as above.

Find d: $$h = 0.10 \text{ m}$$

Substituting into Eq. (1), $$\frac{289}{45}d + 15(0.10) = \frac{70.6}{12}$$ or $d = 0.683$ m ◀



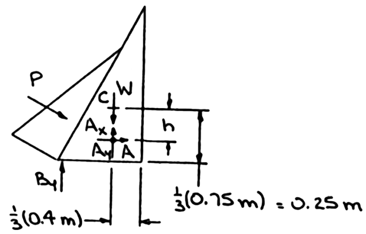
PROBLEM 5.89

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. Determine the distance h if the gate is to open when $d = 0.75$ m.

SOLUTION

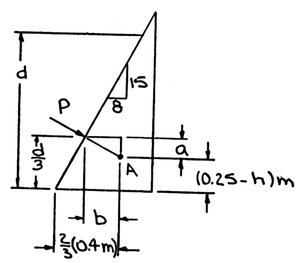
First note that when the gate is about to open (clockwise rotation is impending), $B_y \longrightarrow 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at A. In addition, if it is assumed that the gate is homogeneous, then its center of gravity C coincides with the centroid of the triangular area. Then

$$a = \frac{d}{3} - (0.25 - h)$$



and
$$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now
$$\frac{a}{b} = \frac{8}{15}$$



so that
$$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

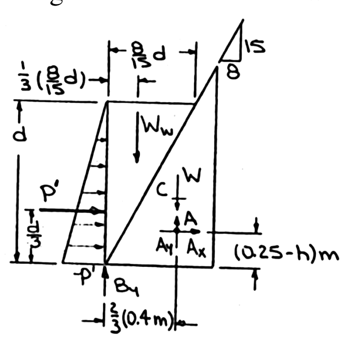
Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \tag{1}$$

Alternative solution:

Consider a free body consisting of a 1-m thick section of the gate and the triangular section BDE of water above the gate.

Now
$$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$
$$= \frac{1}{2}\rho g d^2 \text{ (N)}$$
$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$
$$= \frac{4}{15}\rho g d^2 \text{ (N)}$$



PROBLEM 5.89 (Continued)

Then with $B_y = 0$ (as explained above), we have

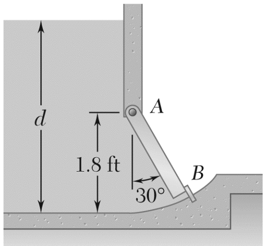
$$+\circlearrowleft \Sigma M_A = 0: \quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $$\frac{289}{45}d + 15h = \frac{70.6}{12}$$

as above.

Find h: $$d = 0.75 \text{ m}$$

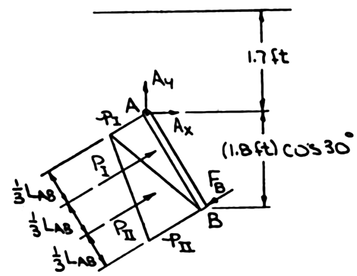
Substituting into Eq. (1), $$\frac{289}{45}(0.75) + 15h = \frac{70.6}{12}$$ or $h = 0.0711 \text{ m}$ ◀



PROBLEM 5.90

The square gate AB is held in the position shown by hinges along its top edge A and by a shear pin at B. For a depth of water $d = 3.5\,\text{ft}$, determine the force exerted on the gate by the shear pin.

SOLUTION



First consider the force of the water on the gate. We have

$$P = \frac{1}{2}Ap$$

$$= \frac{1}{2}A(\gamma h)$$

Then

$$P_{\mathrm{I}} = \frac{1}{2}(1.8\text{ ft})^2(62.4\text{ lb/ft}^3)(1.7\text{ ft})$$

$$= 171.850\text{ lb}$$

$$P_{\mathrm{II}} = \frac{1}{2}(1.8\text{ ft})^2(62.4\text{ lb/ft}^3)\times(1.7 + 1.8\cos 30°)\text{ ft}$$

$$= 329.43\text{ lb}$$

Now

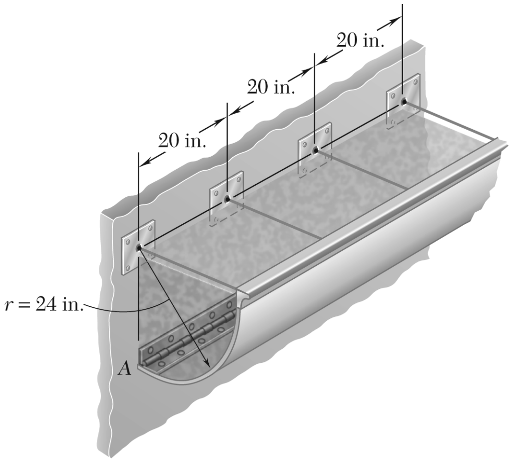
$$\Sigma M_A = 0\colon\quad \left(\frac{1}{3}L_{AB}\right)P_{\mathrm{I}} + \left(\frac{2}{3}L_{AB}\right)P_{\mathrm{II}} - L_{AB}F_B = 0$$

or

$$\frac{1}{3}(171.850\text{ lb}) + \frac{2}{3}(329.43\text{ lb}) - F_B = 0$$

or

$$F_B = 276.90\text{ lb} \qquad \mathbf{F}_B = 277\text{ lb} \measuredangle\ 30.0° \blacktriangleleft$$



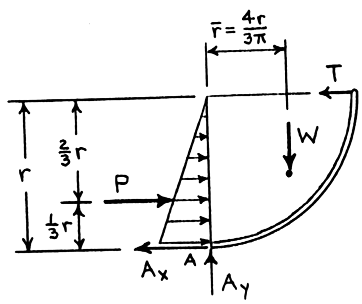
PROBLEM 5.91

A long trough is supported by a continuous hinge along its lower edge and by a series of horizontal cables attached to its upper edge. Determine the tension in each of the cables, at a time when the trough is completely full of water.

SOLUTION

Consider free body consisting of 20-in. length of the trough and water.

$$l = 20\text{-in. length of free body}$$



$$W = \gamma v = \gamma\left[\frac{\pi}{4}r^2 l\right]$$

$$P_A = \gamma r$$

$$P = \frac{1}{2}P_A r l = \frac{1}{2}(\gamma r) r l = \frac{1}{2}\gamma r^2 l$$

$$+\circlearrowleft \Sigma M_A = 0: \quad Tr - Wr - P\left(\frac{1}{3}r\right) = 0$$

$$Tr - \left(\gamma\frac{\pi}{4}r^2 l\right)\left(\frac{4r}{3\pi}\right) - \left(\frac{1}{2}\gamma r^2 l\right)\left(\frac{1}{3}r\right) = 0$$

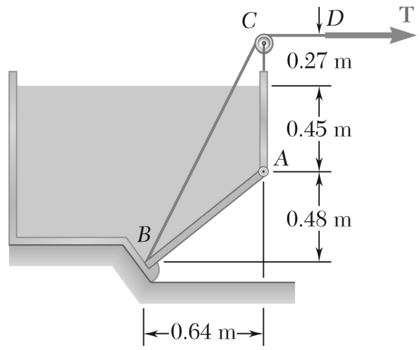
$$T = \frac{1}{3}\gamma r^2 l + \frac{1}{6}\gamma r^2 l = \frac{1}{2}\gamma r^2 l$$

Data: $\gamma = 62.4 \text{ lb/ft}^3 \quad r = \frac{24}{12}\text{ ft} = 2 \text{ ft} \quad l = \frac{20}{12}\text{ ft}$

Then $T = \frac{1}{2}(62.4 \text{ lb/ft}^3)(2 \text{ ft})^2\left(\frac{20}{12}\text{ ft}\right)$

$= 208.00 \text{ lb}$

$T = 208 \text{ lb}$ ◀

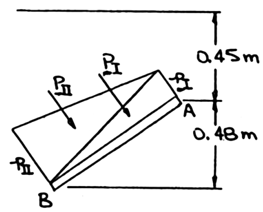


PROBLEM 5.92

A 0.5 × 0.8-m gate AB is located at the bottom of a tank filled with water. The gate is hinged along its top edge A and rests on a frictionless stop at B. Determine the reactions at A and B when cable BCD is slack.

SOLUTION

First consider the force of the water on the gate.



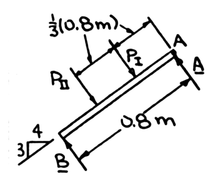
We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho gh)$$

so that $$P_{\mathrm{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$$
$$= 882.9\text{ N}$$

$$P_{\mathrm{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$$
$$= 1824.66\text{ N}$$

Reactions at A and B when $T = 0$:

We have



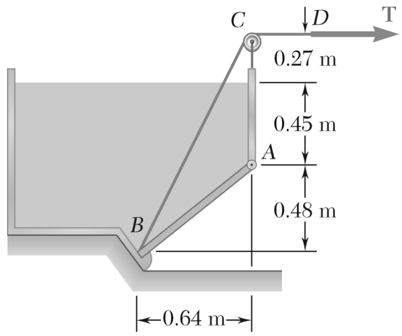
$$+\circlearrowleft \Sigma M_A = 0: \quad \frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N}) - (0.8\text{ m})B = 0$$

or $$B = 1510.74\text{ N}$$

or $$\mathbf{B} = 1511\text{ N} \measuredangle 53.1° \blacktriangleleft$$

$$+\nwarrow \Sigma F = 0: \quad A + 1510.74\text{ N} - 882.9\text{ N} - 1824.66\text{ N} = 0$$

or $$\mathbf{A} = 1197\text{ N} \measuredangle 53.1° \blacktriangleleft$$

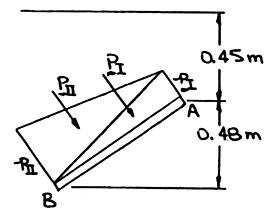


PROBLEM 5.93

A 0.5 × 0.8-m gate *AB* is located at the bottom of a tank filled with water. The gate is hinged along its top edge *A* and rests on a frictionless stop at *B*. Determine the minimum tension required in cable *BCD* to open the gate.

SOLUTION

First consider the force of the water on the gate.



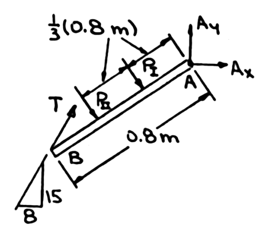
We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g h)$$

so that $$P_{\mathrm{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})]\times[(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$$
$$= 882.9\text{ N}$$

$$P_{\mathrm{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})]\times[(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$$
$$= 1824.66\text{ N}$$

<u>*T* to open gate</u>:

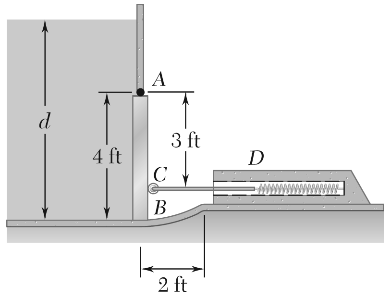
First note that when the gate begins to open, the reaction at $B \longrightarrow 0$.



Then $$+\circlearrowleft \Sigma M_A = 0:\quad \frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N})$$
$$-(0.45+0.27)\text{m}\times\left(\frac{8}{17}T\right) = 0$$

or $$235.44 + 973.152 - 0.33882T = 0$$

or $\mathbf{T} = 3570\text{ N}$ ◀

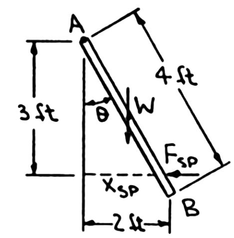


PROBLEM 5.94

A 4×2-ft gate is hinged at A and is held in position by rod CD. End D rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod CD on the gate remains horizontal, determine the minimum depth of water d for which the bottom B of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when B is at the end of the cylindrical portion of the floor.

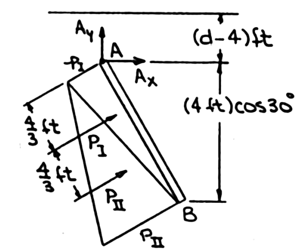


We have $\sin\theta = \dfrac{2}{4} \quad \theta = 30°$

Then $x_{SP} = (3\text{ ft})\tan 30°$

and
$$F_{SP} = kx_{SP}$$
$$= 828 \text{ lb/ft} \times 3 \text{ ft} \times \tan 30°$$
$$= 1434.14 \text{ lb}$$

Assume $d \geq 4$ ft



We have $P = \dfrac{1}{2}Ap = \dfrac{1}{2}A(\gamma h)$

Then
$$P_{\text{I}} = \frac{1}{2}[(4\text{ ft})(2\text{ ft})] \times [(62.4\text{ lb/ft}^3)(d-4)\text{ ft}]$$
$$= 249.6(d-4)\text{ lb}$$
$$P_{\text{II}} = \frac{1}{2}[(4\text{ ft})(2\text{ ft})] \times [(62.4\text{ lb/ft}^3)(d - 4 + 4\cos 30°)]$$
$$= 249.6(d - 0.53590°)\text{ lb}$$

For $d_{\min}$ so that the gate opens, $W = 0$

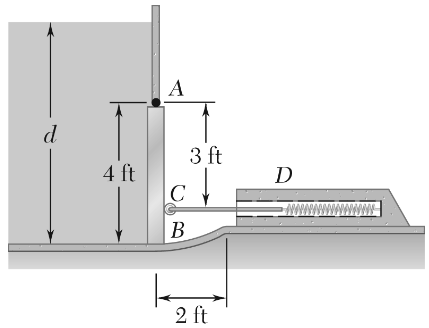
Using the above free-body diagrams of the gate, we have

$$+\circlearrowleft \Sigma M_A = 0: \left(\frac{4}{3}\text{ ft}\right)[249.6(d-4)\text{ lb}] + \left(\frac{8}{3}\text{ ft}\right)[249.6(d - 0.53590)\text{ lb}]$$
$$-(3\text{ ft})(1434.14\text{ lb}) = 0$$

or $(332.8d - 1331.2) + (665.6d - 356.70) - 4302.4 = 0$

or $d = 6.00$ ft

$d \geq 4 \text{ ft} \Rightarrow$ assumption correct $\quad d = 6.00$ ft ◀



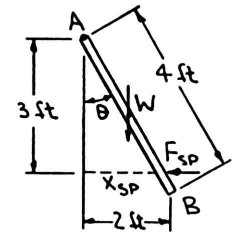
PROBLEM 5.95

Solve Problem 5.94 if the gate weighs 1000 lb.

PROBLEM 5.94 A 4×2-ft gate is hinged at A and is held in position by rod CD. End D rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod CD on the gate remains horizontal, determine the minimum depth of water d for which the bottom B of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when B is at the end of the cylindrical portion of the floor.



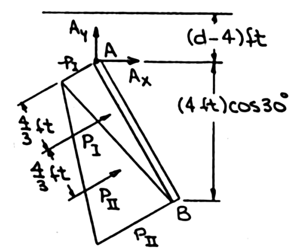
We have $\sin\theta = \dfrac{2}{4} \quad \theta = 30°$

Then $x_{SP} = (3\text{ ft})\tan 30°$

and $F_{SP} = kx_{SP} = 828\text{ lb/ft}\times 3\text{ ft}\times\tan 30°$

$= 1434.14\text{ lb}$

Assume $d \geq 4\text{ ft}$



We have $P = \dfrac{1}{2}Ap = \dfrac{1}{2}A(\gamma h)$

Then $P_{\text{I}} = \dfrac{1}{2}[(4\text{ ft})(2\text{ ft})]\times[(62.4\text{ lb/ft}^3)(d-4)\text{ ft}]$

$= 249.6(d-4)\text{ lb}$

$P_{\text{II}} = \dfrac{1}{2}[(4\text{ ft})(2\text{ ft})]\times[(62.4\text{ lb/ft}^3)(d-4+4\cos 30°)]$

$= 249.6(d-0.53590°)\text{ lb}$

For $d_{\min}$ so that the gate opens, $W = 1000\text{ lb}$

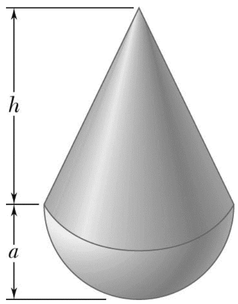
Using the above free-body diagrams of the gate, we have

$$+\circlearrowleft \Sigma M_A = 0:\quad \left(\frac{4}{3}\text{ ft}\right)[249.6(d-4)\text{ lb}] + \left(\frac{8}{3}\text{ ft}\right)[249.6(d-0.53590)\text{ lb}]$$
$$-(3\text{ ft})(1434.14\text{ lb}) - (1\text{ ft})(1000\text{ lb}) = 0$$

or $(332.8d - 1331.2) + (665.6d - 356.70) - 4302.4 - 1000 = 0$

or $d = 7.00\text{ ft}$

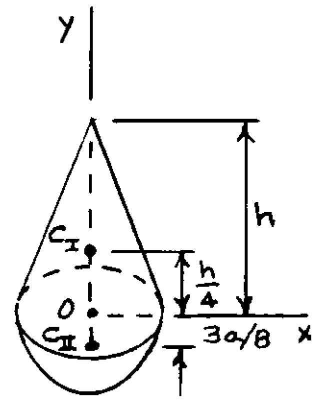
$d \geq 4\text{ ft} \Rightarrow$ assumption correct $\qquad d = 7.00\text{ ft}$ ◀



PROBLEM 5.96

A hemisphere and a cone are attached as shown. Determine the location of the centroid of the composite body when (*a*) $h = 1.5a$, (*b*) $h = 2a$.

SOLUTION



	V	$\bar{y}$	$\bar{y}V$
Cone I	$\frac{1}{3}\pi a^2 h$	$\frac{h}{4}$	$\frac{1}{12}\pi a^2 h^2$
Hemisphere II	$\frac{2}{3}\pi a^3$	$-\frac{3a}{8}$	$-\frac{1}{4}\pi a^4$

$$V = \frac{1}{3}\pi a^2(h + 2a)$$

$$\Sigma \bar{y}V = \frac{1}{12}\pi a^2(h^2 - 3a^2)$$

(*a*) For $h = 1.5a$,

$$V = \frac{1}{3}\pi a^2(1.5a + 2a) = \frac{7}{6}\pi a^2$$

$$\Sigma \bar{y}V = \frac{1}{12}\pi a^2[(1.5a)^2 - 3a^2] = -\frac{1}{16}\pi a^4$$

$$\bar{Y}V = \Sigma \bar{y}V: \quad \bar{Y}\left(\frac{7}{6}\pi a^3\right) = -\frac{1}{16}\pi a^4 \quad \bar{Y} = -\frac{3}{56}a$$

Centroid is 0.0536*a* below base of cone. ◀

PROBLEM 5.96 (Continued)

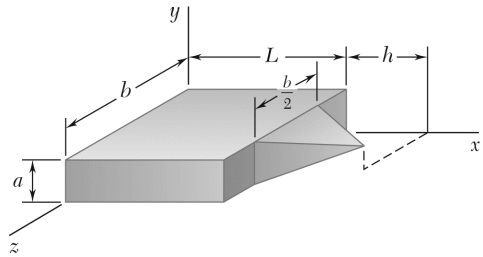
(*b*) For $h = 2a$,

$$V = \frac{1}{3}\pi a^2(2a + 2a) = \frac{4}{3}\pi a^3$$

$$\Sigma \bar{y} V = \frac{1}{12}\pi a^2[(2a)^2 - 3a^2] = \frac{1}{12}\pi a^4$$

$$\bar{Y} V = \Sigma \bar{y} V: \quad \bar{Y}\left(\frac{4}{3}\pi a^3\right) = \frac{1}{12}\pi a^4 \quad \bar{Y} = \frac{1}{16}a$$

Centroid is $0.0625a$ above base of cone. ◀



PROBLEM 5.97

Consider the composite body shown. Determine (*a*) the value of $\bar{x}$ when $h = L/2$, (*b*) the ratio h/L for which $\bar{x} = L$.

