Complex number

→ Counting numbers = natural numbers

$\{1, 2, 3, 4, \ldots\}$ ($\mathbb{N}$)

→ Integers $= \{0, \pm 1, \pm 2, \pm 3\}$ ($\mathbb{Z}$)

Close operation: العملية مغلقة

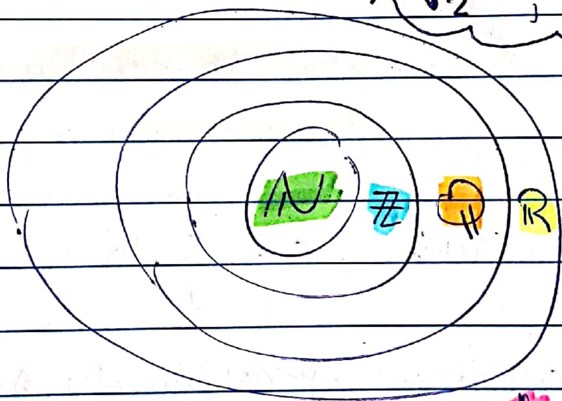
عملية يكون فيها النواتج والعمليات من نفس المجموعة

→ Rational numbers ($\mathbb{Q}$) $= \{ x = \frac{a}{b},\ a, b \in \mathbb{Z} \}$

$\frac{2}{3}$, $\frac{6}{12}$

→ The Real numbers $\mathbb{R}$ "Rational + Non Rational"

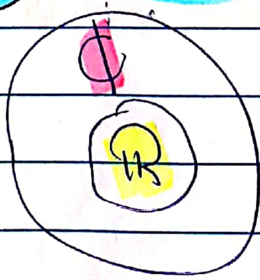
$\sqrt{2}$, π



$\sqrt{-4} \Rightarrow$ complex numbers $\mathbb{C}$

* The Complex numbers is the set of all numbers of the form $z = a + ib$, a, b are real number

$i^2 = -1$, $i = \sqrt{-1}$



Solving Equation

$x + a = 0 \Rightarrow \mathbb{Z}$ "the solution from $\mathbb{Z}$"

$ax + b = 0 \rightarrow \mathbb{Q}$ "the solution from $\mathbb{Q}$"

$ax^2 + bx + c = 0 \Rightarrow$ "Quadratic"

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$

- $< 0 \rightarrow$ No real solutions
- $= 0 \rightarrow$ 1 solution $\rightarrow$ real
- $> 0 \rightarrow$ 2 real solution

Z: Complex number

$z = a + ib \rightarrow$ a = the real part

b = the imaginary part

$i^2 = -1 \rightarrow \sqrt{-1} = i$

$\mathbb{C} = \{ z : z = a + ib \,,\; a, b \in \mathbb{R} \}$

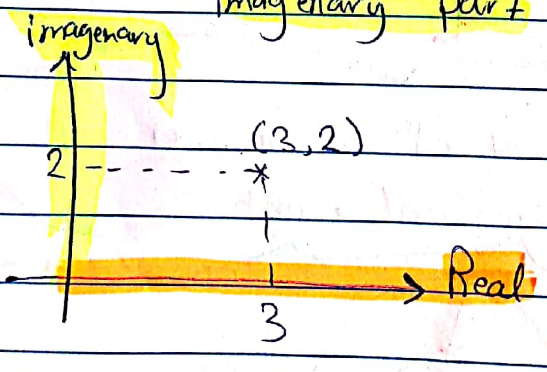
$i = -1$

real part

(a, b) order pair زوج مرتب شكلها ثابت ما بتتغير 8 *

imagenary part

Ex: $3 + 2i$



Notes:

1) $a_1 + b_1 i = a_2 + b_2 i$

$(a_1, b_1) = (a_2, b_2)$

if $a_1 = a_2$

$b_1 = b_2$

Ex: $2+3i = 2+xi$

$\Rightarrow \boxed{x = 3}$

2) $z_1 \pm z_2 = (a_1 + b_1 i) \pm (a_2 + b_2 i)$

$(a_1 + b_1 i) \pm (a_2 + b_2 i)$

$(a_1 \pm a_2) \pm (b_1 + b_2) i$

Ex: $(2-3i) + (4+i) =$

$\boxed{6 - 2i}$

3) $z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= a_1 a_2 + a_1 b_2 i + i b_1 a_2 + i^2 b_1 b_2$

$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$

Ex: $(2+3i)(1-2i) =$

$2 - 4i + 3i + 6$

$= 8 - i$

4) if $z = a + ib$, then the complex Conjugate of z is $\bar{z} = a - ib$

Ex: $z = 2 - 3i$

$\Rightarrow \bar{z} = 2 + 3i$

Ex: $\frac{2+i}{1-i} * \frac{(1+i)}{(1+i)}$

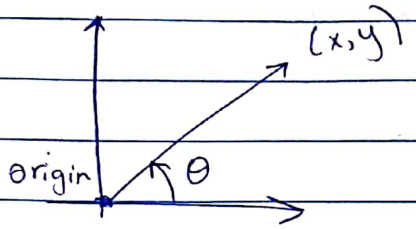
$\frac{(2-1) + (2+1) i}{(1-1) + (1+1)} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3i}{2}$