SOLUTION

	V	$\bar{x}$	$\bar{x}V$
Rectangular prism	Lab	$\frac{1}{2}L$	$\frac{1}{2}L^2ab$
Pyramid	$\frac{1}{3}a\left(\frac{b}{2}\right)h$	$L+\frac{1}{4}h$	$\frac{1}{6}abh\left(L+\frac{1}{4}h\right)$

Then $$\Sigma V = ab\left(L+\frac{1}{6}h\right)$$

$$\Sigma\bar{x}V = \frac{1}{6}ab\left[3L^2 + h\left(L+\frac{1}{4}h\right)\right]$$

Now $$\bar{X}\Sigma V = \Sigma\bar{x}V$$

so that $$\bar{X}\left[ab\left(L+\frac{1}{6}h\right)\right] = \frac{1}{6}ab\left(3L^2 + hL + \frac{1}{4}h^2\right)$$

or $$\bar{X}\left(1+\frac{1}{6}\frac{h}{L}\right) = \frac{1}{6}L\left(3+\frac{h}{L}+\frac{1}{4}\frac{h^2}{L^2}\right) \qquad (1)$$

(*a*) $\bar{X} = ?$ when $h = \frac{1}{2}L.$

Substituting $\frac{h}{L} = \frac{1}{2}$ into Eq. (1),

$$\bar{X}\left[1+\frac{1}{6}\left(\frac{1}{2}\right)\right] = \frac{1}{6}L\left[3+\left(\frac{1}{2}\right)+\frac{1}{4}\left(\frac{1}{2}\right)^2\right]$$

or $$\bar{X} = \frac{57}{104}L$$

$\bar{X} = 0.548L$ ◀

PROBLEM 5.97 (Continued)

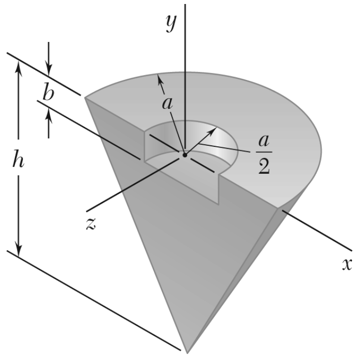
(*b*) $\frac{h}{L} = ?$ when $\bar{X} = L$.

Substituting into Eq. (1),
$$L\left(1+\frac{1}{6}\frac{h}{L}\right)=\frac{1}{6}L\left(3+\frac{h}{L}+\frac{1}{4}\frac{h^2}{L^2}\right)$$

or
$$1+\frac{1}{6}\frac{h}{L}=\frac{1}{2}+\frac{1}{6}\frac{h}{L}+\frac{1}{24}\frac{h^2}{L^2}$$

or
$$\frac{h^2}{L^2}=12$$

$$\frac{h}{L}=2\sqrt{3} \blacktriangleleft$$

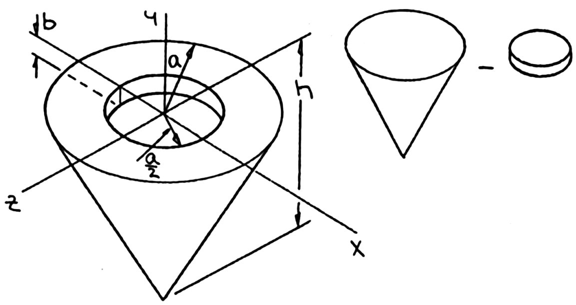


PROBLEM 5.98

Determine the y coordinate of the centroid of the body shown.

SOLUTION

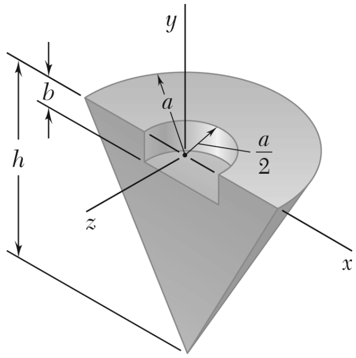
First note that the values of $\overline{Y}$ will be the same for the given body and the body shown below. Then



	V	$\overline{y}$	$\overline{y}V$
Cone	$\frac{1}{3}\pi a^2 h$	$-\frac{1}{4}h$	$-\frac{1}{12}\pi a^2 h^2$
Cylinder	$-\pi\left(\frac{a}{2}\right)^2 b = -\frac{1}{4}\pi a^2 b$	$-\frac{1}{2}b$	$\frac{1}{8}\pi a^2 b^2$
Σ	$\frac{\pi}{12}a^2(4h-3b)$		$-\frac{\pi}{24}a^2(2h^2-3b^2)$

We have $$\overline{Y}\Sigma V = \Sigma \overline{y}V$$

Then $$\overline{Y}\left[\frac{\pi}{12}a^2(4h-3b)\right] = -\frac{\pi}{24}a^2(2h^2-3b^2)$$ or $$\overline{Y} = -\frac{2h^2-3b^2}{2(4h-3b)} \blacktriangleleft$$

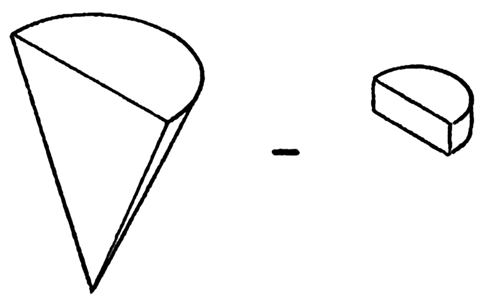


PROBLEM 5.99

Determine the z coordinate of the centroid of the body shown. (*Hint:* Use the result of Sample Problem 5.13.)

SOLUTION

First note that the body can be formed by removing a half cylinder from a half cone, as shown.



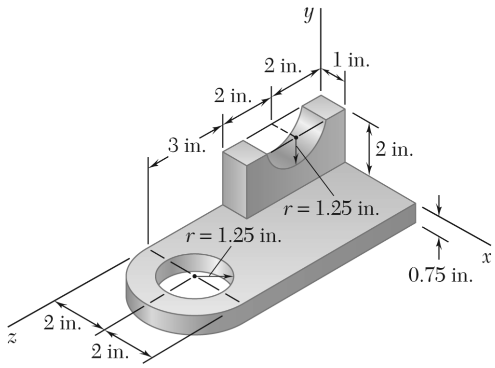
	V	$\bar{z}$	$\bar{z}V$
Half cone	$\frac{1}{6}\pi a^2 h$	$-\frac{a}{\pi}$	$-\frac{1}{6}a^3 h$
Half cylinder	$-\frac{\pi}{2}\left(\frac{a}{2}\right)^2 b = -\frac{\pi}{8}a^2 b$	$-\frac{4}{3\pi}\left(\frac{a}{2}\right) = -\frac{2a}{3\pi}$	$\frac{1}{12}a^3 b$
Σ	$\frac{\pi}{24}a^2(4h-3b)$		$-\frac{1}{12}a^3(2h-b)$

From Sample Problem 5.13:

We have $$\bar{Z}\Sigma V = \Sigma\bar{z}V$$

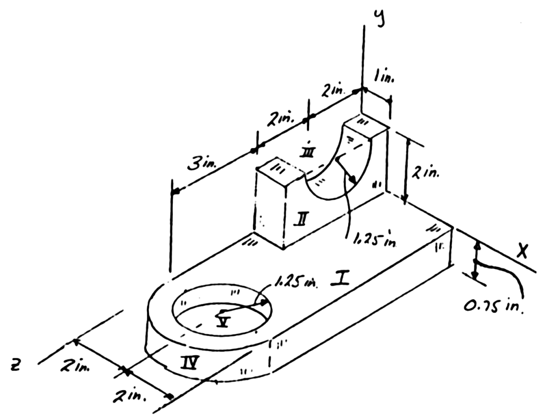
Then $$\bar{Z}\left[\frac{\pi}{24}a^2(4h-3b)\right] = -\frac{1}{12}a^3(2h-b)$$ or $$\bar{Z} = -\frac{a}{\pi}\left(\frac{4h-2b}{4h-3b}\right) \blacktriangleleft$$

PROBLEM 5.100



For the machine element shown, locate the y coordinate of the center of gravity.

SOLUTION



For half-cylindrical hole,

$$r = 1.25 \text{ in.}$$

$$\bar{y}_{\text{III}} = 2 - \frac{4(1.25)}{3\pi}$$

$$= 1.470 \text{ in.}$$

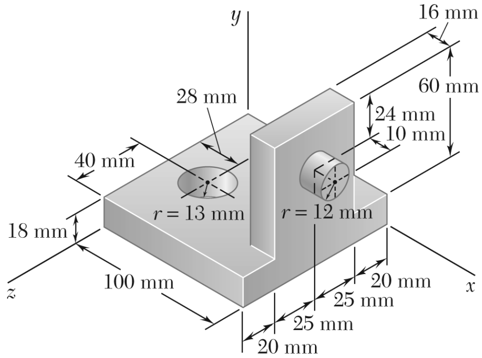
For half-cylindrical plate, $r = 2$ in.

$$\bar{z}_{\text{IV}} = 7 + \frac{4(2)}{3\pi} = 7.85 \text{ in.}$$

		V, in^3	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	Rectangular plate	$(7)(4)(0.75) = 21.0$	–0.375	3.5	–7.875	73.50
II	Rectangular plate	$(4)(2)(1) = 8.0$	1.0	2	8.000	16.00
III	–(Half cylinder)	$-\frac{\pi}{2}(1.25)^2(1) = 2.454$	1.470	2	–3.607	–4.908
IV	Half cylinder	$\frac{\pi}{2}(2)^2(0.75) = 4.712$	–0.375	–7.85	–1.767	36.99
V	–(Cylinder)	$-\pi(1.25)^2(0.75) = -3.682$	–0.375	7	1.381	–25.77
	Σ	27.58			–3.868	95.81

$$\bar{Y}\Sigma V = \Sigma \bar{y} V$$

$$\bar{Y}(27.58 \text{ in}^3) = -3.868 \text{ in}^4 \qquad \bar{Y} = -0.1403 \text{ in.} \blacktriangleleft$$

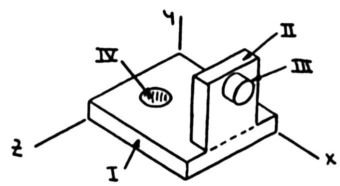


PROBLEM 5.101

For the machine element shown, locate the y coordinate of the center of gravity.

SOLUTION

First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.

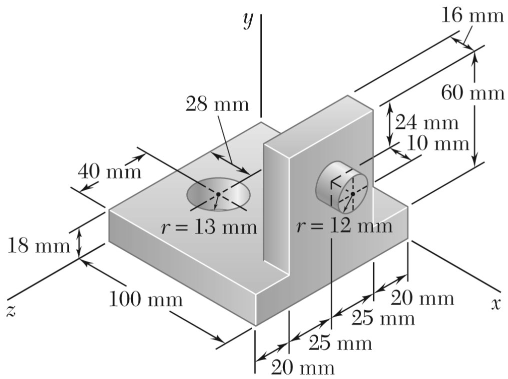


	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(100)(18)(90) = 162{,}000$	50	9	8,100,000	1,458,000
II	$(16)(60)(50) = 48{,}000$	92	48	4,416,000	2,304,000
III	$\pi(12)^2(10) = 4523.9$	105	54	475,010	244,290
IV	$-\pi(13)^2(18) = -9556.7$	28	9	–267,590	–86,010
Σ	204,967.2			12,723,420	3,920,280

We have $$\bar{Y}\Sigma V = \Sigma \bar{y}V$$

$$\bar{Y}(204{,}967.2 \text{ mm}^3) = 3{,}920{,}280 \text{ mm}^4$$

or $\bar{Y} = 19.13$ mm ◀

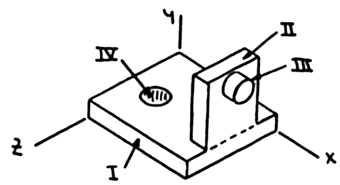


PROBLEM 5.102

For the machine element shown, locate the x coordinate of the center of gravity.

SOLUTION

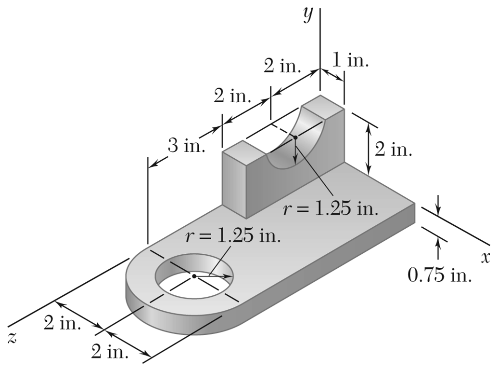
First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.



	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(100)(18)(90) = 162,000$	50	9	8,100,000	1,458,000
II	$(16)(60)(50) = 48,000$	92	48	4,416,000	2,304,000
III	$\pi(12)^2(10) = 4523.9$	105	54	475,010	244,290
IV	$-\pi(13)^2(18) = -9556.7$	28	9	–267,590	–86,010
Σ	204,967.2			12,723,420	3,920,280

We have $$\bar{X}\Sigma V = \Sigma \bar{x}V$$

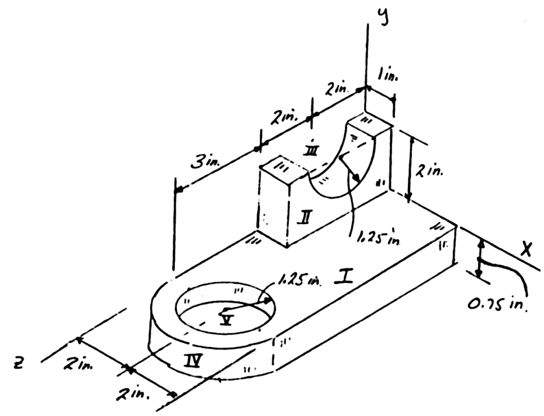
$$\bar{X}(204,967.2 \text{ mm}^3) = 12,723,420 \text{ mm}^4$$ $\bar{X} = 62.1$ mm ◀



PROBLEM 5.103

For the machine element shown, locate the z coordinate of the center of gravity.

SOLUTION



For half-cylindrical hole,

$$r = 1.25 \text{ in.}$$

$$\bar{y}_{\text{III}} = 2 - \frac{4(1.25)}{3\pi}$$

$$= 1.470 \text{ in.}$$

For half-cylindrical plate, $r = 2$ in.

$$\bar{z}_{\text{IV}} = 7 + \frac{4(2)}{3\pi} = 7.85 \text{ in.}$$

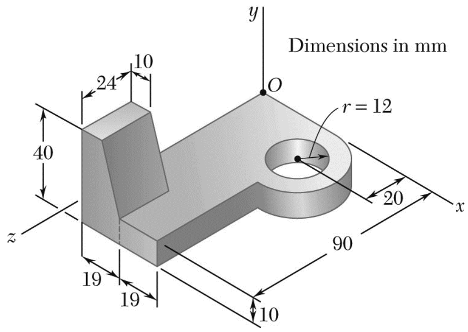
		V, in^3	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	Rectangular plate	$(7)(4)(0.75) = 21.0$	–0.375	3.5	–7.875	73.50
II	Rectangular plate	$(4)(2)(1) = 8.0$	1.0	2	8.000	16.00
III	–(Half cylinder)	$-\frac{\pi}{2}(1.25)^2(1) = 2.454$	1.470	2	–3.607	–4.908
IV	Half cylinder	$\frac{\pi}{2}(2)^2(0.75) = 4.712$	–0.375	–7.85	–1.767	36.99
V	–(Cylinder)	$-\pi(1.25)^2(0.75) = -3.682$	–0.375	7	1.381	–25.77
	Σ	27.58			–3.868	95.81

Now

$$\bar{Z}\Sigma V = \bar{z}V$$

$$\bar{Z}(27.58 \text{ in}^3) = 95.81 \text{ in}^4$$

$\bar{Z} = 3.47$ in. ◀

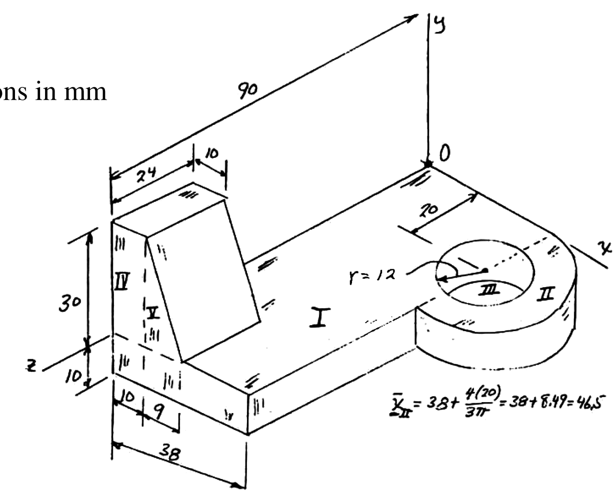


PROBLEM 5.104

For the machine element shown, locate the x coordinate of the center of gravity.

SOLUTION

Dimensions in mm



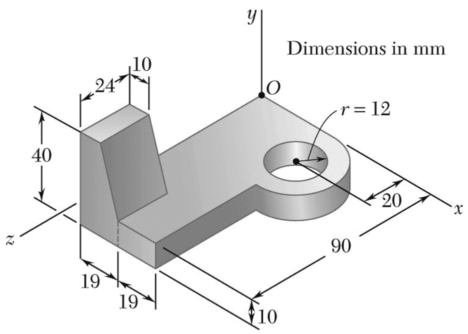
		V, mm^3	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(10)(90)(38) = 34.2\times10^3$	19	45	649.8×10^3	1539×10^3
II	Half cylinder	$\frac{\pi}{2}(20)^2(10) = 6.2832\times10^3$	46.5	20	292.17×10^3	125.664×10^3
III	–(Cylinder)	$-\pi(12)^2(10) = -4.5239\times10^3$	38	20	-171.908×10^3	-90.478×10^3
IV	Rectangular prism	$(30)(10)(24) = 7.2\times10^3$	5	78	36×10^3	561.6×10^3
V	Triangular prism	$\frac{1}{2}(30)(9)(24) = 3.24\times10^3$	13	78	42.12×10^3	252.72×10^3
	Σ	46.399×10^3			848.18×10^3	2388.5×10^3

$$\bar{X}\Sigma V = \Sigma\bar{x}V$$

$$\bar{X} = \frac{\Sigma\bar{x}V}{\Sigma V} = \frac{848.18\times10^3\ \text{mm}^4}{46.399\times10^3\ \text{mm}^3}$$

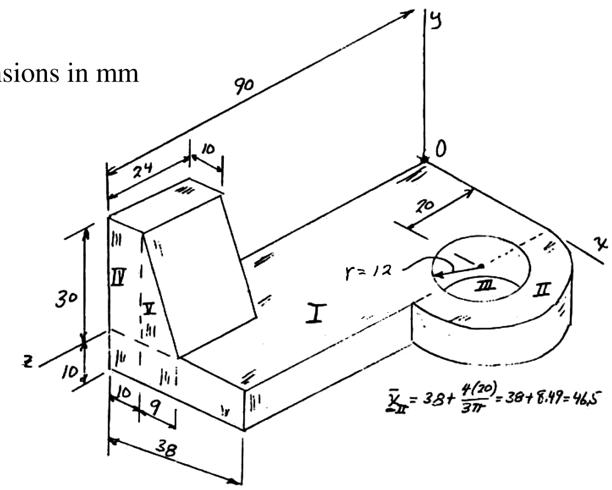
$\bar{X} = 18.28$ mm ◀

PROBLEM 5.105



For the machine element shown, locate the z coordinate of the center of gravity.