Ex: $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{(a_2 - b_2 i)}{(a_2 - b_2 i)}$

$\Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Argan Diagrams

$$z = x + iy \;=\; (x, y)$$



$\overrightarrow{OP}$ = Vector

مقدار اتجاه

θ = argument of z (arg Z)

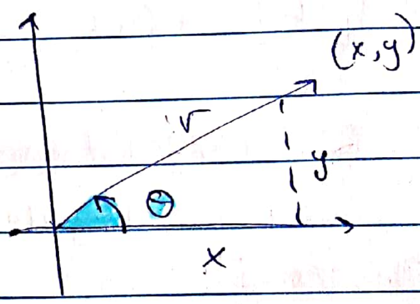
Absolut value = $|z| = |\overrightarrow{OP}|$

$|z| = \sqrt{x^2 + y^2}$ ⇒ مقدار

$\tan\theta = \dfrac{y}{x}$ ⇒ الاتجاه

→ $\cos\theta = \dfrac{x}{r}$ ⇒ $x = r\cos\theta$

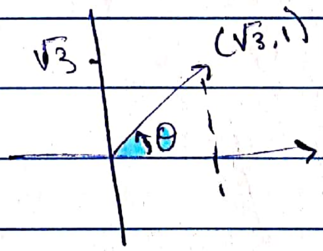
→ $\sin\theta = \dfrac{y}{r}$ ⇒ $y = r\sin\theta$



$z = x + iy$

$z = (r\cos\theta + ri\sin\theta)$

$z = r(\cos\theta + i\sin\theta)$



Ex. $z = 1 + \sqrt{3}i$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\tan\theta = \dfrac{\sqrt{3}}{1} \rightarrow \theta = 60^\circ$

$z = 2(\cos 60 + i\sin 60)$

⇒ $z = 2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$

Note : If $z = x + iy$

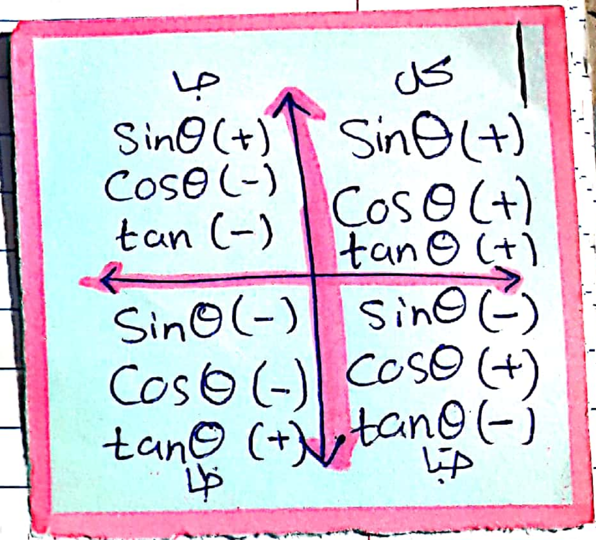
$\bar{z} = x - iy$

then $\Rightarrow z.\bar{z} = (x+iy)\,(x-iy)$

$\Rightarrow x^2 + y^2$

$\Rightarrow |z|^2$

$\Rightarrow \boxed{z.\bar{z} = |z|^2}$



Euleris formula :-

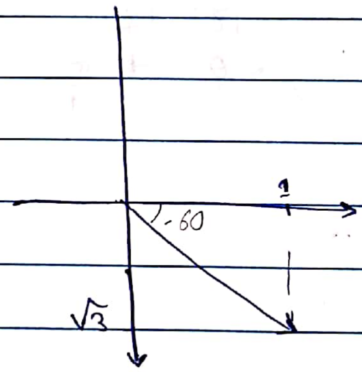
$\rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

Ex : $e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

If $z = x + iy \rightarrow z = r\,[\cos\theta + i\sin\theta]$

$\Rightarrow \boxed{z = re^{i\theta}}$

Ex : $z = 1 - \sqrt{3}\, i$



$r = 2$

$\theta = -60 = -\frac{\pi}{3}$

$z = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$

$= 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$\rightarrow z = 2e^{i\frac{\pi}{3}}$

$$\boxed{z = 2e^{-i\frac{\pi}{3}}}$$

*** If** $z_1 = x_1 + iy_1 \rightarrow z_1 = r_1 e^{i\theta_1}$

$z_2 = x_2 + iy_2 \rightarrow z_2 = r_2 e^{i\theta_2}$

then (1) $z_1 \cdot z_2 = (r_1 e^{i\theta_2}) \times (r_2 e^{i\theta_2})$

$= r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$

(2) $\dfrac{z_1}{z_2} = \left(\dfrac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$