SOLUTION



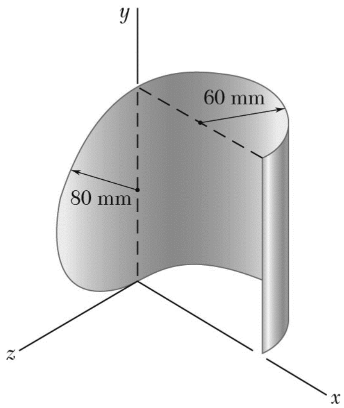
		V, mm^3	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(10)(90)(38) = 34.2\times10^3$	19	45	649.8×10^3	1539×10^3
II	Half cylinder	$\frac{\pi}{2}(20)^2(10) = 6.2832\times10^3$	46.5	20	292.17×10^3	125.664×10^3
III	–(Cylinder)	$-\pi(12)^2(10) = -4.5239\times10^3$	38	20	-171.908×10^3	-90.478×10^3
IV	Rectangular prism	$(30)(10)(24) = 7.2\times10^3$	5	78	36×10^3	561.6×10^3
V	Triangular prism	$\frac{1}{2}(30)(9)(24) = 3.24\times10^3$	13	78	42.12×10^3	252.72×10^3
	Σ	46.399×10^3			848.18×10^3	2388.5×10^3

$$\bar{Z}\Sigma V = \Sigma\bar{z}V$$

$$\bar{Z} = \frac{\Sigma\bar{z}V}{\Sigma V} = \frac{2388.5\times10^3\,\text{mm}^4}{46.399\times10^3\,\text{mm}^3}$$

$\bar{Z} = 51.5$ mm ◀

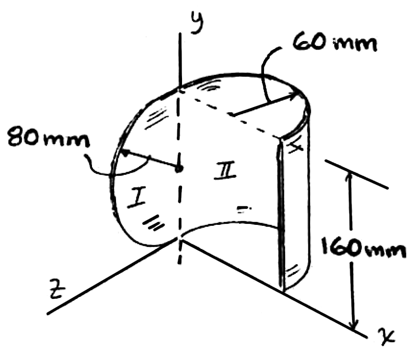
PROBLEM 5.106



Locate the center of gravity of the sheet-metal form shown.

SOLUTION

By symmetry, $\bar{Y} = 80.0 \text{ mm}$ ◀



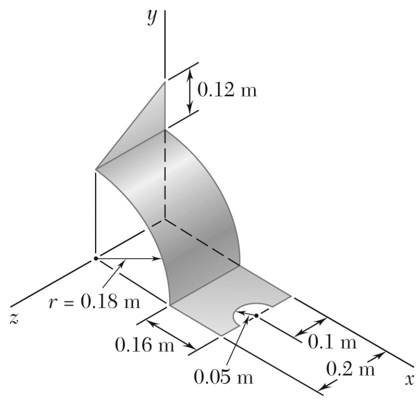
$$\bar{z}_{\mathrm{I}} = \frac{4(80)}{3\pi} = 33.953 \text{ mm}$$

$$\bar{z}_{\mathrm{II}} = -\frac{2(60)}{\pi} = -38.197 \text{ mm}$$

	A, mm^2	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}A$, mm^3	$\bar{z}A$, mm^3
I	$\frac{\pi}{2}(80)^2 = 10{,}053$	0	33.953	0	341.33×10^3
II	$\pi(60)(160) = 30{,}159$	60	−38.197	1809.54×10^3	-1151.98×10^3
Σ	40,212			1809.54×10^3	-810.65×10^3

$\bar{X}\Sigma A = \Sigma\bar{x}A$: $\quad \bar{X}(40{,}212) = 1809.54\times10^3$ $\qquad \bar{X} = 45.0 \text{ mm}$ ◀

$\bar{Z}\Sigma A = \Sigma\bar{z}A$: $\quad \bar{Z}(40{,}212) = -810.65\times10^3$ $\qquad \bar{Z} = -20.2 \text{ mm}$ ◀

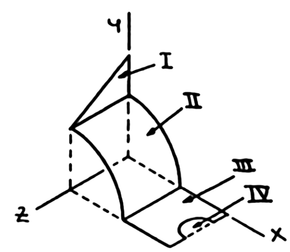


PROBLEM 5.107

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area.



$$\bar{y}_{\mathrm{I}} = 0.18 + \frac{1}{3}(0.12) = 0.22 \text{ m}$$

$$\bar{z}_{\mathrm{I}} = \frac{1}{3}(0.2 \text{ m})$$

$$\bar{x}_{\mathrm{II}} = \bar{y}_{\mathrm{II}} = \frac{2 \times 0.18}{\pi} = \frac{0.36}{\pi} \text{ m}$$

$$\bar{x}_{\mathrm{IV}} = 0.34 - \frac{4 \times 0.05}{3\pi}$$
$$= 0.31878 \text{ m}$$

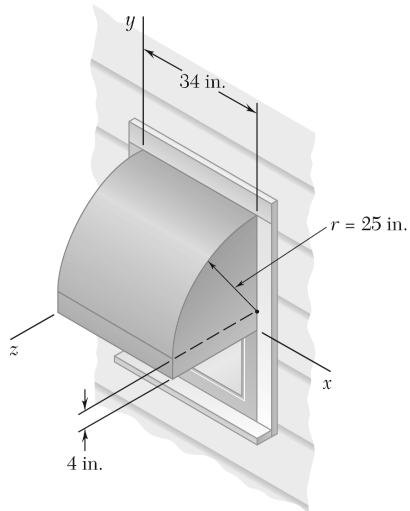
	A, m^2	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}A$, m^3	$\bar{y}A$, m^3	$\bar{z}A$, m^3
I	$\frac{1}{2}(0.2)(0.12) = 0.012$	0	0.22	$\frac{0.2}{3}$	0	0.00264	0.0008
II	$\frac{\pi}{2}(0.18)(0.2) = 0.018\pi$	$\frac{0.36}{\pi}$	$\frac{0.36}{\pi}$	0.1	0.00648	0.00648	0.005655
III	$(0.16)(0.2) = 0.032$	0.26	0	0.1	0.00832	0	0.0032
IV	$-\frac{\pi}{2}(0.05)^2 = -0.00125\pi$	0.31878	0	0.1	−0.001258	0	−0.000393
Σ	0.096622				0.013542	0.00912	0.009262

PROBLEM 5.107 (Continued)

We have $\bar{X}\Sigma V = \Sigma\bar{x}V$: $\bar{X}(0.096622\text{ m}^2) = 0.013542\text{ m}^3$ or $\bar{X} = 0.1402\text{ m}$ ◀

$\bar{Y}\Sigma V = \Sigma\bar{y}V$: $\bar{Y}(0.096622\text{ m}^2) = 0.00912\text{ m}^3$ or $\bar{Y} = 0.0944\text{ m}$ ◀

$\bar{Z}\Sigma V = \Sigma\bar{z}V$: $\bar{Z}(0.096622\text{ m}^2) = 0.009262\text{ m}^3$ or $\bar{Z} = 0.0959\text{ m}$ ◀

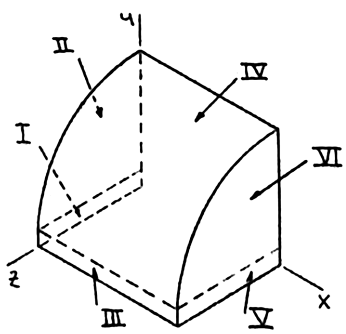


PROBLEM 5.108

A window awning is fabricated from sheet metal of uniform thickness. Locate the center of gravity of the awning.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the awning coincides with the centroid of the corresponding area.



$$\bar{y}_{II} = \bar{y}_{VI} = 4 + \frac{(4)(25)}{3\pi} = 14.6103 \text{ in.}$$

$$\bar{z}_{II} = \bar{z}_{VI} = \frac{(4)(25)}{3\pi} = \frac{100}{3\pi} \text{ in.}$$

$$\bar{y}_{IV} = 4 + \frac{(2)(25)}{\pi} = 19.9155 \text{ in.}$$

$$\bar{z}_{IV} = \frac{(2)(25)}{\pi} = \frac{50}{\pi} \text{ in.}$$

$$A_{II} = A_{VI} = \frac{\pi}{4}(25)^2 = 490.87 \text{ in}^2$$

$$A_{IV} = \frac{\pi}{2}(25)(34) = 1335.18 \text{ in}^2$$

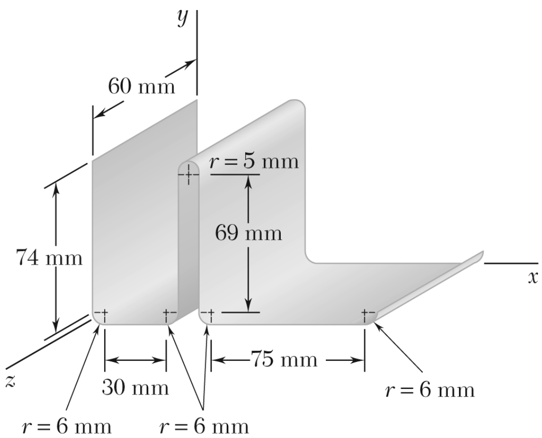
	A, in^2	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}A$, in^3	$\bar{z}A$, in^3
I	$(4)(25) = 100$	2	12.5	200	1250
II	490.87	14.6103	$\frac{100}{3\pi}$	7171.8	5208.3
III	$(4)(34) = 136$	2	25	272	3400
IV	1335.18	19.9155	$\frac{50}{\pi}$	26,591	21,250
V	$(4)(25) = 100$	2	12.5	200	1250
VI	490.87	14.6103	$\frac{100}{3\pi}$	7171.8	5208.3
Σ	2652.9			41,607	37,567

PROBLEM 5.108 (Continued)

Now, symmetry implies $\bar{X} = 17.00 \text{ in.}$ ◀

and $\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(2652.9 \text{ in}^2) = 41{,}607 \text{ in}^3$ or $\bar{Y} = 15.68 \text{ in.}$ ◀

$\bar{Z}\Sigma A = \Sigma \bar{z}A$: $\bar{Z}(2652.9 \text{ in}^2) = 37{,}567 \text{ in}^3$ or $\bar{Z} = 14.16 \text{ in.}$ ◀



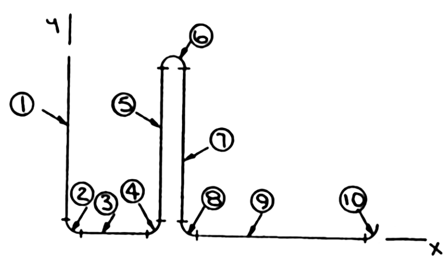
PROBLEM 5.109

A thin sheet of plastic of uniform thickness is bent to form a desk organizer. Locate the center of gravity of the organizer.

SOLUTION

First assume that the plastic is homogeneous so that the center of gravity of the organizer will coincide with the centroid of the corresponding area. Now note that symmetry implies

$\bar{Z} = 30.0\text{ mm}$ ◀



$$\bar{x}_2 = 6 - \frac{2\times 6}{\pi} = 2.1803\text{ mm}$$

$$\bar{x}_4 = 36 + \frac{2\times 6}{\pi} = 39.820\text{ mm}$$

$$\bar{x}_8 = 58 - \frac{2\times 6}{\pi} = 54.180\text{ mm}$$

$$\bar{x}_{10} = 133 + \frac{2\times 6}{\pi} = 136.820\text{ mm}$$

$$\bar{y}_2 = \bar{y}_4 = \bar{y}_8 = \bar{y}_{10} = 6 - \frac{2\times 6}{\pi} = 2.1803\text{ mm}$$

$$\bar{y}_6 = 75 + \frac{2\times 5}{\pi} = 78.183\text{ mm}$$

$$A_2 = A_4 = A_8 = A_{10} = \frac{\pi}{2}\times 6\times 60 = 565.49\text{ mm}^2$$

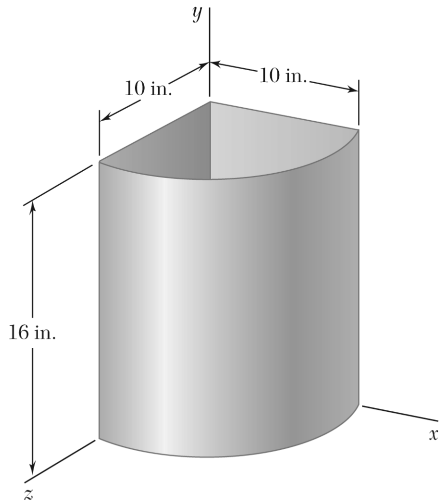
$$A_6 = \pi\times 5\times 60 = 942.48\text{ mm}^2$$

PROBLEM 5.109 (Continued)

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(74)(60) = 4440$	0	43	0	190,920
2	565.49	2.1803	2.1803	1233	1233
3	$(30)(60) = 1800$	21	0	37,800	0
4	565.49	39.820	2.1803	22,518	1233
5	$(69)(60) = 4140$	42	40.5	173,880	167,670
6	942.48	47	78.183	44,297	73,686
7	$(69)(60) = 4140$	52	40.5	215,280	167,670
8	565.49	54.180	2.1803	30,638	1233
9	$(75)(60) = 4500$	95.5	0	429,750	0
10	565.49	136.820	2.1803	77,370	1233
Σ	22,224.44			1,032,766	604,878

We have $\bar{X}\Sigma A = \Sigma \bar{x}A$: $\bar{X}(22,224.44 \text{ mm}^2) = 1,032,766 \text{ mm}^3$ or $\bar{X} = 46.5$ mm ◀

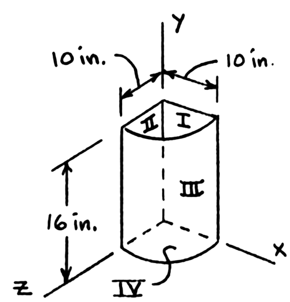
$\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(22,224.44 \text{ mm}^2) = 604,878 \text{ mm}^3$ or $\bar{Y} = 27.2$ mm ◀



PROBLEM 5.110

A wastebasket, designed to fit in the corner of a room, is 16 in. high and has a base in the shape of a quarter circle of radius 10 in. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

SOLUTION



By symmetry, $\bar{X} = \bar{Z}$

For III (Cylindrical surface), $\bar{x} = \dfrac{2r}{\pi} = \dfrac{2(10)}{\pi} = 6.3662 \text{ in.}$

$$A = \frac{\pi}{2} rh = \frac{\pi}{2}(10)(16) = 251.33 \text{ in}^2$$

For IV (Quarter-circle bottom), $\bar{x} = \dfrac{4r}{3\pi} = \dfrac{4(10)}{3\pi} = 4.2441 \text{ in.}$

$$A = \frac{\pi}{4} r^2 = \frac{\pi}{4}(10)^2 = 78.540 \text{ in}^2$$

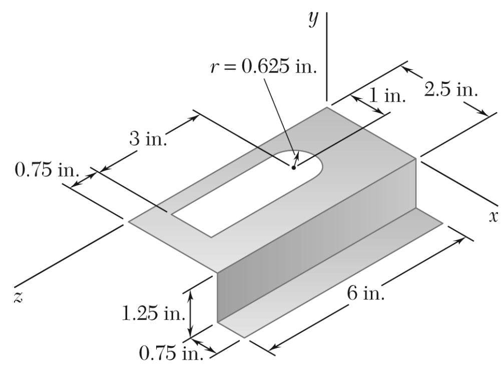
	A, in^2	$\bar{x}$, in.	$\bar{x}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
I	$(10)(16) = 160$	5	8	800	1280
II	$(10)(16) = 160$	0	8	0	1280
III	251.33	6.3662	8	1600.0	2010.6
IV	78.540	4.2441	0	333.33	0
Σ	649.87			2733.3	4570.6

$\bar{X}\Sigma A = \Sigma \bar{x}A$: $\bar{X}(649.87 \text{ in}^2) = 2733.3 \text{ in}^3$

$\bar{X} = 4.2059 \text{ in.}$ $\qquad \bar{X} = \bar{Z} = 4.21 \text{ in.}$ ◀

$\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(649.87 \text{ in}^2) = 4570.6 \text{ in}^3$

$\bar{Y} = 7.0331 \text{ in.}$ $\qquad \bar{Y} = 7.03 \text{ in.}$ ◀

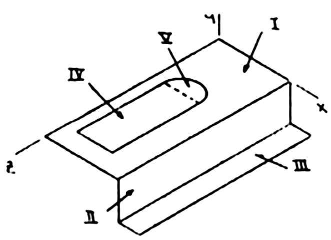


PROBLEM 5.111

A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the bracket coincides with the centroid of the corresponding area. Then (see diagram)



$$\bar{z}_V = 2.25 - \frac{4(0.625)}{3\pi}$$
$$= 1.98474 \text{ in.}$$
$$A_V = -\frac{\pi}{2}(0.625)^2$$
$$= -0.61359 \text{ in}^2$$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3	$\bar{z}A$, in^3
I	$(2.5)(6) = 15$	1.25	0	3	18.75	0	45
II	$(1.25)(6) = 7.5$	2.5	–0.625	3	18.75	–4.6875	22.5
III	$(0.75)(6) = 4.5$	2.875	–1.25	3	12.9375	–5.625	13.5
IV	$-\left(\frac{5}{4}\right)(3) = -3.75$	1.0	0	3.75	3.75	0	–14.0625
V	-0.61359	1.0	0	1.98474	0.61359	0	–1.21782
Σ	22.6364				46.0739	10.3125	65.7197

We have $\bar{X}\Sigma A = \Sigma \bar{x}A$

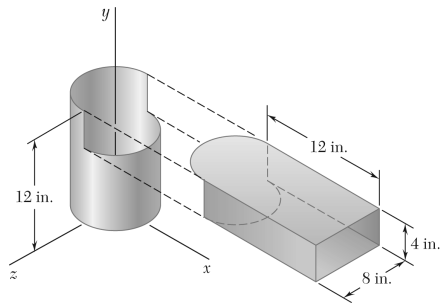
$$\bar{X}(22.6364 \text{ in}^2) = 46.0739 \text{ in}^3 \qquad \text{or} \qquad \bar{X} = 2.04 \text{ in.} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(22.6364 \text{ in}^2) = -10.3125 \text{ in}^3 \qquad \text{or} \quad \bar{Y} = -0.456 \text{ in.} \blacktriangleleft$$

$$\bar{Z}\Sigma A = \Sigma \bar{z}A$$

$$\bar{Z}(22.6364 \text{ in}^2) = 65.7197 \text{ in}^3 \qquad \text{or} \qquad \bar{Z} = 2.90 \text{ in.} \blacktriangleleft$$



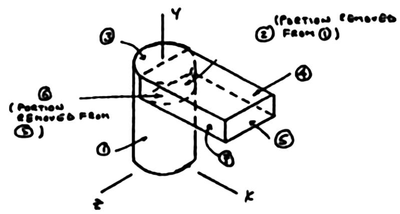
PROBLEM 5.112

An 8-in.-diameter cylindrical duct and a 4 × 8-in. rectangular duct are to be joined as indicated. Knowing that the ducts were fabricated from the same sheet metal, which is of uniform thickness, locate the center of gravity of the assembly.

SOLUTION

Assume that the body is homogeneous so that its center of gravity coincides with the centroid of the area.

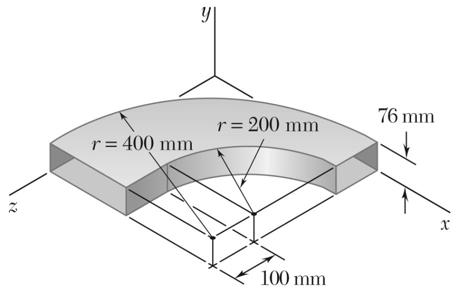
By symmetry, $\bar{z} = 0$.



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\pi(8)(12) = 96\pi$	0	6	0	576π
2	$-\frac{\pi}{2}(8)(4) = -16\pi$	$\frac{2(4)}{\pi} = \frac{8}{\pi}$	10	-128	-160π
3	$\frac{\pi}{2}(4)^2 = 8\pi$	$-\frac{4(4)}{3\pi} = -\frac{16}{3\pi}$	12	-42.667	96π
4	$(8)(12) = 96$	6	12	576	1152
5	$(8)(12) = 96$	6	8	576	768
6	$-\frac{\pi}{2}(4)^2 = -8\pi$	$\frac{4(4)}{3\pi} = \frac{16}{3\pi}$	8	-42.667	-64π
7	$(4)(12) = 48$	6	10	288	480
8	$(4)(12) = 48$	6	10	288	480
Σ	539.33			1514.6	4287.4

Then $$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{1514.67}{539.33} \text{ in.}$$ or $\bar{X} = 2.81$ in. ◀

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{4287.4}{539.33} \text{ in.}$$ or $\bar{Y} = 7.95$ in. ◀



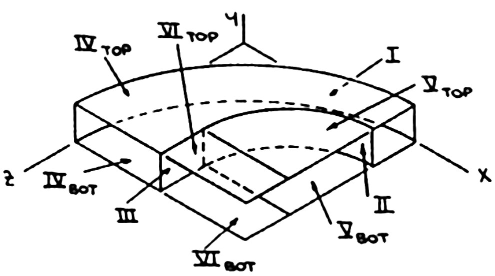
PROBLEM 5.113

An elbow for the duct of a ventilating system is made of sheet metal of uniform thickness. Locate the center of gravity of the elbow.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the duct coincides with the centroid of the corresponding area. Also, note that the shape of the duct implies

$$\overline{Y} = 38.0 \text{ mm} \blacktriangleleft$$



Note that
$$\bar{x}_{\text{I}} = \bar{z}_{\text{I}} = 400 - \frac{2}{\pi}(400) = 145.352 \text{ mm}$$

$$\bar{x}_{\text{II}} = 400 - \frac{2}{\pi}(200) = 272.68 \text{ mm}$$

$$\bar{z}_{\text{II}} = 300 - \frac{2}{\pi}(200) = 172.676 \text{ mm}$$

$$\bar{x}_{\text{IV}} = \bar{z}_{\text{IV}} = 400 - \frac{4}{3\pi}(400) = 230.23 \text{ mm}$$

$$\bar{x}_{\text{V}} = 400 - \frac{4}{3\pi}(200) = 315.12 \text{ mm}$$

$$\bar{z}_{\text{V}} = 300 - \frac{4}{3\pi}(200) = 215.12 \text{ mm}$$

Also note that the corresponding top and bottom areas will contribute equally when determining $\bar{x}$ and $\bar{z}$.

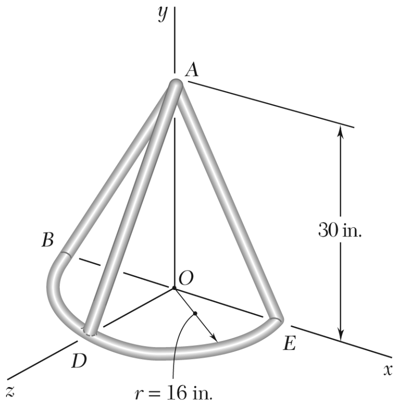
Thus,

	A, mm^2	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}A$, mm^3	$\bar{z}A$, mm^3
I	$\frac{\pi}{2}(400)(76) = 47,752$	145.352	145.352	6,940,850	6,940,850
II	$\frac{\pi}{2}(200)(76) = 23,876$	272.68	172.676	6,510,510	4,122,810
III	$100(76) = 7600$	200	350	1,520,000	2,660,000
IV	$2\left(\frac{\pi}{4}\right)(400)^2 = 251,327$	230.23	230.23	57,863,020	57,863,020
V	$-2\left(\frac{\pi}{4}\right)(200)^2 = -62,832$	315.12	215.12	–19,799,620	–13,516,420
VI	$-2(100)(200) = -40,000$	300	350	–12,000,000	–14,000,000
Σ	227,723			41,034,760	44,070,260

PROBLEM 5.113 (Continued)

We have $\bar{X}\Sigma A = \Sigma\bar{x}A$: $\bar{X}(227,723 \text{ mm}^2) = 41,034,760 \text{ mm}^3$ or $\bar{X} = 180.2 \text{ mm}$ ◀

$\bar{Z}\Sigma A = \Sigma\bar{z}A$: $\bar{Z}(227,723 \text{ mm}^2) = 44,070,260 \text{ mm}^3$ or $\bar{Z} = 193.5 \text{ mm}$ ◀

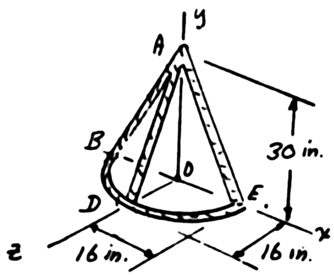


PROBLEM 5.114

Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

By symmetry, $\bar{X} = 0$ ◀



	L, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}L$, in^2	$\bar{z}L$, in^2
AB	$\sqrt{30^2+16^2}=34$	15	0	510	0
AD	$\sqrt{30^2+16^2}=34$	15	8	510	272
AE	$\sqrt{30^2+16^2}=34$	15	0	510	0
BDE	$\pi(16)=50.265$	0	$\dfrac{2(16)}{\pi}=10.186$	0	512
Σ	152.265			1530	784

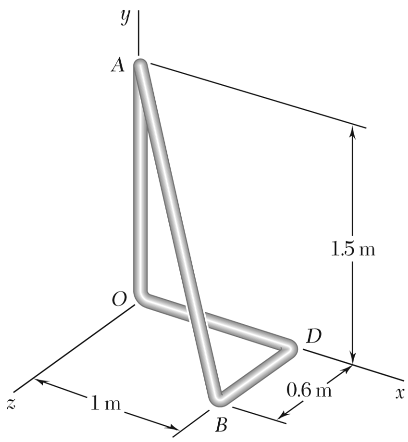
$$\bar{Y}\Sigma L = \Sigma\bar{y}L: \quad \bar{Y}(152.265\text{ in.}) = 1530\text{ in}^2$$

$$\bar{Y} = 10.048\text{ in.} \qquad \bar{Y} = 10.05\text{ in.} ◀$$

$$\bar{Z}\Sigma L = \Sigma\bar{z}L: \quad \bar{Z}(152.265\text{ in.}) = 784\text{ in}^2$$

$$\bar{Z} = 5.149\text{ in.} \qquad \bar{Z} = 5.15\text{ in.} ◀$$

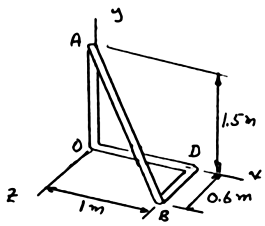
PROBLEM 5.115



Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

Uniform rod:



$$AB^2 = (1\text{ m})^2 + (0.6\text{ m})^2 + (1.5\text{ m})^2$$

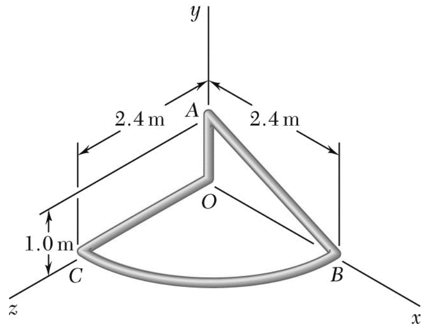
$$AB = 1.9\text{ m}$$

	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	ΣL, m
AB	1.9	0.5	0.75	0.3	0.95	1.425	0.57
BD	0.6	1.0	0	0.3	0.60	0	0.18
DO	1.0	0.5	0	0	0.50	0	0
OA	1.5	0	0.75	0	0	1.125	0
Σ	5.0				2.05	2.550	0.75

$\bar{X}\Sigma L = \Sigma \bar{x} L$: $\quad \bar{X}(5.0\text{ m}) = 2.05\text{ m}^2$ $\qquad \bar{X} = 0.410\text{ m}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y} L$: $\quad \bar{Y}(5.0\text{ m}) = 2.55\text{ m}^2$ $\qquad \bar{Y} = 0.510\text{ m}$ ◀

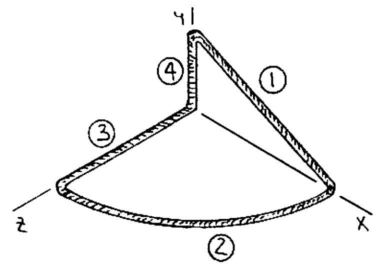
$\bar{Z}\Sigma L = \Sigma \bar{z} L$: $\quad \bar{Z}(5.0\text{ m}) = 0.75\text{ m}^2$ $\qquad \bar{Z} = 0.1500\text{ m}$ ◀



PROBLEM 5.116

A thin steel wire of uniform cross section is bent into the shape shown. Locate its center of gravity.

SOLUTION



First assume that the wire is homogeneous so that its center of gravity will coincide with the centroid of the corresponding line.

$$\bar{x}_2 = \bar{z}_2 = \frac{2\times 2.4}{\pi} = \frac{4.8}{\pi}\text{ m}$$

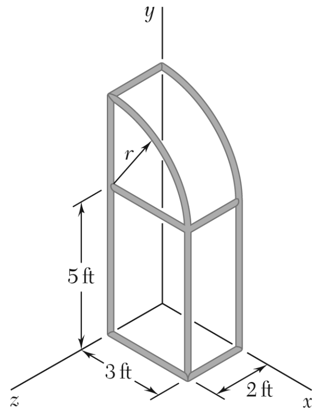
	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	$\bar{z}L$, m^2
1	2.6	1.2	0.5	0	3.12	1.3	0
2	$\frac{\pi}{2}\times 2.4 = 1.2\pi$	$\frac{4.8}{\pi}$	0	$\frac{4.8}{\pi}$	5.76	0	5.76
3	2.4	0	0	1.2	0	0	2.88
4	1.0	0	0.5	0	0	0.5	0
Σ	9.7699				8.88	1.8	8.64

We have $\bar{X}\Sigma L = \Sigma\bar{x}L$: $\bar{X}(9.7699\text{ m}) = 8.88\text{ m}^2$ or $\bar{X} = 0.909$ m ◀

$\bar{Y}\Sigma L = \Sigma\bar{y}L$: $\bar{Y}(9.7699\text{ m}) = 1.8\text{ m}^2$ or $\bar{Y} = 0.1842$ m ◀

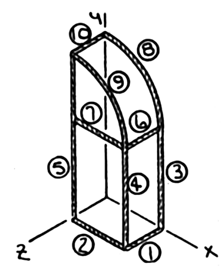
$\bar{Z}\Sigma L = \Sigma\bar{z}L$: $\bar{Z}(9.7699\text{ m}) = 8.64\text{ m}^2$ or $\bar{Z} = 0.884$ m ◀

PROBLEM 5.117



The frame of a greenhouse is constructed from uniform aluminum channels. Locate the center of gravity of the portion of the frame shown.

SOLUTION



First assume that the channels are homogeneous so that the center of gravity of the frame will coincide with the centroid of the corresponding line.

$$\bar{x}_8 = \bar{x}_9 = \frac{2\times 3}{\pi} = \frac{6}{\pi}\text{ ft}$$

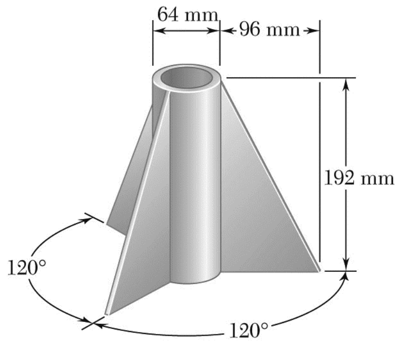
$$\bar{y}_8 = \bar{y}_9 = 5 + \frac{2\times 3}{\pi} = 6.9099\text{ ft}$$

	L, ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$\bar{x}L$, ft^2	$\bar{y}L$, ft^2	$\bar{z}L$, ft^2
1	2	3	0	1	6	0	2
2	3	1.5	0	2	4.5	0	6
3	5	3	2.5	0	15	12.5	0
4	5	3	2.5	2	15	12.5	10
5	8	0	4	2	0	32	16
6	2	3	5	1	6	10	2
7	3	1.5	5	2	4.5	15	6
8	$\frac{\pi}{2}\times 3 = 4.7124$	$\frac{6}{\pi}$	6.9099	0	9	32.562	0
9	$\frac{\pi}{2}\times 3 = 4.7124$	$\frac{6}{\pi}$	6.9099	2	9	32.562	9.4248
10	2	0	8	1	0	16	2
Σ	39.4248				69	163.124	53.4248

We have $\bar{X}\Sigma L = \Sigma\bar{x}L$: $\bar{X}(39.4248\text{ ft}) = 69\text{ ft}^2$ or $\bar{X} = 1.750\text{ ft}$ ◀

$\bar{Y}\Sigma L = \Sigma\bar{y}L$: $\bar{Y}(39.4248\text{ ft}) = 163.124\text{ ft}^2$ or $\bar{Y} = 4.14\text{ ft}$ ◀

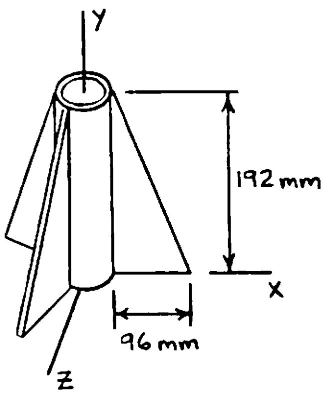
$\bar{Z}\Sigma L = \Sigma\bar{z}L$: $\bar{Z}(39.4248\text{ ft}) = 53.4248\text{ ft}^2$ or $\bar{Z} = 1.355\text{ ft}$ ◀



PROBLEM 5.118

Three brass plates are brazed to a steel pipe to form the flagpole base shown. Knowing that the pipe has a wall thickness of 8 mm and that each plate is 6 mm thick, determine the location of the center of gravity of the base. (Densities: brass = 8470 kg/m^3; steel = 7860 kg/m^3.)

SOLUTION



Since brass plates are equally spaced, we note that the center of gravity lies on the y-axis.

Thus, $\bar{x} = \bar{z} = 0$ ◀

Steel pipe: $V = \dfrac{\pi}{4}[(0.064\text{ m})^2 - (0.048\text{ m})^2](0.192\text{ m})$

$= 270.22\times10^{-6}\text{ m}^3$

$m = \rho V = (7860\text{ kg/m}^3)(270.22\times10^{-6}\text{ m}^3)$

$= 2.1239\text{ kg}$

Each brass plate: $V = \dfrac{1}{2}(0.096\text{ m})(0.192\text{ m})(0.006\text{ m}) = 55.296\times10^{-6}\text{ m}^3$

$m = \rho V = (8470\text{ kg/m}^3)(55.296\times10^{-6}\text{ m}^3) = 0.46836\text{ kg}$

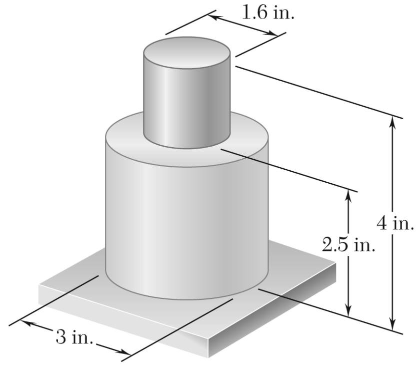
Flagpole base:

$$\Sigma m = 2.1239\text{ kg} + 3(0.46836\text{ kg}) = 3.5290\text{ kg}$$

$$\Sigma \bar{y} m = (0.096\text{ m})(2.1239\text{ kg}) + 3[(0.064\text{ m})(0.46836\text{ kg})] = 0.29382\text{ kg}\cdot\text{m}$$

$$\bar{Y}\Sigma m = \Sigma \bar{y} m: \qquad \bar{Y}(3.5290\text{ kg}) = 0.29382\text{ kg}\cdot\text{m}$$

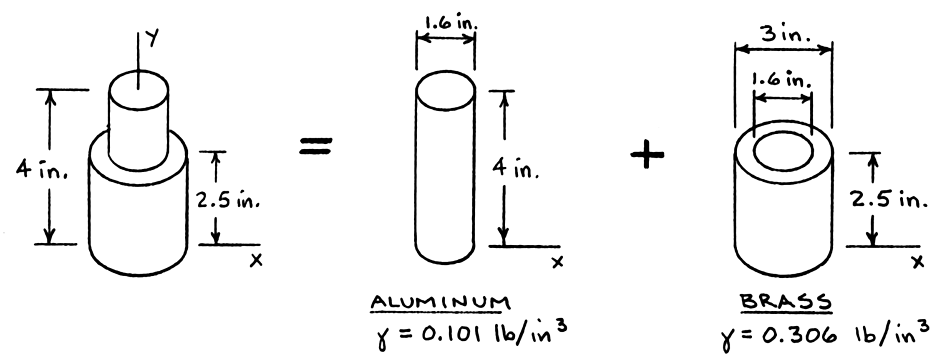
$\bar{Y} = 0.083259\text{ m}$ $\qquad$ $\bar{Y} = 83.3\text{ mm}$ above the base ◀



PROBLEM 5.119

A brass collar, of length 2.5 in., is mounted on an aluminum rod of length 4 in. Locate the center of gravity of the composite body. (Specific weights: brass = 0.306 lb/in^3, aluminum = 0.101 lb/in^3)

SOLUTION



Aluminum rod: $W = \gamma V$

$$= (0.101\ \text{lb/in}^3)\left[\frac{\pi}{4}(1.6\ \text{in.})^2(4\ \text{in.})\right]$$

$$= 0.81229\ \text{lb}$$

Brass collar: $W = \gamma V$

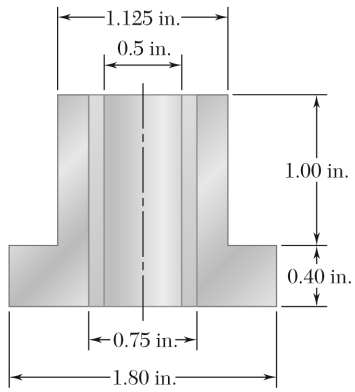
$$= (0.306\ \text{lb/in.}^3)\frac{\pi}{4}[(3\ \text{in.})^2 - (1.6\ \text{in.})^2](2.5\ \text{in.})$$

$$= 3.8693\ \text{lb}$$

Component	W(lb)	$\bar{y}$ (in.)	$\bar{y}W$ (lb · in.)
Rod	0.81229	2	1.62458
Collar	3.8693	1.25	4.8366
Σ	4.6816		6.4612

$$\bar{Y}\Sigma W = \Sigma \bar{y} W: \quad \bar{Y}(4.6816\ \text{lb}) = 6.4612\ \text{lb} \cdot \text{in.}$$

$$\bar{Y} = 1.38013\ \text{in.} \qquad \bar{Y} = 1.380\ \text{in.} \blacktriangleleft$$

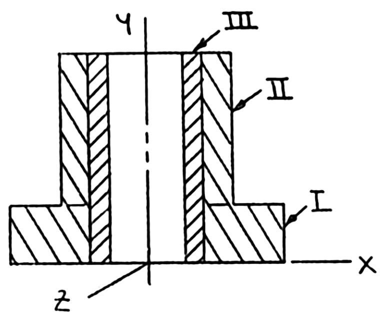


PROBLEM 5.120

A bronze bushing is mounted inside a steel sleeve. Knowing that the specific weight of bronze is 0.318 lb/in^3 and of steel is 0.284 lb/in^3, determine the location of the center of gravity of the assembly.