(3) $z = x + iy = re^{i\theta}$

$z^n = (re^{i\theta})^n$

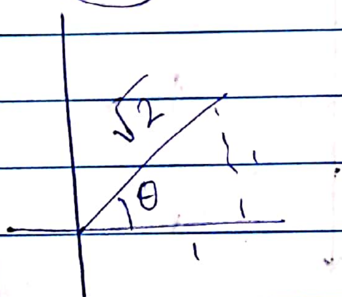
$z^n = r^n e^{i\theta n}$

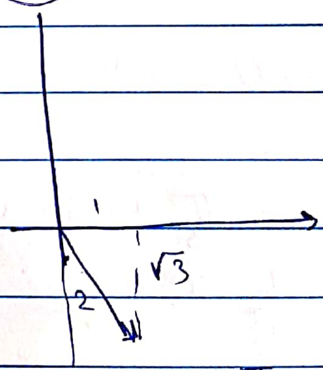
Ex: $z_1 = 1 + i$

$z_2 = \sqrt{3} - i$

z_1 $\rightarrow |z_1| = \sqrt{2}$, $\rightarrow \theta_1 = 45 = \frac{\pi}{4}$

z_2 $\rightarrow r = 2$, $\rightarrow \theta_1 = -\frac{\pi}{6}$





$\rightarrow z_1 \cdot z_2 = (\sqrt{2} \, e^{i\frac{\pi}{4}})(2 e^{-i\frac{\pi}{6}})$

$= 2\sqrt{2} \; e^{i(\frac{\pi}{4} - \frac{\pi}{6})} = 2\sqrt{2} \; e^{i\frac{\pi}{12}}$

$\rightarrow \dfrac{z_1}{z_2} = \dfrac{\sqrt{2} \, e^{i\frac{\pi}{4}}}{2 \, e^{i\frac{\pi}{6}}} = \dfrac{\sqrt{2}}{2} \, e^{i(\frac{5\pi}{12})}$

$\rightarrow z_2^6 = 2^6 (e^{-i\frac{\pi}{6}})^6 = 64 \times e^{-i\pi}$

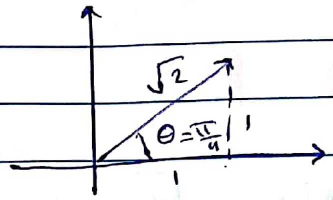
$= 64 [\cos -\pi + i \sin -\pi]$

$= 64 [-1 + i 0 \times 1] = -64$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Ex: $1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$



$r = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$z = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$z^{10} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{10}$

$= 32\, e^{i\frac{\pi 5}{2}}$

$= 32\left[\cos\frac{\pi 5}{2} + i\sin\frac{\pi 5}{2}\right]$

✱ Note: when you solve aproblem in this section ($\sin\theta$ & $\cos\theta$) always prove togather.

Complex Roots:

$$\sqrt[n]{z} = w \Leftrightarrow w^n = z$$

$\Rightarrow \rho^n e^{in\alpha} = re^{i\theta}$

$\rho^n = r \Rightarrow \rho = \sqrt[n]{r}$

$\Rightarrow \dfrac{n\alpha = \theta + 2k\pi}{n} \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}$

$k = 0, \pm 1, \ldots$

$k = 0 \Rightarrow w_0 \Rightarrow \alpha = \frac{\theta}{n}$

$k = 0 \Rightarrow w_0 \Rightarrow \alpha = \frac{\theta}{n} \Rightarrow w_0 = \sqrt[n]{r}\, e^{i(\frac{\theta}{n})}$

$k = 1 \Rightarrow w_1 \Rightarrow w_1 = \sqrt[n]{r}\, e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})}$

$w_{k-1} = \sqrt[n]{r}\, \exp i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)$

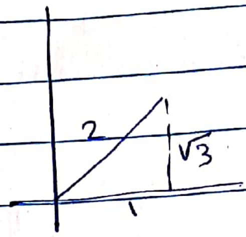
✱ Note:
$\exp = e$
$\exp(f(x)) = e^{f(x)}$

$$k = 0, \ldots, n-1$$

Ex: find the 2 aquared roots of $z = 1+\sqrt{3}\,i$

$r = |z| = 2$

$\theta = \frac{\pi}{3}$



$\sqrt{z} = \sqrt{2e^{i\frac{\pi}{3}}}$

$w_k = \sqrt{r} \cdot \exp i \left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)$

$w_k = \sqrt{r} \; \exp i \left(\frac{\pi}{6} + k\pi\right) \qquad k = 0,1$

$k=0 \Rightarrow w_0 = \sqrt{2}\, \exp i \left(\frac{\pi}{6}\right) = \sqrt{2}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

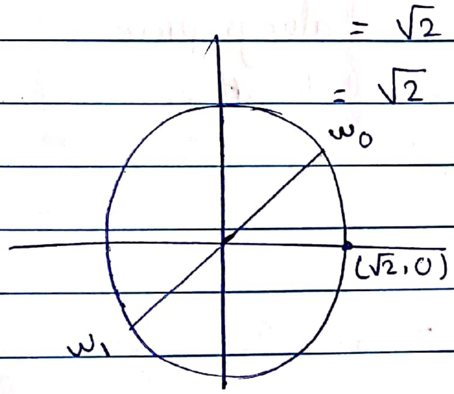
$= \sqrt{2}\left[\frac{\sqrt{3}}{2} + i/2\right]$

$k=1 \Rightarrow w_1 = \sqrt{2}\, \exp i \left(\frac{\pi}{6} + \pi\right)$

$= \sqrt{2}\, \exp i \left(\frac{7\pi}{6}\right)$

$= \sqrt{2}\left[\cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}\right]$

$= \sqrt{2}\left[-\frac{\sqrt{3}}{2} - i/2\right]$



The fundmetal theorm

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z,\ a_0 \Rightarrow a_n \neq 0$

$p(z)$ = Complex ploynomial of degree

$p(z) = 0$ has exacty "n" roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* Note: the number of roots equal n

Ch. 1 : Functions

1.1 Def: A function from a set D to a set R is a rule that assigns a uniqe element $y \in R$ for each element $x \in D$.

$y = f(x) \rightarrow \quad f: D \rightarrow R$

$\quad x \rightarrow y = f(x)$

* in puts → The domain → x → independent
* out puts → The rang → y → dependent

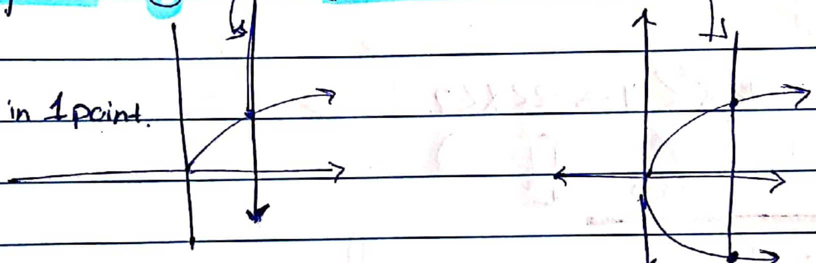
$$y = f(x)$$

Graph :

① << vertical line test >>

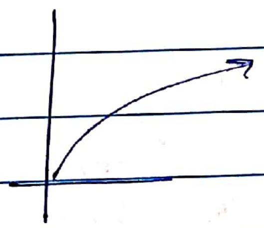
* cross in 1 point.

* cross in 2 or more points.



Function   not a function

①



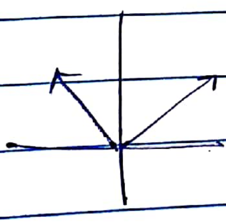
$f(x) = \sqrt{x}$

$D: x \geq 0$

$R: [0, \infty) : y \geq 0$

② $f(x) = |x|$ Absolute value

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



$D = \mathbb{R}$

$R = [0, \infty)$

Note $\sqrt{x^2} = |x|$

1.2 ⇒ Trigonomtre functions

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x = \frac{\sin x}{\cos x}$
4. $\csc x = \frac{1}{\sin x}$
5. $\sec x = \frac{1}{\cos x}$
6. $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$.

Remark:

$\sin(x + 2\pi) = \sin x$.

$\cos(x + 2\pi) = \cos x$.

$\sec(x + 2\pi) = \sec x$.

$\vdots \qquad \vdots$

perodic function

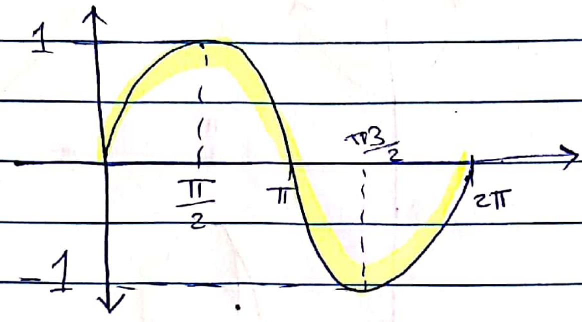
*Def: the function f(x) is a periodic function with period p f(x)+p.