SOLUTION

First, note that symmetry implies $\bar{X} = \bar{Z} = 0$ ◀



Now $W = (\rho g)V$

$$\bar{y}_{\mathrm{I}} = 0.20 \text{ in.} \quad W_{\mathrm{I}} = (0.284 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(1.8^2 - 0.75^2)\text{ in}^2](0.4 \text{ in.})\right\} = 0.23889 \text{ lb}$$

$$\bar{y}_{\mathrm{II}} = 0.90 \text{ in.} \quad W_{\mathrm{II}} = (0.284 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(1.125^2 - 0.75^2)\text{ in}^2](1 \text{ in.})\right\} = 0.156834 \text{ lb}$$

$$\bar{y}_{\mathrm{III}} = 0.70 \text{ in.} \quad W_{\mathrm{III}} = (0.318 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(0.75^2 - 0.5^2)\text{ in}^2](1.4 \text{ in.})\right\} = 0.109269 \text{ lb}$$

We have $\bar{Y}\Sigma W = \Sigma \bar{y} W$

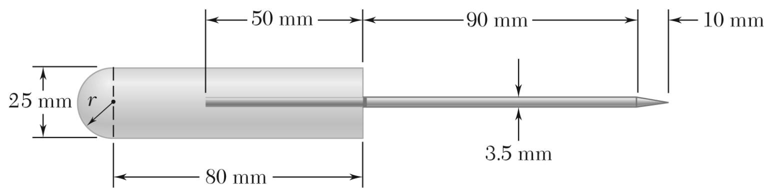
$$\bar{Y} = \frac{(0.20 \text{ in.})(0.23889 \text{ lb}) + (0.90 \text{ in.})(0.156834 \text{ lb}) + (0.70 \text{ in.})(0.109269 \text{ lb})}{0.23889 \text{ lb} + 0.156834 \text{ lb} + 0.109269 \text{ lb}}$$

or $\bar{Y} = 0.526$ in. ◀

(above base)

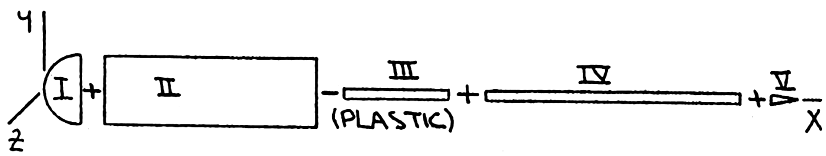
PROBLEM 5.121

A scratch awl has a plastic handle and a steel blade and shank. Knowing that the density of plastic is 1030 kg/m^3 and of steel is 7860 kg/m^3, locate the center of gravity of the awl.



SOLUTION

First, note that symmetry implies $\bar{Y} = \bar{Z} = 0$ ◀



$$\bar{x}_{\text{I}} = \frac{5}{8}(12.5 \text{ mm}) = 7.8125 \text{ mm}$$

$$W_{\text{I}} = (1030 \text{ kg/m}^3)\left(\frac{2\pi}{3}\right)(0.0125 \text{ m})^3$$

$$= 4.2133 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{II}} = 52.5 \text{ mm}$$

$$W_{\text{II}} = (1030 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.025 \text{ m})^2(0.08 \text{ m})$$

$$= 40.448 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{III}} = 92.5 \text{ mm} - 25 \text{ mm} = 67.5 \text{ mm}$$

$$W_{\text{III}} = -(1030 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035 \text{ m})^2(0.05 \text{ m})$$

$$= -0.49549 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{IV}} = 182.5 \text{ mm} - 70 \text{ mm} = 112.5 \text{ mm}$$

$$W_{\text{IV}} = (7860 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035 \text{ m})^2(0.14 \text{ m})^2 = 10.5871 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{V}} = 182.5 \text{ mm} + \frac{1}{4}(10 \text{ mm}) = 185 \text{ mm}$$

$$W_{\text{V}} = (7860 \text{ kg/m}^3)\left(\frac{\pi}{3}\right)(0.00175 \text{ m})^2(0.01 \text{ m}) = 0.25207 \times 10^{-3} \text{ kg}$$

PROBLEM 5.121 (Continued)

	W, kg	$\bar{x}$, mm	$\bar{x}W$, kg · mm
I	4.123×10^{-3}	7.8125	32.916×10^{-3}
II	40.948×10^{-3}	52.5	2123.5×10^{-3}
III	-0.49549×10^{-3}	67.5	-33.447×10^{-3}
IV	10.5871×10^{-3}	112.5	1191.05×10^{-3}
V	0.25207×10^{-3}	185	46.633×10^{-3}
Σ	55.005×10^{-3}		3360.7×10^{-3}

We have $\bar{X}\Sigma W = \Sigma\bar{x}W$: $\bar{X}(55.005\times10^{-3}\text{ kg}) = 3360.7\times10^{-3}\text{ kg}\cdot\text{mm}$

or $\bar{X} = 61.1\text{ mm}$ ◀

(from the end of the handle)

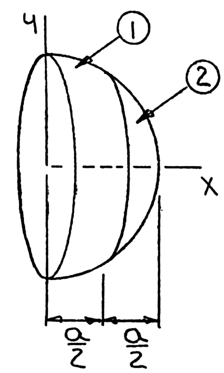
PROBLEM 5.122

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A hemisphere

SOLUTION

Choose as the element of volume a disk of radius r and thickness dx. Then



$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $x^2 + y^2 = a^2$ so that $r^2 = a^2 - x^2$ and then

$$dV = \pi(a^2 - x^2)dx$$

Component 1:

$$V_1 = \int_0^{a/2} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_0^{a/2}$$

$$= \frac{11}{24}\pi a^3$$

and

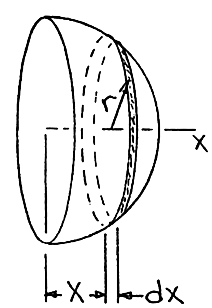
$$\int_1 \bar{x}_{EL} dV = \int_0^{a/2} x\left[\pi(a^2 - x^2)dx\right]$$

$$= \pi\left[a^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{a/2}$$

$$= \frac{7}{64}\pi a^4$$

Now $\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV$: $\quad \bar{x}_1\left(\frac{11}{24}\pi a^3\right) = \frac{7}{64}\pi a^4$ or $\bar{x}_1 = \frac{21}{88}a$ ◀

Component 2:



$$V_2 = \int_{a/2}^{a} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a}$$

$$= \pi\left\{\left[a^2(a) - \frac{a^3}{3}\right] - \left[a^2\left(\frac{a}{2}\right) - \frac{\left(\frac{a}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^3$$

PROBLEM 5.122 (Continued)

and
$$\int_2 \bar{x}_{EL}\,dV = \int_{a/2}^{a} x\left[\pi(a^2 - x^2)dx\right] = \pi\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{a/2}^{a}$$
$$= \pi\left\{\left[a^2\frac{(a)^2}{2} - \frac{(a)^4}{4}\right] - \left[a^2\frac{\left(\frac{a}{2}\right)^2}{2} - \frac{\left(\frac{a}{2}\right)^4}{4}\right]\right\}$$
$$= \frac{9}{64}\pi a^4$$

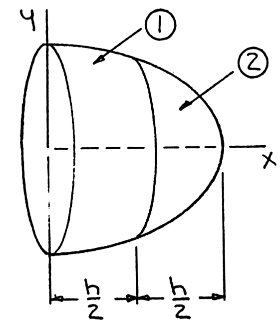
Now
$$\bar{x}_2 V_2 = \int_2 \bar{x}_{EL}\,dV: \quad \bar{x}_2\left(\frac{5}{24}\pi a^3\right) = \frac{9}{64}\pi a^4$$
or $\bar{x}_2 = \dfrac{27}{40}a$ ◀

PROBLEM 5.123

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A semiellipsoid of revolution

SOLUTION



Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $\dfrac{x^2}{h^2} + \dfrac{y^2}{a^2} = 1$ so that

$$r^2 = \frac{a^2}{h^2}(h^2 - x^2)$$

and then $$dV = \pi \frac{a^2}{h^2}(h^2 - x^2)dx$$

Component 1: $$V_1 = \int_0^{h/2} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_0^{h/2}$$

$$= \frac{11}{24}\pi a^2 h$$

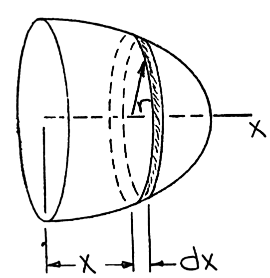
and $$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi \frac{a^2}{h^2}(h^2 - x^2)dx\right]$$

$$= \pi \frac{a^2}{h^2}\left[h^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{h/2}$$

$$= \frac{7}{64}\pi a^2 h^2$$

Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{11}{24}\pi a^2 h\right) = \frac{7}{64}\pi a^2 h^2$$

or $\bar{x}_1 = \dfrac{21}{88}h$ ◀



Component 2: $$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h^2}\left\{\left[h^2(h) - \frac{(h)^3}{3}\right] - \left[h^2\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^2 h$$

PROBLEM 5.123 (Continued)

and
$$\int_2 \bar{x}_{EL}\,dV = \int_{h/2}^{h} x\left[\pi\frac{a^2}{h^2}(h^2 - x^2)\,dx\right]$$
$$= \pi\frac{a^2}{h^2}\left[h^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{h/2}^{h}$$
$$= \pi\frac{a^2}{h^2}\left\{\left[h^2\frac{(h)^2}{2} - \frac{(h)^4}{4}\right] - \left[h^2\frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(\frac{h}{2}\right)^4}{4}\right]\right\}$$
$$= \frac{9}{64}\pi a^2 h^2$$

Now
$$\bar{x}_2 V_2 = \int_2 \bar{x}_{EL}\,dV: \quad \bar{x}_2\left(\frac{5}{24}\pi a^2 h\right) = \frac{9}{64}\pi a^2 h^2$$

or $\bar{x}_2 = \dfrac{27}{40}h$ ◀

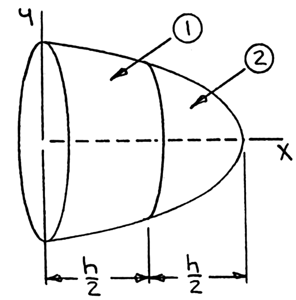
PROBLEM 5.124

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A paraboloid of revolution

SOLUTION

Choose as the element of volume a disk of radius r and thickness dx. Then



$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $x = h - \dfrac{h}{a^2}y^2$ so that $r^2 = \dfrac{a^2}{h}(h - x)$.

and then $$dV = \pi \frac{a^2}{h}(h - x)dx$$

Component 1: $$V_1 = \int_0^{h/2} \pi \frac{a^2}{h}(h - x)dx$$

$$= \pi \frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_0^{h/2}$$

$$= \frac{3}{8}\pi a^2 h$$

and $$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi \frac{a^2}{h}(h - x)dx\right]$$

$$= \pi \frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{h/2} = \frac{1}{12}\pi a^2 h^2$$

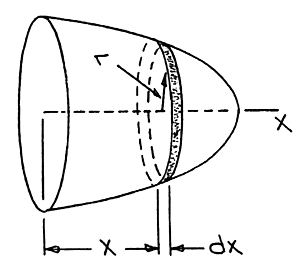
Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{3}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$ or $\bar{x}_1 = \dfrac{2}{9}h$ ◀

Component 2:

$$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h}(h - x)dx = \pi \frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h}\left\{\left[h(h) - \frac{(h)^2}{2}\right] - \left[h\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^2}{2}\right]\right\}$$

$$= \frac{1}{8}\pi a^2 h$$



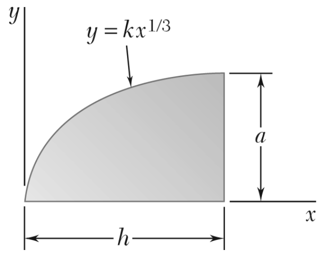
PROBLEM 5.124 (Continued)

and $$\int_2 \overline{x}_{EL} dV = \int_{h/2}^{h} x\left[\pi \frac{a^2}{h}(h-x)dx\right] = \pi \frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h}\left\{\left[h\frac{(h)^2}{2} - \frac{(h)^3}{3}\right] - \left[h\frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{1}{12}\pi a^2 h^2$$

Now $$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL} dV: \quad \overline{x}_2\left(\frac{1}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$ or $$\overline{x}_2 = \frac{2}{3}h \blacktriangleleft$$



PROBLEM 5.125

Locate the centroid of the volume obtained by rotating the shaded area about the x-axis.

SOLUTION

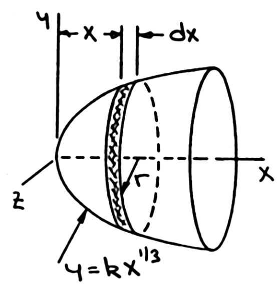
First note that symmetry implies $\qquad \bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad x_{EL} = x$$

Now $\qquad r = kx^{1/3}$



so that $\qquad dV = \pi k^2 x^{2/3} dx$

at $x = h,\ y = a,$ $\qquad a = kh^{1/3}$

or $\qquad k = \dfrac{a}{h^{1/3}}$

Then $\qquad dV = \pi \dfrac{a^2}{h^{2/3}} x^{2/3} dx$

and

$$V = \int_0^h \pi \frac{a^2}{h^{2/3}} x^{2/3} dx$$

$$= \pi \frac{a^2}{h^{2/3}} \left[\frac{3}{5} x^{5/3} \right]_0^h$$

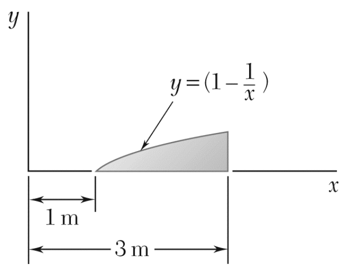
$$= \frac{3}{5} \pi a^2 h$$

Also

$$\int \bar{x}_{EL} dV = \int_0^h x \left(\pi \frac{a^2}{h^{2/3}} x^{2/3} dx \right) = \pi \frac{a^2}{h^{2/3}} \left[\frac{3}{8} x^{8/3} \right]_0^h$$

$$= \frac{3}{8} \pi a^2 h^2$$

Now $\qquad \bar{x}V = \int \bar{x} dV: \quad \bar{x} \left(\dfrac{3}{5} \pi a^2 h \right) = \dfrac{3}{8} \pi a^2 h^2$ $\qquad$ or $\bar{x} = \dfrac{5}{8} h$ ◀



PROBLEM 5.126

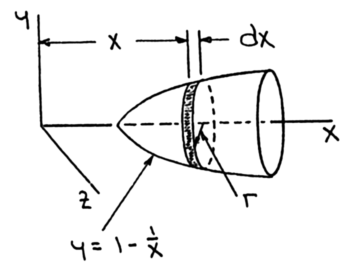
Locate the centroid of the volume obtained by rotating the shaded area about the x-axis.

SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx.



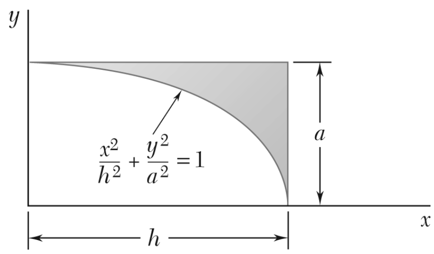
Then $$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now $r = 1 - \dfrac{1}{x}$ so that $$dV = \pi\left(1 - \frac{1}{x}\right)^2 dx$$
$$= \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

Then $$V = \int_1^3 \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx = \pi\left[x - 2\ln x - \frac{1}{x}\right]_1^3$$
$$= \pi\left[\left(3 - 2\ln 3 - \frac{1}{3}\right) - \left(1 - 2\ln 1 - \frac{1}{1}\right)\right]$$
$$= (0.46944\pi)\ \text{m}^3$$

and $$\int \bar{x}_{EL} dV = \int_1^3 x\left[\pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx\right] = \pi\left[\frac{x^2}{2} - 2x + \ln x\right]_1^3$$
$$= \pi\left\{\left[\frac{3^2}{2} - 2(3) + \ln 3\right] - \left[\frac{1^3}{2} - 2(1) + \ln 1\right]\right\}$$
$$= (1.09861\pi)\ \text{m}$$

Now $$\bar{x}V = \int \bar{x}_{EL} dV: \quad \bar{x}(0.46944\pi\ \text{m}^3) = 1.09861\pi\ \text{m}^4$$ or $\bar{x} = 2.34$ m ◀



PROBLEM 5.127

Locate the centroid of the volume obtained by rotating the shaded area about the line $x = h$.

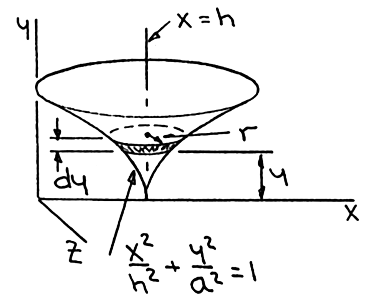
SOLUTION

First, note that symmetry implies $\bar{x} = h$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dy, \quad \bar{y}_{EL} = y$$



Now $x^2 = \frac{h^2}{a^2}(a^2 - y^2)$ so that $r = h - \frac{h}{a}\sqrt{a^2 - y^2}$.

Then
$$dV = \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

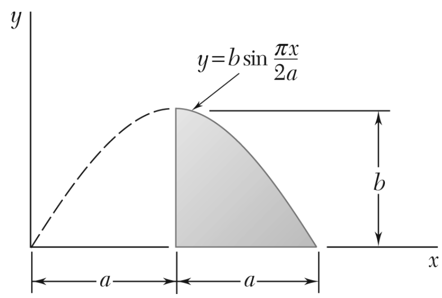
and
$$V = \int_0^a \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

Let
$$y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta$$

Then
$$V = \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left(a - \sqrt{a^2 - a^2\sin^2\theta}\right)^2 a\cos\theta\, d\theta$$
$$= \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left[a^2 - 2a(a\cos\theta) + (a^2 - a^2\sin^2\theta)\right] a\cos\theta\, d\theta$$
$$= \pi a h^2 \int_0^{\pi/2}(2\cos\theta - 2\cos^2\theta - \sin^2\theta\cos\theta)\, d\theta$$
$$= \pi a h^2\left[2\sin\theta - 2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$$
$$= \pi a h^2\left[2 - 2\left(\frac{\frac{\pi}{2}}{2}\right) - \frac{1}{3}\right]$$
$$= 0.095870\pi a h^2$$

and
$$\int \overline{y}_{EL}\,dV = \int_0^a y\left[\pi\frac{h^2}{a^2}\left(a-\sqrt{a^2-y^2}\right)^2 dy\right]$$
$$= \pi\frac{h^2}{a^2}\int_0^a \left(2a^2y - 2ay\sqrt{a^2-y^2} - y^3\right)dy$$
$$= \pi\frac{h^2}{a^2}\left[a^2y^2 + \frac{2}{3}a(a^2-y^2)^{3/2} - \frac{1}{4}y^4\right]_0^a$$
$$= \pi\frac{h^2}{a^2}\left\{\left[a^2(a)^2 - \frac{1}{4}a^4\right] - \left[\frac{2}{3}a(a^2)^{3/2}\right]\right\}$$
$$= \frac{1}{12}\pi a^2h^2$$

Now
$$\overline{y}V = \int \overline{y}_{EL}\,dV: \quad \overline{y}(0.095870\pi ah^2) = \frac{1}{12}\pi a^2h^2$$
or $\overline{y} = 0.869a$ ◀



PROBLEM 5.128*

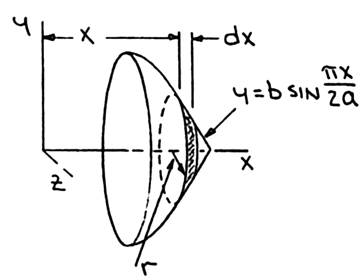
Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the x-axis.

SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx.