→ $f(x) = \sin x$

$D = (-\infty, \infty)$

$R = [-1, 1]$

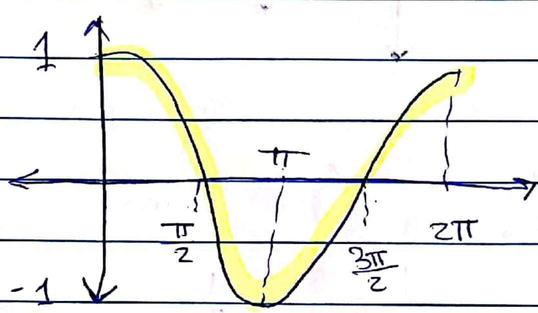
* sin x is periodic with 2x



→ $f(x) = \cos x$

$D = (-\infty, \infty)$

$R = [-1, 1]$



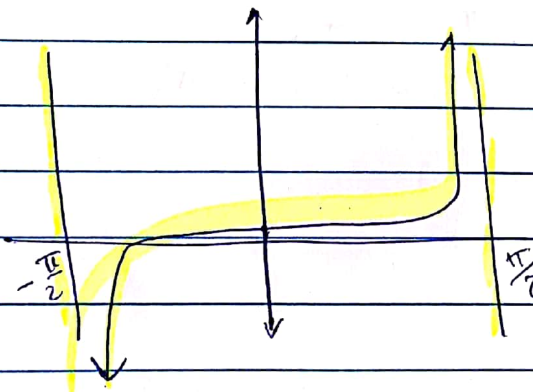
→ $f(x) = \tan x$ ⇒ D: $\cos \neq 0$

$\tan x = \frac{\sin x}{\cos x}$

$x \notin \frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \dots$

$R = (-\infty, \infty)$

period $= \pi$

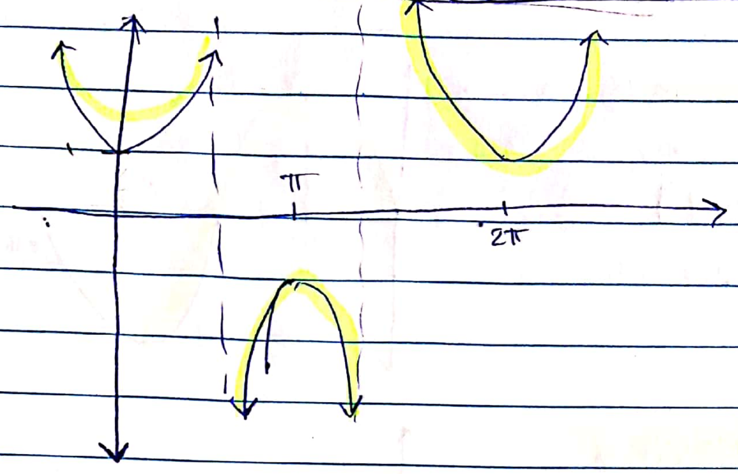


Nots: the most ugly Graphic I have drowen it

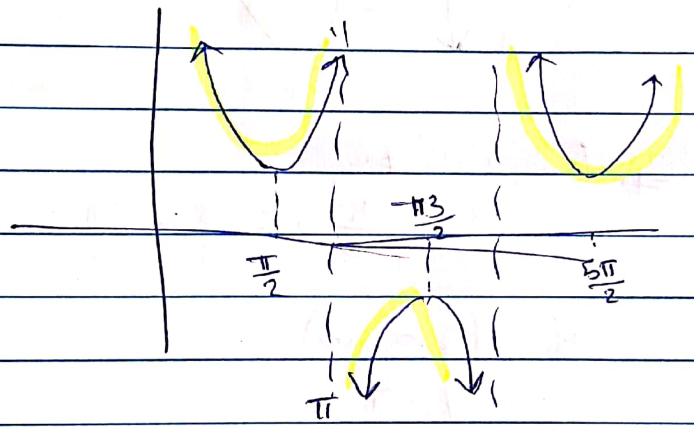
④ $y = \sec x = \dfrac{1}{\cos x}$

D: $\cos\theta \neq 0 \Rightarrow \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$

R: $\mathbb{R} \,/\, (-1,1)$ $\quad \theta \neq \frac{\pi}{2} \pm n\pi, \; n = 0, \pm 1, \pm 2 \ldots$



⑤ $y = \csc x = \dfrac{1}{\sin x}$



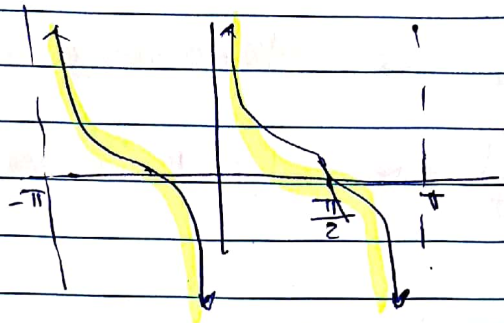
D: $\sin x \neq 0 \quad x \neq 0, \pm\pi, \pm 2\pi \ldots$

R: $\mathbb{R} \setminus (-1,1)$ $\quad$ أي $\quad$ R: $[1,\infty) \cup (-\infty,-1]$

(6) $y = \cot x = \dfrac{\cos x}{\sin x}$

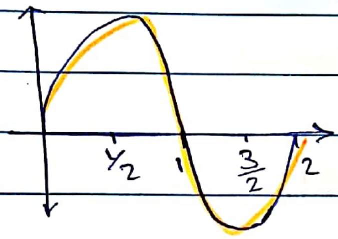
Domain : $\sin x \neq 0 \Rightarrow x \neq 0, \pm\pi, \pm 2\pi$

R : $(-\infty, \infty)$



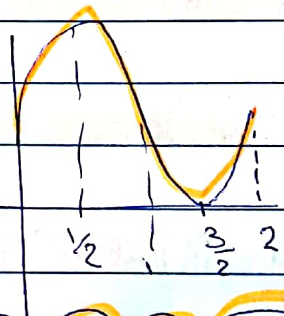
Ex: Graph $y = \sin \pi x$

$\frac{2\pi}{\pi} = 2$



$*$ period $= \dfrac{2\pi}{x \text{ dolas}} \rightarrow$ cos, sin

Ex: $y = \sin(\pi x) + 1$

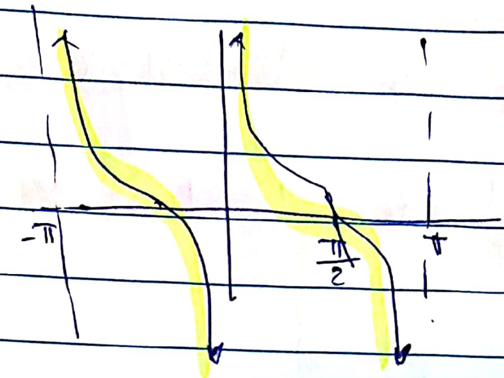


$*$ Period $= \dfrac{\pi}{x \text{ dolas}} \rightarrow \tan$

⑥ $y = \cot x = \dfrac{\cos x}{\sin x}$

Domain : $\sin x \neq 0 \Rightarrow x \neq 0, \pm\pi, \pm 2\pi$

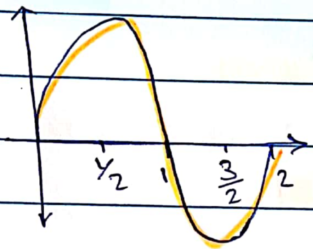
R : $(-\infty, \infty)$



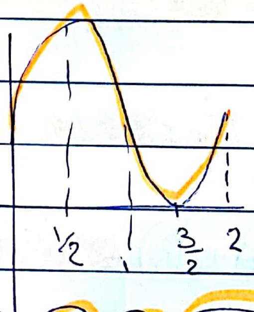
Ex: Graph $y = \sin \pi x$

* period $= \dfrac{2\pi}{x \text{ dolas}}$ → cos, sin

$\dfrac{2\pi}{\pi} = 2$



Ex. $y = \sin(\pi x) + 1$



* Period $= \dfrac{\pi}{x \text{ dolas}}$ → tan

Trigonometric Identites

① $\sin^2 x + \cos^2 x = 1$

$\frac{s^2}{s^2} + \frac{c^2}{s^2} = \frac{1}{s^2} \Rightarrow 1 + \cot^2 x = \csc^2 x$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2} \Rightarrow 1 + \tan^2 x = \sec^2 x$

(2) $\sin 2x = 2(\cos x)(\sin x)$

(3) $\cos 2x = \cos^2 x - \sin^2 x$

(4) $\cos^2 x = \frac{1 + \cos 2x}{2}$

(5) $\sin^2 x = \frac{1 - \cos 2x}{2}$

(6) $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$

(7) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

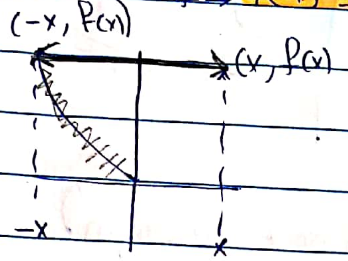
(8) $\sin x + \pi/2 = \cos x$

Example: $\sin(x + \frac{\pi}{2}) = \sin x \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos x$
$= \cos x$

Prove $\sin(x + \frac{\pi}{2}) = \cos x$

1.3 ⇒ Even and odd functions

⇒ Def : $f(x)$ is an even function if → $f(-x) = f(x)$ symmetry about y-axis.

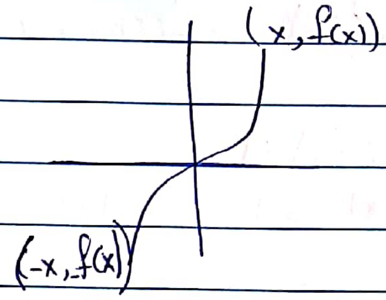


$f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

an even function

⇒ $f(x)$ is an odd function If $f(-x) = -f(x)$



⇒ If $f(-x) \neq f(x) \neq -f(x)$ then $f(x)$ is neither even nor odd.