Then $$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now $$r = b\sin\frac{\pi x}{2a}$$

so that $$dV = \pi b^2 \sin^2\frac{\pi x}{2a} dx$$

Then
$$V = \int_a^{2a} \pi b^2 \sin^2\frac{\pi x}{2a} dx$$
$$= \pi b^2 \left[\frac{x}{2} - \frac{\sin\frac{\pi x}{2a}}{2\frac{\pi}{a}}\right]_a^{2a}$$
$$= \pi b^2\left[\left(\tfrac{2a}{2}\right) - \left(\tfrac{a}{2}\right)\right]$$
$$= \frac{1}{2}\pi a b^2$$

and $$\int \bar{x}_{EL} dV = \int_a^{2a} x\left(\pi b^2 \sin^2\frac{\pi x}{2a} dx\right)$$

Use integration by parts with

$$u = x \qquad dV = \sin^2\frac{\pi x}{2a}$$

$$du = dx \qquad V = \frac{x}{2} - \frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}$$

PROBLEM 5.128* (Continued)

Then $$\int \bar{x}_{EL}dV = \pi b^2\left\{\left[x\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)dx\right\}$$

$$= \pi b^2\left\{\left[2a\left(\frac{2a}{2}\right)-a\left(\frac{a}{2}\right)\right]-\left[\frac{1}{4}x^2+\frac{a^2}{2\pi^2}\cos\frac{\pi x}{a}\right]_a^{2a}\right\}$$

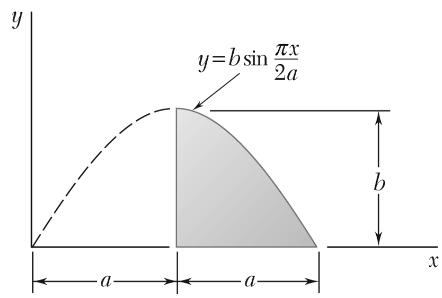
$$= \pi b^2\left\{\left(\frac{3}{2}a^2\right)-\left[\frac{1}{4}(2a)^2+\frac{a^2}{2\pi^2}-\frac{1}{4}(a)^2+\frac{a^2}{2\pi^2}\right]\right\}$$

$$= \pi a^2b^2\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$

$$= 0.64868\pi a^2b^2$$

Now $$\bar{x}V = \int \bar{x}_{EL}dV: \quad \bar{x}\left(\frac{1}{2}\pi ab^2\right) = 0.64868\pi a^2b^2$$

or $\bar{x} = 1.297a$ ◀



PROBLEM 5.129*

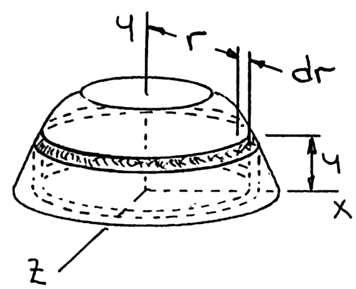
Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the y-axis. (*Hint:* Use a thin cylindrical shell of radius r and thickness dr as the element of volume.)

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a cylindrical shell of radius r and thickness dr.



Then $$dV = (2\pi r)(y)(dr), \quad \bar{y}_{EL} = \frac{1}{2}y$$

Now $$y = b\sin\frac{\pi r}{2a}$$

so that $$dV = 2\pi br\sin\frac{\pi r}{2a}dr$$

Then $$V = \int_a^{2a} 2\pi br\sin\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = rd \qquad dv = \sin\frac{\pi r}{2a}dr$$

$$du = dr \qquad v = -\frac{2a}{\pi}\cos\frac{\pi r}{2a}$$

Then $$V = 2\pi b\left\{\left[(r)\left(-\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)dr\right\}$$

$$= 2\pi b\left\{-\frac{2a}{\pi}[(2a)(-1)] + \left[\frac{4a^2}{\pi^2}\sin\frac{\pi r}{2a}\right]_a^{2a}\right\}$$

$$V = 2\pi b\left(\frac{4a^2}{\pi} - \frac{4a^2}{\pi^2}\right)$$

$$= 8a^2b\left(1 - \frac{1}{\pi}\right)$$

$$= 5.4535a^2b$$

PROBLEM 5.129* (Continued)

Also $$\int \bar{y}_{EL}\,dV = \int_a^{2a}\left(\tfrac{1}{2}b\sin\frac{\pi r}{2a}\right)\left(2\pi br\sin\frac{\pi r}{2a}dr\right)$$

$$= \pi b^2\int_a^{2a} r\sin^2\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = r \qquad dv = \sin^2\frac{\pi r}{2a}dr$$

$$du = dr \qquad v = \frac{r}{2} - \frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}$$

Then $$\int \bar{y}_{EL}\,dV = \pi b^2\left\{\left[(r)\left(\frac{r}{2} - \frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{r}{2} - \frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)dr\right\}$$

$$= \pi b^2\left\{\left[(2a)\left(\frac{2a}{2}\right) - (a)\left(\frac{a}{2}\right)\right] - \left[\frac{r^2}{4} + \frac{a^2}{2\pi^2}\cos\frac{\pi r}{a}\right]_a^{2a}\right\}$$

$$= \pi b^2\left\{\frac{3}{2}a^2 - \left[\frac{(2a)^2}{4} + \frac{a^2}{2\pi^2} - \frac{(a)^2}{4} + \frac{a^2}{2\pi^2}\right]\right\}$$

$$= \pi a^2b^2\left(\frac{3}{4} - \frac{1}{\pi^2}\right)$$

$$= 2.0379a^2b^2$$

Now $$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}(5.4535a^2b) = 2.0379a^2b^2$$

or $\bar{y} = 0.374b$ ◀

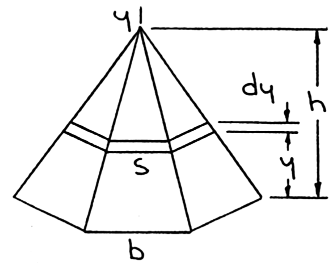
PROBLEM 5.130*

Show that for a regular pyramid of height h and n sides $(n = 3, 4, \ldots)$ the centroid of the volume of the pyramid is located at a distance $h/4$ above the base.

SOLUTION

Choose as the element of a horizontal slice of thickness dy. For any number N of sides, the area of the base of the pyramid is given by

$$A_{\text{base}} = kb^2$$



where $k = k(N)$; see note below. Using similar triangles, we have

$$\frac{s}{b} = \frac{h-y}{h}$$

or $$s = \frac{b}{h}(h-y)$$

Then $$dV = A_{\text{slice}}dy = ks^2dy = k\frac{b^2}{h^2}(h-y)^2 dy$$

and $$V = \int_0^h k\frac{b^2}{h^2}(h-y)^2 dy = k\frac{b^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h$$

$$= \frac{1}{3}kb^2h$$

Also, $$\overline{y}_{EL} = y$$

so that $$\int \overline{y}_{EL}dV = \int_0^h y\left[k\frac{b^2}{h^2}(h-y)^2 dy\right] = k\frac{b^2}{h^2}\int_0^h (h^2y - 2hy^2 + y^3)\,dy$$

$$= k\frac{b^2}{h^2}\left[\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right]_0^h = \frac{1}{12}kb^2h^2$$

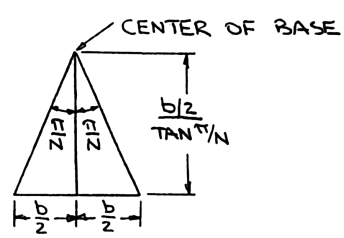
Now $$\overline{y}V = \int \overline{y}_{EL}dV: \quad \overline{y}\left(\frac{1}{3}kb^2h\right) = \frac{1}{12}kb^2h^2$$

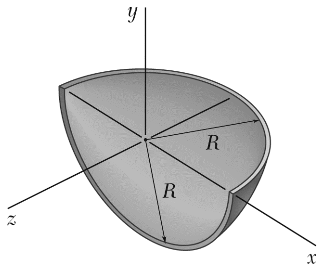
or $\overline{y} = \frac{1}{4}h$ Q.E.D. ◀

Note: $$A_{\text{base}} = N\left(\frac{1}{2}\times b\times\frac{\frac{b}{2}}{\tan\frac{\pi}{N}}\right)$$

$$= \frac{N}{4\tan\frac{\pi}{N}}b^2$$

$$= k(N)b^2$$





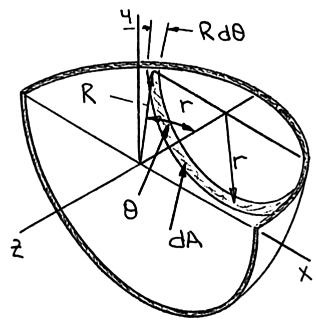
PROBLEM 5.131

Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius R.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

The element of area dA of the shell shown is obtained by cutting the shell with two planes parallel to the xy plane. Now



$$dA = (\pi r)(R\,d\theta)$$

$$\bar{y}_{EL} = -\frac{2r}{\pi}$$

where $r = R\sin\theta$

so that $dA = \pi R^2 \sin\theta\, d\theta$

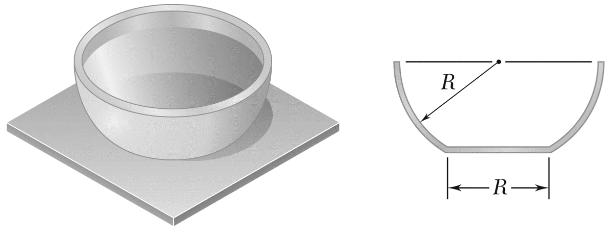
$$\bar{y}_{EL} = -\frac{2R}{\pi}\sin\theta$$

Then
$$A = \int_0^{\pi/2} \pi R^2 \sin\theta\, d\theta = \pi R^2[-\cos\theta]_0^{\pi/2}$$
$$= \pi R^2$$

and
$$\int \bar{y}_{EL}\, dA = \int_0^{\pi/2} \left(-\frac{2R}{\pi}\sin\theta\right)(\pi R^2 \sin\theta\, d\theta)$$
$$= -2R^3\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$
$$= -\frac{\pi}{2}R^3$$

Now $\bar{y}A = \int \bar{y}_{EL}\, dA$: $\quad \bar{y}(\pi R^2) = -\frac{\pi}{2}R^3$ or $\bar{y} = -\frac{1}{2}R$ ◀

Symmetry implies $\bar{z} = \bar{y}$ $\qquad \bar{z} = -\frac{1}{2}R$ ◀



PROBLEM 5.132

The sides and the base of a punch bowl are of uniform thickness t. If $t << R$ and $R = 250$ mm, determine the location of the center of gravity of (a) the bowl, (b) the punch.

SOLUTION

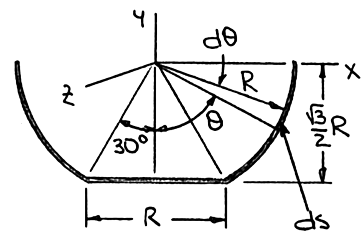
(a) Bowl:

First note that symmetry implies $\qquad \bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

for the coordinate axes shown below. Now assume that the bowl may be treated as a shell; the center of gravity of the bowl will coincide with the centroid of the shell. For the walls of the bowl, an element of area is obtained by rotating the arc ds about the y-axis. Then

$$dA_{\text{wall}} = (2\pi R \sin\theta)(R\,d\theta)$$



and
$$(\bar{y}_{EL})_{\text{wall}} = -R\cos\theta$$

Then
$$A_{\text{wall}} = \int_{\pi/6}^{\pi/2} 2\pi R^2 \sin\theta\,d\theta$$
$$= 2\pi R^2[-\cos\theta]_{\pi/6}^{\pi/2}$$
$$= \pi\sqrt{3}R^2$$

and
$$\bar{y}_{\text{wall}} A_{\text{wall}} = \int (\bar{y}_{EL})_{\text{wall}}\, dA$$
$$= \int_{\pi/6}^{\pi/2} (-R\cos\theta)(2\pi R^2 \sin\theta\,d\theta)$$
$$= \pi R^3[\cos^2\theta]_{\pi/6}^{\pi/2}$$
$$= -\frac{3}{4}\pi R^3$$

By observation,
$$A_{\text{base}} = \frac{\pi}{4}R^2, \quad \bar{y}_{\text{base}} = -\frac{\sqrt{3}}{2}R$$

Now
$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

or
$$\bar{Y}\left(\pi\sqrt{3}R^2 + \frac{\pi}{4}R^2\right) = -\frac{3}{4}\pi R^3 + \frac{\pi}{4}R^2\left(-\frac{\sqrt{3}}{2}R\right)$$

or
$$\bar{Y} = -0.48763R \qquad R = 250 \text{ mm}$$

$\bar{Y} = -121.9$ mm ◀

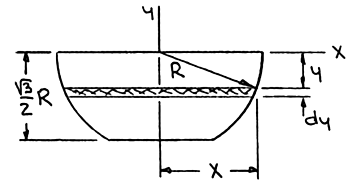
(*b*) Punch:

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

and that because the punch is homogeneous, its center of gravity will coincide with the centroid of the corresponding volume. Choose as the element of volume a disk of radius x and thickness dy. Then

$$dV = \pi x^2 dy, \quad \bar{y}_{EL} = y$$



Now $x^2 + y^2 = R^2$

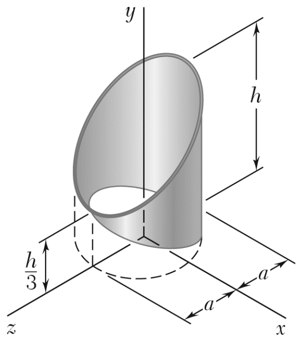
so that $dV = \pi(R^2 - y^2)dy$

Then
$$V = \int_{-\sqrt{3}/2R}^{0} \pi(R^2 - y^2)\,dy$$
$$= \pi\left[R^2 y - \frac{1}{3}y^3\right]_{-\sqrt{3}/2R}^{0}$$
$$= -\pi\left[R^2\left(-\frac{\sqrt{3}}{2}R\right) - \frac{1}{3}\left(-\frac{\sqrt{3}}{2}R\right)^3\right] = \frac{3}{8}\pi\sqrt{3}R^3$$

and
$$\int \bar{y}_{EL}\,dV = \int_{-\sqrt{3}/2R}^{0} (y)\left[\pi\left(R^2 - y^2\right)dy\right]$$
$$= \pi\left[\frac{1}{2}R^2 y^2 - \frac{1}{4}y^4\right]_{-\sqrt{3}/2R}^{0}$$
$$= -\pi\left[\frac{1}{2}R^2\left(-\frac{\sqrt{3}}{2}R\right)^2 - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}R\right)^4\right] = -\frac{15}{64}\pi R^4$$

Now
$$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}\left(\frac{3}{8}\pi\sqrt{3}R^3\right) = -\frac{15}{64}\pi R^4$$

or
$$\bar{y} = -\frac{5}{8\sqrt{3}}R \quad R = 250 \text{ mm} \qquad \bar{y} = -90.2 \text{ mm} \;◀$$



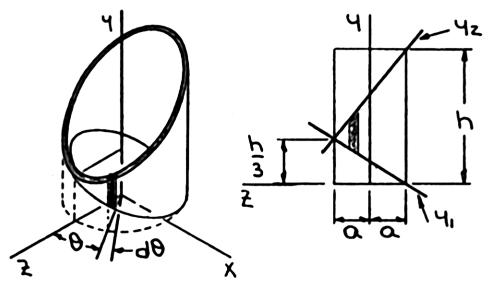
PROBLEM 5.133

Locate the centroid of the section shown, which was cut from a thin circular pipe by two oblique planes.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

Assume that the pipe has a uniform wall thickness t and choose as the element of volume a vertical strip of width $ad\theta$ and height $(y_2 - y_1)$. Then



$$dV = (y_2 - y_1)ta\,d\theta, \quad \bar{y}_{EL} = \frac{1}{2}(y_1 + \bar{y}_2)\bar{z}_{EL} = z$$

Now
$$y_1 = \frac{\frac{h}{3}}{2a}z + \frac{h}{6} \qquad y_2 = -\frac{\frac{2h}{3}}{2a}z + \frac{2}{3}h$$

$$= \frac{h}{6a}(z+a) \qquad = \frac{h}{3a}(-z+2a)$$

and
$$z = a\cos\theta$$

Then
$$(y_2 - y_1) = \frac{h}{3a}(-a\cos\theta + 2a) - \frac{h}{6a}(a\cos\theta + a)$$

$$= \frac{h}{2}(1-\cos\theta)$$

and
$$(y_1 + y_2) = \frac{h}{6a}(a\cos\theta + a) + \frac{h}{3a}(-a\cos\theta + 2a)$$

$$= \frac{h}{6}(5-\cos\theta)$$

$$dV = \frac{aht}{2}(1-\cos\theta)d\theta \quad \bar{y}_{EL} = \frac{h}{12}(5-\cos\theta), \quad \bar{z}_{EL} = a\cos\theta$$

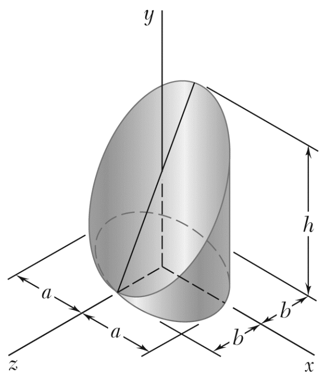
Then $$V = 2\int_0^{\pi} \frac{aht}{2}(1-\cos\theta)d\theta = aht[\theta - \sin\theta]_0^{\pi}$$
$$= \pi aht$$

and $$\int \overline{y}_{EL} dV = 2\int_0^{\pi} \frac{h}{12}(5-\cos\theta)\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$
$$= \frac{ah^2t}{12}\int_0^{\pi}(5 - 6\cos\theta + \cos^2\theta)d\theta$$
$$= \frac{ah^2t}{12}\left[5\theta - 6\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$
$$= \frac{11}{24}\pi ah^2t$$

$$\int \overline{z}_{EL} dV = 2\int_0^{\pi} a\cos\theta\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$
$$= a^2ht\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$
$$= -\frac{1}{2}\pi a^2ht$$

Now $$\overline{y}V = \int \overline{y}_{EL} dV: \quad \overline{y}(\pi aht) = \frac{11}{24}\pi ah^2t$$ or $\overline{y} = \frac{11}{24}h$ ◀

and $$\overline{z}V = \int \overline{z}_{EL} dV: \quad \overline{z}(\pi aht) = -\frac{1}{2}\pi a^2ht$$ or $\overline{z} = -\frac{1}{2}a$ ◀

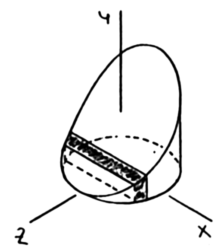


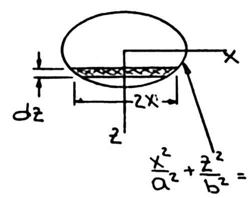
PROBLEM 5.134*

Locate the centroid of the section shown, which was cut from an elliptical cylinder by an oblique plane.

SOLUTION

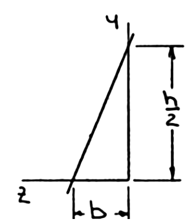
First note that symmetry implies $x = 0$ ◀





Choose as the element of volume a vertical slice of width zx, thickness dz, and height y. Then

$$dV = 2xy\,dz, \quad \bar{y}_{EL} = \frac{1}{24}, \quad \bar{z}_{EL} = z$$



Now
$$x = \frac{a}{b}\sqrt{b^2 - z^2}$$

and
$$y = -\frac{h/2}{b}z + \frac{h}{2} = \frac{h}{2b}(b - z)$$

Then
$$V = \int_{-b}^{b}\left(2\frac{a}{b}\sqrt{b^2 - z^2}\right)\left[\frac{h}{2b}(b - z)\right]dz$$

Let
$$z = b\sin\theta \quad dz = b\cos\theta\, d\theta$$

Then
$$V = \frac{ah}{b^2}\int_{\pi/2}^{\pi/2}(b\cos\theta)[b(1-\sin\theta)]b\cos\theta\, d\theta$$

$$= abh\int_{-\pi/2}^{\pi/2}(\cos^2\theta - \sin\theta\cos^2\theta)\, d\theta$$

$$= abh\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2}$$

$$V = \frac{1}{2}\pi abh$$

PROBLEM 5.134* (Continued)

and $$\int \bar{y}_{EL}dV = \int_{-b}^{b}\left[\frac{1}{2}\times\frac{h}{2b}(b-z)\right]\left\{\left(2\frac{a}{b}\sqrt{b^2-z^2}\right)\left[\frac{h}{2b}(b-z)\right]dz\right\}$$

$$=\frac{1}{4}\frac{ah^2}{b^3}\int_{-b}^{b}(b-z)^2\sqrt{b^2-z^2}\,dz$$

Let $$z=b\sin\theta \quad dz=b\cos\theta\, d\theta$$

Then $$\int \bar{y}_{EL}dV = \frac{1}{4}\frac{ah^2}{b^3}\int_{-\pi/2}^{\pi/2}[b(1-\sin\theta)]^2(b\cos\theta)\times(b\cos\theta\, d\theta)$$

$$=\frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}(\cos^2\theta-2\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta)\,d\theta$$

Now $$\sin^2\theta=\frac{1}{2}(1-\cos 2\theta) \quad \cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

so that $$\sin^2\theta\cos^2\theta=\frac{1}{4}(1-\cos^2 2\theta)$$

Then $$\int \bar{y}_{EL}dV = \frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}\left[\cos^2\theta-2\sin\theta\cos^2\theta+\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$

$$=\frac{1}{4}abh^2\left[\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)+\frac{1}{3}\cos^3\theta+\frac{1}{4}\theta-\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2}$$

$$=\frac{5}{32}\pi abh^2$$

Also, $$\int \bar{z}_{EL}dV = \int_{-b}^{b} z\left\{2\frac{a}{b}\sqrt{a^2-z^2}\left[\frac{h}{2b}(b-z)\right]dz\right\}$$

$$=\frac{ah}{b^2}\int_{-b}^{b} z(b-z)\sqrt{b^2-z^2}\,dz$$

Let $$z=b\sin\theta \quad dz=b\cos\theta\, d\theta$$

Then $$\int \bar{z}_{EL}dV = \frac{ah}{b^2}\int_{-\pi/2}^{\pi/2}(b\sin\theta)[b(1-\sin\theta)](b\cos\theta)\times(b\cos\theta\, d\theta)$$

$$=ab^2h\int_{-\pi/2}^{\pi/2}(\sin\theta\cos^2\theta-\sin^2\theta\cos^2\theta)d\theta$$

Using $\sin^2\theta\cos^2\theta=\frac{1}{4}(1-\cos^2 2\theta)$ from above,

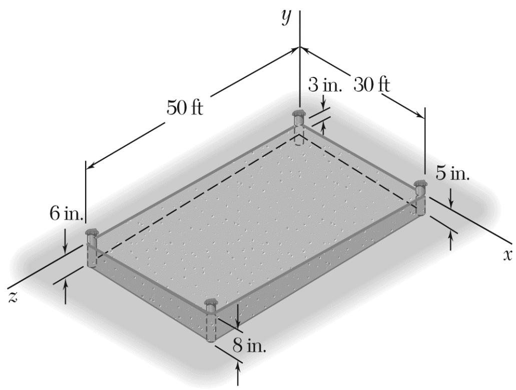
$$\int z_{EL}dV = ab^2h\int_{-\pi/2}^{\pi/2}\left[\sin\theta\cos^2\theta-\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$

$$=ab^2h\left[-\frac{1}{3}\cos^3\theta-\frac{1}{4}\theta+\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2}=-\frac{1}{8}\pi ab^2h$$

PROBLEM 5.134* (Continued)

Now $\quad \bar{y}V = \int \bar{y}_{EL}dV: \quad \bar{y}\left(\frac{1}{2}\pi abh\right) = \frac{5}{32}\pi abh^2 \quad$ or $\quad \bar{y} = \frac{5}{16}h$ ◀

and $\quad \bar{z}V = \int \bar{z}_{EL}dV: \quad \bar{z}\left(\frac{1}{2}\pi abh\right) = -\frac{1}{8}\pi ab^2h \quad$ or $\quad \bar{z} = -\frac{1}{4}b$ ◀

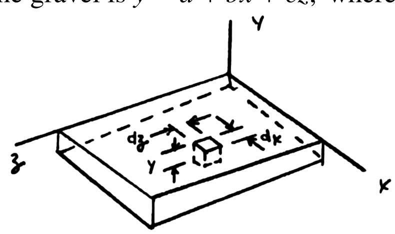


PROBLEM 5.135

After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 3 in. of gravel beneath the slab. Determine the volume of gravel needed and the x coordinate of the centroid of the volume of the gravel. (*Hint:* The bottom surface of the gravel is an oblique plane, which can be represented by the equation $y = a + bx + cz$.)

SOLUTION

The centroid can be found by integration. The equation for the bottom of the gravel is $y = a + bx + cz$, where the constants a, b, and c can be determined as follows:



For $x = 0$ and $z = 0$, $y = -3$ in., and therefore,

$$-\frac{3}{12}\text{ ft} = a, \quad \text{or} \quad a = -\frac{1}{4}\text{ ft}$$

For $x = 30$ ft and $z = 0$, $y = -5$ in., and therefore,

$$-\frac{5}{12}\text{ ft} = -\frac{1}{4}\text{ ft} + b(30\text{ ft}), \quad \text{or} \quad b = -\frac{1}{180}$$

For $x = 0$ and $z = 50$ ft, $y = -6$ in., and therefore,

$$-\frac{6}{12}\text{ ft} = -\frac{1}{4}\text{ ft} + c(50\text{ ft}), \quad \text{or} \quad c = -\frac{1}{200}$$

Therefore,

$$y = -\frac{1}{4}\text{ ft} - \frac{1}{180}x - \frac{1}{200}z$$

Now

$$\bar{x} = \frac{\int x_{EL}\,dV}{V}$$

A volume element can be chosen as

$$dV = |y|\,dx\,dz$$

or

$$dV = \frac{1}{4}\left(1 + \frac{1}{45}x + \frac{1}{50}z\right)dx\,dz$$

and

$$\bar{x}_{EL} = x$$

Then $$\int x_{EL}\,dV = \int_0^{50}\int_0^{30} \frac{x}{4}\left(1+\frac{1}{45}x+\frac{1}{50}z\right)dx\,dz$$

$$= \frac{1}{4}\int_0^{50}\left[\frac{x^2}{2}+\frac{1}{135}x^3+\frac{z}{100}x^2\right]_0^{30} dz$$

$$= \frac{1}{4}\int_0^{50}(650+9z)\,dz$$

$$= \frac{1}{4}\left[650\,z+\frac{9}{2}z^2\right]_0^{50}$$

$$= 10937.5 \text{ ft}^4$$

The volume is $$V\int dV = \int_0^{50}\int_0^{30}\frac{1}{4}\left(1+\frac{1}{45}x+\frac{1}{50}z\right)dx\,dz$$

$$= \frac{1}{4}\int_0^{50}\left[x+\frac{1}{90}x^2+\frac{z}{50}x\right]_0^{30} dz$$

$$= \frac{1}{4}\int_0^{50}\left(40+\frac{3}{5}z\right)dz$$

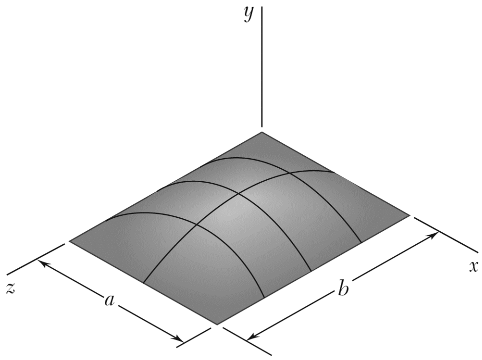
$$= \frac{1}{4}\left[40z+\frac{3}{10}z^2\right]_0^{50}$$

$$= 687.50 \text{ ft}^3$$

Then $$\overline{x} = \frac{\int \overline{x}_{EL}\,dV}{V} = \frac{10937.5 \text{ ft}^4}{687.5 \text{ ft}^3} = 15.9091 \text{ ft}$$

Therefore, $V = 688 \text{ ft}^3$ ◀

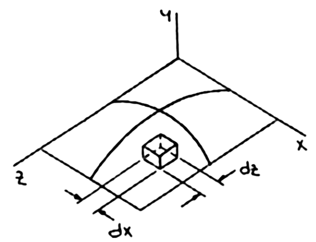
$\overline{x} = 15.91 \text{ ft}$ ◀



PROBLEM 5.136

Determine by direct integration the location of the centroid of the volume between the xz plane and the portion shown of the surface $y = 16h(ax - x^2)(bz - z^2)/a^2b^2$.

SOLUTION



First note that symmetry implies $\bar{x} = \dfrac{a}{2}$ ◀

$\bar{z} = \dfrac{b}{2}$ ◀

Choose as the element of volume a filament of base $dx \times dz$ and height y. Then

$$dV = y\,dx\,dz, \quad \bar{y}_{EL} = \frac{1}{2}y$$

or
$$dV = \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)dx\,dz$$

Then
$$V = \int_0^b \int_0^a \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)dx\,dz$$

$$V = \frac{16h}{a^2b^2}\int_0^b (bz - z^2)\left[\frac{a}{z}x^2 - \frac{1}{3}x^3\right]_0^a dz$$

$$= \frac{16h}{a^2b^2}\left[\frac{a}{2}(a^2) - \frac{1}{3}(a)^3\right]\left[\frac{b}{2}z^2 - \frac{1}{3}z^3\right]_0^b$$

$$= \frac{8ah}{3b^2}\left[\frac{b}{2}(b)^2 - \frac{1}{3}(b)^3\right]$$

$$= \frac{4}{9}abh$$

PROBLEM 5.136 (Continued)

and $$\int \bar{y}_{EL}dV = \int_0^b\int_0^a \frac{1}{2}\left[\frac{16h}{a^2b^2}(ax-x^2)(bz-z^2)\right]\left[\frac{16h}{a^2b^2}(ax-x^2)(bz-z^2)dx\,dz\right]$$

$$= \frac{128h^2}{a^4b^4}\int_0^b\int_0^a (a^2x^2-2ax^3+x^4)(b^2z^2-2bz^3+z^4)dx\,dz$$

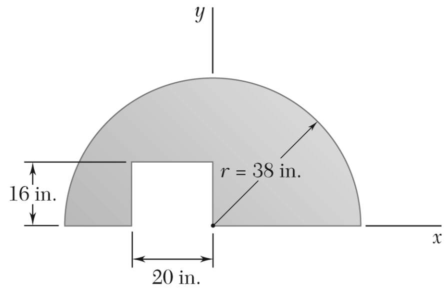
$$= \frac{128h^2}{a^2b^4}\int_0^b (b^2z^2-2bz^3+z^4)\left[\frac{a^2}{3}x^3-\frac{a}{2}x^4+\frac{1}{5}x^5\right]_0^a dz$$

$$= \frac{128h^2}{a^4b^4}\left[\frac{a^2}{3}(a)^3-\frac{a}{2}(a)^4+\frac{1}{5}(a)^5\right]\left[\frac{b^2}{3}z^3-\frac{b}{z}z^4+\frac{1}{5}z^5\right]_0^b$$

$$= \frac{64ah^2}{15b^4}\left[\frac{b^3}{3}(b)^3-\frac{b}{2}(b)^4+\frac{1}{5}(b)^5\right] = \frac{32}{225}abh^2$$

Now $$\bar{y}V = \int \bar{y}_{EL}dV: \quad \bar{y}\left(\frac{4}{9}abh\right) = \frac{32}{225}abh^2$$

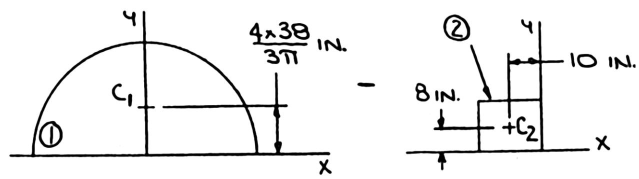
or $\bar{y} = \dfrac{8}{25}h$ ◀



PROBLEM 5.137

Locate the centroid of the plane area shown.

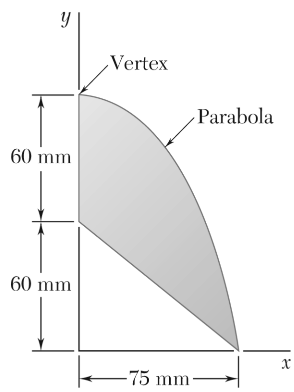
SOLUTION



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{\pi}{2}(38)^2 = 2268.2$	0	16.1277	0	36,581
2	$-20\times16 = 320$	-10	8	3200	-2560
Σ	1948.23			3200	34,021

Then $$\bar{X} = \frac{\Sigma\bar{x}A}{\Sigma A} = \frac{3200}{1948.23}$$ $\bar{X} = 1.643$ in. ◀

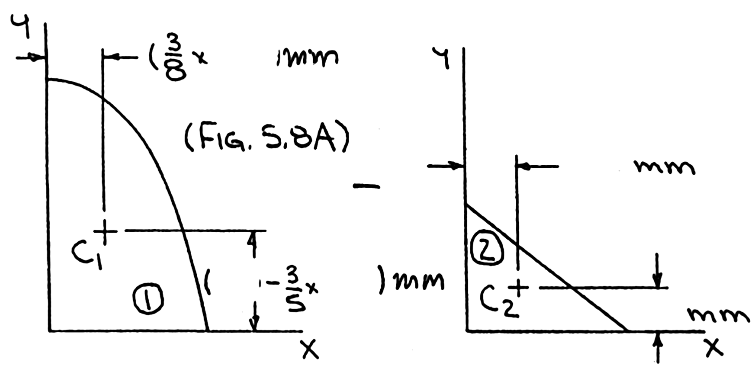
$$\bar{Y} = \frac{\Sigma\bar{y}A}{\Sigma A} = \frac{34{,}021}{1948.23}$$ $\bar{Y} = 17.46$ in. ◀



PROBLEM 5.138

Locate the centroid of the plane area shown.

SOLUTION



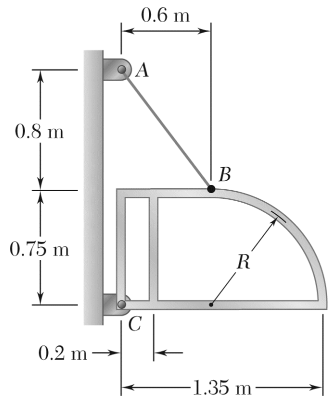
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}(75)(120) = 6000$	28.125	48	168,750	288,000
2	$-\frac{1}{2}(75)(60) = -2250$	25	20	–56,250	–45,000
Σ	3750			112,500	243,000

Then $$\bar{X}\Sigma A = \Sigma \bar{x}A$$

$$\bar{X}(3750 \text{ mm}^2) = 112{,}500 \text{ mm}^3 \qquad \text{or} \quad \bar{X} = 30.0 \text{ mm} \blacktriangleleft$$

and $$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(3750 \text{ mm}^2) = 243{,}000 \text{ mm}^3 \qquad \text{or} \quad \bar{Y} = 64.8 \text{ mm} \blacktriangleleft$$



PROBLEM 5.139

The frame for a sign is fabricated from thin, flat steel bar stock of mass per unit length 4.73 kg/m. The frame is supported by a pin at C and by a cable AB. Determine (*a*) the tension in the cable, (*b*) the reaction at C.

SOLUTION

First note that because the frame is fabricated from uniform bar stock, its center of gravity will coincide with the centroid of the corresponding line.

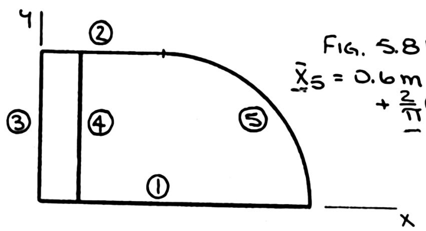
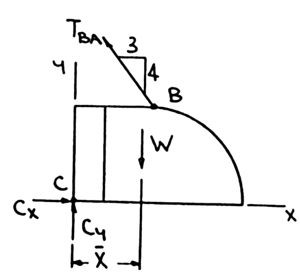


FIG. 5.8B
$\bar{X}_5 = 0.6\text{ m} + \frac{2}{\pi}(0.75\text{ m})$

	L, m	$\bar{x}$, m	$\bar{x}L$, m^2
1	1.35	0.675	0.91125
2	0.6	0.3	0.18
3	0.75	0	0
4	0.75	0.2	0.15
5	$\frac{\pi}{2}(0.75) = 1.17810$	1.07746	1.26936
Σ	4.62810		2.5106

Then
$$\bar{X}\Sigma L = \Sigma \bar{x} L$$
$$\bar{X}(4.62810) = 2.5106$$

or
$$\bar{X} = 0.54247 \text{ m}$$



The free-body diagram of the frame is then

where
$$W = (m'\Sigma L)g$$
$$= 4.73 \text{ kg/m} \times 4.62810 \text{ m} \times 9.81 \text{ m/s}^2$$
$$= 214.75 \text{ N}$$

PROBLEM 5.139 (Continued)

Equilibrium then requires

(*a*) $\Sigma M_C = 0$: $(1.55 \text{ m})\left(\frac{3}{5}T_{BA}\right) - (0.54247 \text{ m})(214.75 \text{ N}) = 0$

or $T_{BA} = 125.264 \text{ N}$ or $T_{BA} = 125.3 \text{ N}$ ◀

(*b*) $\Sigma F_x = 0$: $C_x - \frac{3}{5}(125.264 \text{ N}) = 0$

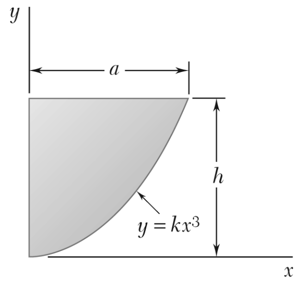
or $\mathbf{C}_x = 75.158 \text{ N} \longrightarrow$

$\Sigma F_y = 0$: $C_y + \frac{4}{5}(125.264 \text{ N}) - (214.75 \text{ N}) = 0$

or $\mathbf{C}_y = 114.539 \text{ N} \uparrow$

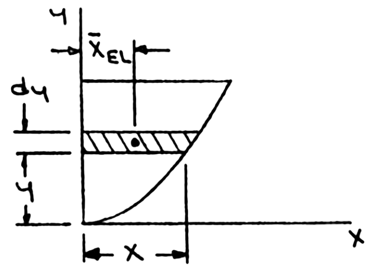
Then $\mathbf{C} = 137.0 \text{ N} \measuredangle 56.7°$ ◀

PROBLEM 5.140



Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

SOLUTION



For the element (EL) shown,

at $x = a,\ y = h,\quad h = ka^3$ or $k = \dfrac{h}{a^3}$

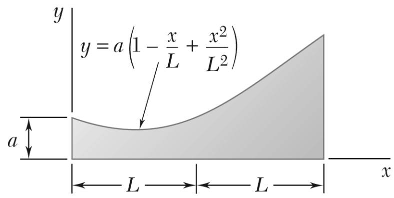
Then
$$x = \frac{a}{h^{1/3}} y^{1/3}$$

Now
$$dA = x\,dy = \frac{a}{h^{1/3}} y^{1/3} dy$$
$$\bar{x}_{EL} = \frac{1}{2}x = \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3}$$
$$\bar{y}_{EL} = y$$

Then
$$A = \int dA = \int_0^h \frac{a}{h^{1/3}} y^{1/3} dy = \frac{3}{4}\frac{a}{h^{1/3}}\left(y^{4/3}\right)\Big|_0^h = \frac{3}{4}ah$$

and
$$\int \bar{x}_{EL}\,dA = \int_0^h \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3}\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{1}{2}\frac{a}{h^{2/3}}\left(\frac{3}{5} y^{5/3}\right)\Big|_0^h = \frac{3}{10}a^2 h$$
$$\int \bar{y}_{EL}\,dA = \int_0^h y\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{a}{h^{1/3}}\left(\frac{3}{7} y^{7/3}\right)\Big|_0^h = \frac{3}{7}ah^2$$

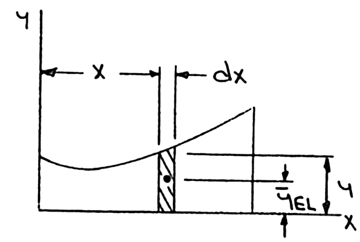
Hence
$$\bar{x}A = \int \bar{x}_{EL}\,dA:\quad \bar{x}\left(\frac{3}{4}ah\right) = \frac{3}{10}a^2 h \qquad \bar{x} = \frac{2}{5}a \blacktriangleleft$$
$$\bar{y}A = \int \bar{y}_{EL}\,dA:\quad \bar{y}\left(\frac{3}{4}ah\right) = \frac{3}{7}ah^2 \qquad \bar{y} = \frac{4}{7}h \blacktriangleleft$$



PROBLEM 5.141

Determine by direct integration the centroid of the area shown.