Ex: $f(x) = \frac{x}{x^2+1} \Rightarrow f(-x) = \frac{-x}{x^2+1} = -f(x)$ Odd

Ex: $f(x) = x^3 + x^2 + 1$

$f(-x) = -x^3 + x^2 + 1 \neq f(x)$

$\neq -f(x)$ neither even nor odd

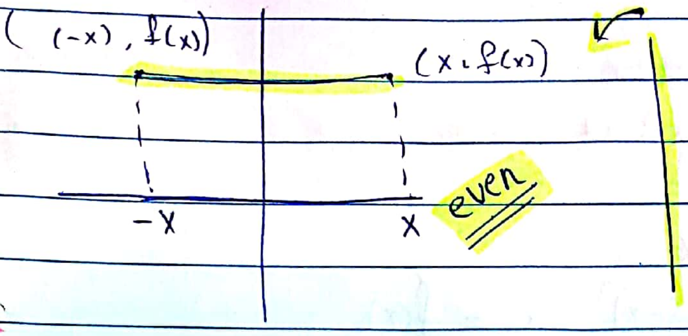
Ex: $y = \cos x \Rightarrow f(x) = f(-x)$ even

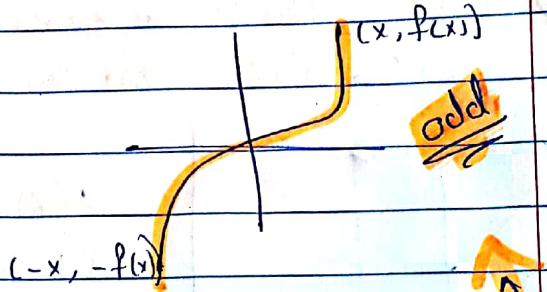
Ex: $y = \sin x \Rightarrow f(x) = -f(-x)$ odd

Ex: $\tan x = \frac{\sin x}{\cos x}$ $\frac{\to \text{even}}{\to \text{odd}}$ odd

Ex: $|x|$ even

* Note : $f(-x) = f(x) \Rightarrow$ even $\Rightarrow$ symmetry about y-axis





* $f(-x) = -f(x) \Rightarrow$ odd $\Rightarrow$ symmetry about the origin

* $f(-x) \neq f(x) \neq -f(x) \Rightarrow$ neither even nor odd

Ex: $f(x) = |x|$

$f(-x) = |-x| = |x| = f(x) \Rightarrow$ even

Ex: (1) If $f(x)$ is even, $g(x)$ is odd?

$(f.g)$ is ?? $\Rightarrow (f.g)(-x) = f(-x) \,.\, g(-x)$

$= f(x) * -g(x)$

$= -f(x) * g(x)$

$= -f(x)$ odd

(2) $fog(x)$?

$f(g(-x)) = f(-g(x)) = f(g(x)) = fog(x)$

even

Limits and Continuity

11 ⇒ **Note:** ① If $p(x)$ is a polynomial then $\lim_{x \to x_0} p(x) = p(x_0)$

Ex: $\lim_{x \to -1} x^3 + 2x^2 + 10 = 11$

* if $R(x) = \frac{p(x)}{q(x)} \Rightarrow q(x) \neq 0$ Rational function

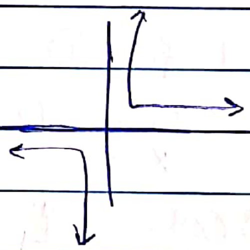
then $\lim_{x \to x_0} p(x) = \frac{p(x_0)}{q(x_0)} \Rightarrow q(x_0) \neq 0$

Ex: $\lim_{x \to -1} \frac{x+2}{x-1} = -\frac{1}{2}$

Ex: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4$

Indeterminate form

Ex: $f(x) = \frac{1}{x}$



$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to 0^+} f(x) = \frac{1}{0^+} = +\infty$

$\lim_{x \to 0^-} f(x) = \frac{1}{0^-} = -\infty$

Ex: $\lim_{x \to -1} \frac{1}{1+x}$

$\lim_{x \to -1^+} = \frac{1}{1+x} = \frac{1}{0^+} = +\infty$

$\lim_{x \to -1^-} = \frac{1}{1+x} = \frac{1}{0^-} = -\infty$

Ex: $\lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1} \times \frac{(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)}$

$\lim_{x \to -1} \frac{x^2+8-9}{x+1\ (\sqrt{x^2+8}+3)} = \lim_{x \to -1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+8}+3)}$