SOLUTION



We have

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)$$

$$dA = y\,dx = a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx$$

Then

$$A = \int dA = \int_0^{2L} a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx = a\left[x - \frac{x^2}{2L} + \frac{x^3}{3L^2}\right]_0^{2L}$$

$$= \frac{8}{3}aL$$

and

$$\int \bar{x}_{EL}\,dA = \int_0^{2L} x\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right] = a\left[\frac{x^2}{2} - \frac{x^3}{3L} + \frac{x^4}{4L^2}\right]_0^{2L}$$

$$= \frac{10}{3}aL^2$$

$$\int \bar{y}_{EL}\,dA = \int_0^{2L} \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right]$$

$$= \frac{a^2}{2}\int_0^{EL}\left(1 - 2\frac{x}{L} + 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right)dx$$

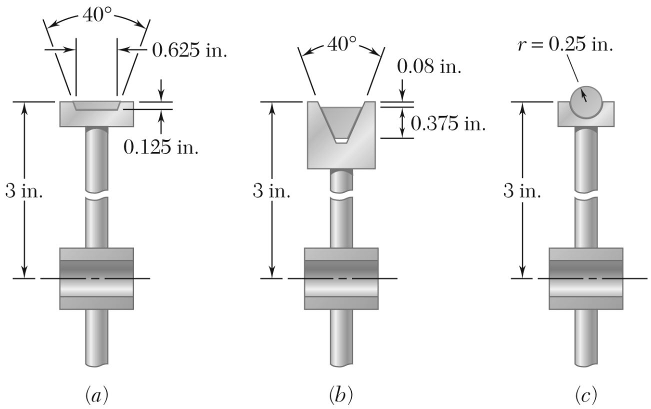
$$= \frac{a^2}{2}\left[x - \frac{x^2}{L} + \frac{x^3}{L^2} - \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^{2L}$$

$$= \frac{11}{5}a^2L$$

Hence,

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{8}{3}aL\right) = \frac{10}{3}aL^2 \qquad \bar{x} = \frac{5}{4}L \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{1}{8}a\right) = \frac{11}{5}a^2 \qquad \bar{y} = \frac{33}{40}a \blacktriangleleft$$



PROBLEM 5.142

Three different drive belt profiles are to be studied. If at any given time each belt makes contact with one-half of the circumference of its pulley, determine the *contact area* between the belt and the pulley for each design.

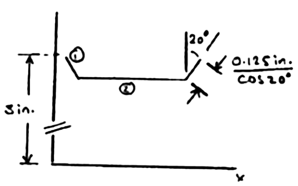
SOLUTION

SOLUTION

Applying the first theorem of Pappus-Guldinus, the contact area A_C of a belt is given by

$$A_C = \pi \bar{y} L = \pi \Sigma \bar{y} L$$

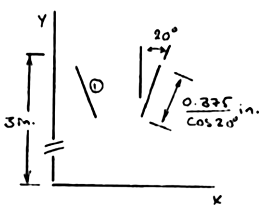
where the individual lengths are the lengths of the belt cross section that are in contact with the pulley.



(*a*) $A_C = \pi[2(\bar{y}_1 L_1) + \bar{y}_2 L_2]$

$$= \pi\left\{2\left[\left(3 - \frac{0.125}{2}\right)\text{in.}\right]\left[\frac{0.125\text{ in.}}{\cos 20°}\right] + [(3-0.125)\text{ in.}](0.625\text{ in.})\right\}$$

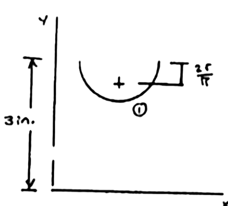
or $A_C = 8.10\text{ in}^2$ ◀



(*b*) $A_C = \pi[2(\bar{y}_1 L_1)]$

$$= 2\pi\left[\left(3 - 0.08 - \frac{0.375}{2}\right)\text{in.}\right]\left(\frac{0.375\text{ in.}}{\cos 20°}\right)$$

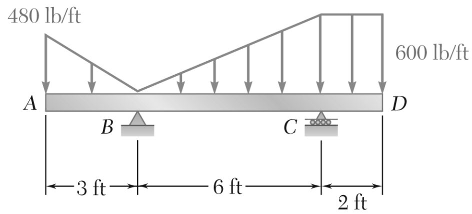
or $A_C = 6.85\text{ in}^2$ ◀



(*c*) $A_C = \pi[2(\bar{y}_1 L_1)]$

$$= \pi\left[\left(3 - \frac{2(0.25)}{\pi}\right)\text{in.}\right][\pi(0.25\text{ in.})]$$

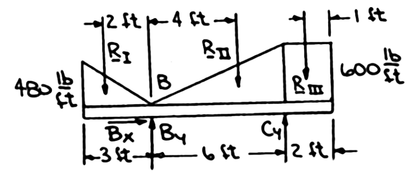
or $A_C = 7.01\text{ in}^2$ ◀



PROBLEM 5.143

Determine the reactions at the beam supports for the given loading.

SOLUTION



We have

$$R_{\mathrm{I}} = \frac{1}{2}(3\text{ ft})(480\text{ lb/ft}) = 720\text{ lb}$$

$$R_{\mathrm{II}} = \frac{1}{2}(6\text{ ft})(600\text{ lb/ft}) = 1800\text{ lb}$$

$$R_{\mathrm{III}} = (2\text{ ft})(600\text{ lb/ft}) = 1200\text{ lb}$$

Then

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (2\text{ ft})(720\text{ lb}) - (4\text{ ft})(1800\text{ lb}) + (6\text{ ft})C_y - (7\text{ ft})(1200\text{ lb}) = 0$$

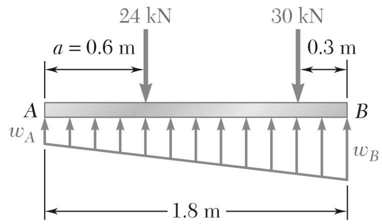
or

$$C_y = 2360\text{ lb} \qquad \mathbf{C} = 2360\text{ lb}\uparrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -720\text{ lb} + B_y - 1800\text{ lb} + 2360\text{ lb} - 1200\text{ lb} = 0$$

or

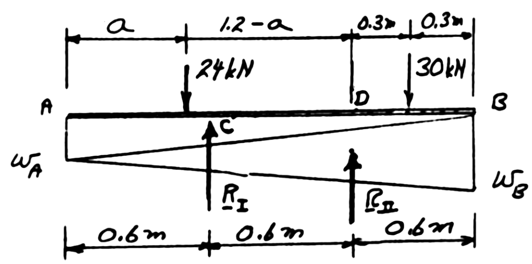
$$B_y = 1360\text{ lb} \qquad \mathbf{B} = 1360\text{ lb}\uparrow \blacktriangleleft$$



PROBLEM 5.144

The beam *AB* supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of ω_A and ω_B corresponding to equilibrium.

SOLUTION



$$R_{\mathrm{I}} = \frac{1}{2}\omega_A(1.8 \text{ m}) = 0.9\omega_A$$

$$R_{\mathrm{II}} = \frac{1}{2}\omega_B(1.8 \text{ m}) = 0.9\omega_B$$

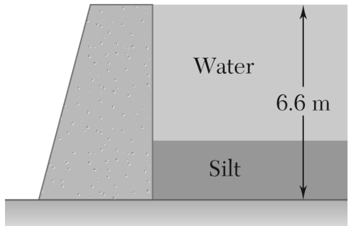
$$+\circlearrowleft \Sigma M_D = 0: \quad (24 \text{ kN})(1.2 - a) - (30 \text{ kN})(0.3 \text{ m}) - (0.9\omega_A)(0.6 \text{ m}) = 0 \qquad (1)$$

For $a = 0.6$ m,

$$24(1.2 - 0.6) - (30)(0.3) - 0.54\omega_a = 0$$

$$14.4 - 9 - 0.54\omega_A = 0 \qquad \omega_A = 10.00 \text{ kN/m} \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -24 \text{ kN} - 30 \text{ kN} + 0.9(10 \text{ kN/m}) + 0.9\omega_B = 0 \qquad \omega_B = 50.0 \text{ kN/m} \blacktriangleleft$$

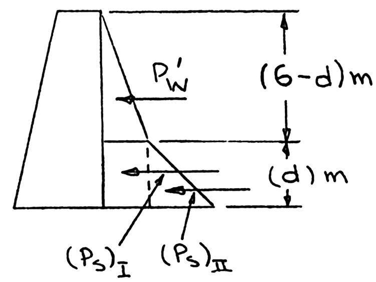


PROBLEM 5.145

The base of a dam for a lake is designed to resist up to 120 percent of the horizontal force of the water. After construction, it is found that silt (that is equivalent to a liquid of density $\rho_s = 1.76 \times 10^3 \text{ kg/m}^3$) is settling on the lake bottom at the rate of 12 mm/year. Considering a 1-m-wide section of dam, determine the number of years until the dam becomes unsafe.

SOLUTION

First determine force on dam without the silt,



$$P_w = \frac{1}{2} A_{p_w} = \frac{1}{2} A(\rho g h)$$

$$= \frac{1}{2}[(6.6 \text{ m})(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 \text{ m})]$$

$$= 213.66 \text{ kN}$$

$$P_{\text{allow}} = 1.2 P_w = (1.5)(213.66 \text{ kN}) = 256.39 \text{ kN}$$

Next determine the force P' on the dam face after a depth d of silt has settled.

We have
$$P'_w = \frac{1}{2}[(6.6 - d) \text{ m} \times (1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 - d) \text{ m}]$$

$$= 4.905(6.6 - d)^2 \text{ kN}$$

$$(P_s)_{\text{I}} = [d(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 - d) \text{ m}]$$

$$= 9.81(6.6d - d^2) \text{ kN}$$

$$(P_s)_{\text{II}} = \frac{1}{2}[d(1 \text{ m})][(1.76 \times 10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(d) \text{ m}]$$

$$= 8.6328 d^2 \text{ kN}$$

$$P' = P'_w + (P_s)_{\text{I}} + (P_s)_{\text{II}} = [4.905(43.560 - 13.2000d + d^2) + 9.81(6.6d - d^2) + 8.6328d^2] \text{ kN}$$

$$= [3.7278d^2 + 213.66] \text{ kN}$$

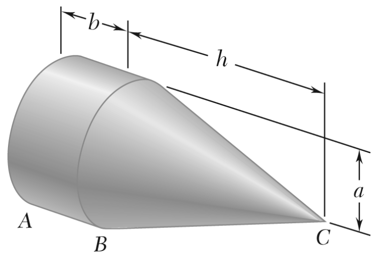
Now it's required that $P' = P_{\text{allow}}$ to determine the maximum value of d.

$$(3.7278d^2 + 213.66) \text{ kN} = 256.39 \text{ kN}$$

or
$$d = 3.3856 \text{ m}$$

Finally,
$$3.3856 \text{ m} = 12 \times 10^{-3} \frac{\text{m}}{\text{year}} \times N$$

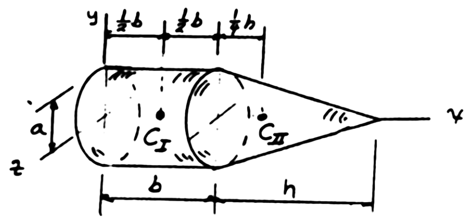
or $N = 282$ years ◀



PROBLEM 5.146

Determine the location of the centroid of the composite body shown when (*a*) $h = 2b$, (*b*) $h = 2.5b$.

SOLUTION



	V	$\bar{x}$	$\bar{x}V$
Cylinder I	$\pi a^2 b$	$\frac{1}{2}b$	$\frac{1}{2}\pi a^2 b^2$
Cone II	$\frac{1}{3}\pi a^2 h$	$b + \frac{1}{4}h$	$\frac{1}{3}\pi a^2 h\left(b + \frac{1}{4}h\right)$

$$V = \pi a^2\left(b + \frac{1}{3}h\right)$$

$$\Sigma \bar{x}V = \pi a^2\left(\frac{1}{2}b^2 + \frac{1}{3}hb + \frac{1}{12}h^2\right)$$

(*a*) For $h = 2b$,

$$V = \pi a^2\left[b + \frac{1}{3}(2b)\right] = \frac{5}{3}\pi a^2 b$$

$$\Sigma \bar{x}V = \pi a^2\left[\frac{1}{2}b^2 + \frac{1}{3}(2b)b + \frac{1}{12}(2b)^2\right]$$

$$= \pi a^2 b^2\left[\frac{1}{2} + \frac{2}{3} + \frac{1}{3}\right] = \frac{3}{2}\pi a^2 b^2$$

$$\bar{X}V = \Sigma \bar{x}V: \quad \bar{X}\left(\frac{5}{3}\pi a^2 b\right) = \frac{3}{2}\pi a^2 b^2 \qquad \bar{X} = \frac{9}{10}b$$

Centroid is $\frac{1}{10}b$ to left of base of cone. ◀

PROBLEM 5.146 (Continued)

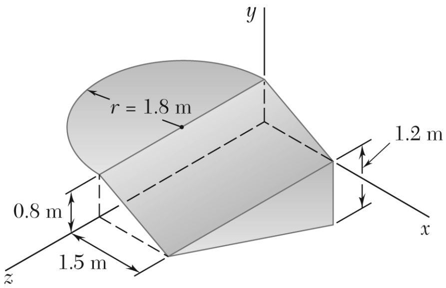
(*b*) For $h = 2.5b$, $$V = \pi a^2\left[b + \frac{1}{3}(2.5b)\right] = 1.8333\pi a^2 b$$

$$\Sigma \bar{x} V = \pi a^2\left[\frac{1}{2}b^2 + \frac{1}{3}(2.5b)b + \frac{1}{12}(2.5b)^2\right]$$
$$= \pi a^2 b^2[0.5 + 0.8333 + 0.52083]$$
$$= 1.85416\pi a^2 b^2$$

$$\bar{X}V = \Sigma\bar{x}V: \quad \bar{X}(1.8333\pi a^2 b) = 1.85416\pi a^2 b^2 \quad \bar{X} = 1.01136b$$

Centroid is $0.01136b$ to right of base of cone. ◀

Note: Centroid is at base of cone for $h = \sqrt{6}b = 2.449b$.

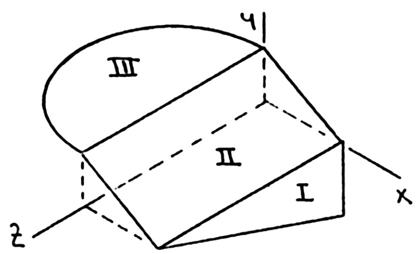


PROBLEM 5.147

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area.



$$\bar{y}_{\mathrm{I}} = -\frac{1}{3}(1.2) = -0.4 \text{ m}$$

$$\bar{z}_{\mathrm{I}} = \frac{1}{3}(3.6) = 1.2 \text{ m}$$

$$\bar{x}_{\mathrm{III}} = -\frac{4(1.8)}{3\pi} = -\frac{2.4}{\pi} \text{ m}$$

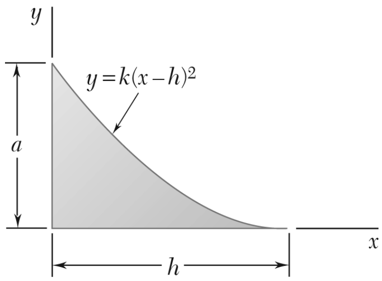
	A, m^2	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}A$, m^3	$\bar{y}A$, m^3	$\bar{z}A$, m^3
I	$\frac{1}{2}(3.6)(1.2) = 2.16$	1.5	−0.4	1.2	3.24	−0.864	2.592
II	$(3.6)(1.7) = 6.12$	0.75	0.4	1.8	4.59	2.448	11.016
III	$\frac{\pi}{2}(1.8)^2 = 5.0894$	$-\frac{2.4}{\pi}$	0.8	1.8	−3.888	4.0715	9.1609
Σ	13.3694				3.942	5.6555	22.769

We have

$\bar{X}\Sigma V = \Sigma \bar{x} V$: $\bar{X}(13.3694 \text{ m}^2) = 3.942 \text{ m}^3$ or $\bar{X} = 0.295$ m ◀

$\bar{Y}\Sigma V = \Sigma \bar{y} V$: $\bar{Y}(13.3694 \text{ m}^2) = 5.6555 \text{ m}^3$ or $\bar{Y} = 0.423$ m ◀

$\bar{Z}\Sigma V = \Sigma \bar{z} V$: $\bar{Z}(13.3694 \text{ m}^2) = 22.769 \text{ m}^3$ or $\bar{Z} = 1.703$ m ◀



PROBLEM 5.148

Locate the centroid of the volume obtained by rotating the shaded area about the x-axis.

SOLUTION

First note that symmetry implies $\qquad \bar{y} = 0$ ◀

and $\qquad \bar{z} = 0$ ◀

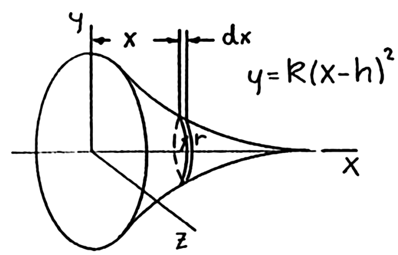
We have $\qquad y = k(X - h)^2$

At $x = 0$, $y = a$, $\qquad a = k(-h)^2$

or $\qquad k = \dfrac{a}{h^2}$

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{X}_{EL} = x$$



Now $\qquad r = \dfrac{a}{h^2}(x-h)^2$

so that $\qquad dV = \pi \dfrac{a^2}{h^4}(x-h)^4 dx$

Then
$$V = \int_0^h \pi \frac{a^2}{h^4}(x-h)^4 dx = \frac{\pi}{5}\frac{a^2}{h^4}[(x-h)^5]_0^h$$
$$= \frac{1}{5}\pi a^2 h$$

and
$$\int \bar{x}_{EL} dV = \int_0^h x\left[\pi \frac{a^2}{h^4}(x-h)^4 dx\right]$$
$$= \pi \frac{a^2}{h^4}\int_0^h (x^5 - 4hx^4 + 6h^2x^3 - 4h^3x^2 + h^4x)dx$$
$$= \pi \frac{a^2}{h^4}\left[\frac{1}{6}x^6 - \frac{4}{5}hx^5 + \frac{3}{2}h^2x^4 - \frac{4}{3}h^3x^3 + \frac{1}{2}h^4x^2\right]_0^h$$
$$= \frac{1}{30}\pi a^2 h^2$$

Now $\qquad \bar{x}V = \int \bar{x}_{EL} dV: \quad \bar{x}\left(\dfrac{\pi}{5}a^2 h\right) = \dfrac{\pi}{30}a^2h^2 \qquad$ or $\quad \bar{x} = \dfrac{1}{6}h$ ◀