$= \frac{-2}{6} = -\frac{1}{3}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ex. $\lim_{x \to 0} \frac{2 * \sin 2x}{2 * x} = 2$

Theorem The Sandwich theorem

If ① $g(x) \le f(x) \le h(x)$, for all x in an open interval containing:

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} f(x) = L$

$(g(x) \le f(x) \le h(x))$

$\downarrow \quad \downarrow \quad \downarrow$

$L \quad L \quad L$

Ex: show that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

$\lim_{x \to \infty} \frac{\sin x}{x} = 0$

$x / (-1 \le \sin x \le 1)$

$\downarrow$

$\lim_{x \to \infty} \left(\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \right)$

$\downarrow$ zero $\quad \downarrow$ zero $\quad \downarrow$ zero $\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0$

Ex: $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} * \frac{1+\sqrt{x}}{1+\sqrt{x}} = 1/2$

Ex: $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin 2\theta} \times \frac{(1+\cos\theta)}{(1+\cos\theta)}$

$\lim_{\theta\to 0} \frac{1-\cos^2\theta}{2\sin\theta \underbrace{\cos\theta}_{1}} * \frac{}{(1+\cos\theta)}$

$\lim_{\theta\to 0} \frac{\sin\theta^2}{2\cancel{\sin\theta}(1+\cos\theta)} = \frac{0}{4} = 0$

(j) $\lim_{t\to 3^+} \frac{\lfloor t \rfloor}{t}$

$\lim_{t\to 3^-} \frac{\lfloor t \rfloor}{t} = \frac{3}{3} = 1$ | $\lim_{t\to 3^-} \frac{\lfloor t \rfloor}{t} = \frac{2}{3}$

Example

$\lfloor 1.7 \rfloor = 1$

$\lceil 1.7 \rceil = 2$

2.2 ⇒ Continuity الاتصال

* **Def**: A function $f(x)$ is continuous at a point $x = x_0$ if:

1) $f(x_0)$ exists.
2) $\lim_{x \to x_0} f(x)$ = exists ⇒ $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$
3) $\lim_{x \to x_0} f(x) = f(x_0)$

* Not continuous = Discontinuous.

Ex: $f(x) = \frac{|x|}{x}$ $x \neq 0$

→ $\lim_{x \to 0^+} f(x) = 1$

→ $\lim_{x \to 0^-} f(x) = -1$

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

* Note: $f(x)$ is a continuous function if it is cont at each point of its domain.

* $f(x)$ is defined on $[a,b]$

$a < c < b \Rightarrow \lim_{x \to c} f(x) = f(c)$

$x = b \Rightarrow \lim_{x \to b^-} f(x) = f(b)$

$x = a \Rightarrow \lim_{x \to a^+} f(x) = f(a)$

* Note: (1) polynomial continuous.

$\lim_{x \to c} p(x) = p(c)$

② Ruhmal function

$$R(x) = \frac{p(x)}{q(x)} \quad \text{cont. for all such } q(x) \neq 0$$

③ $\sin x$, $\cos x$, $|x|$ cont for all x

$\Rightarrow \tan x = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0$

$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots \left\{ \frac{\pi}{2} \pm n\pi, \ n = 0, 1, 2 \right.$

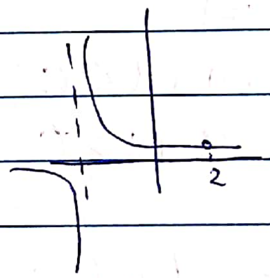
Ex: $f(x) = \dfrac{x-2}{x^2-4}$ $\quad x \neq 2, -2$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \frac{1}{4}$$

$$\frac{x-2}{x^2-4} \neq \frac{1}{x+2} \Rightarrow f(x) = g(x)$$

$$D(f) \neq D(g)$$

$$F(x) = \begin{cases} \dfrac{x-2}{x^2-4}, & x \neq 2 \\ \dfrac{1}{4}, & x = 2 \end{cases}$$

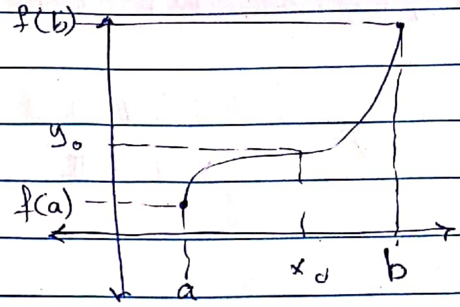


$F(x)$ contiuous at $x = 2$

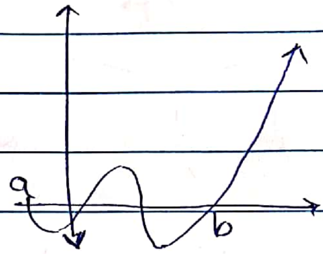
Removable discontinuity.

Theorem: The Intermediate value (IVP)

* If $f(x)$ is continuous on $[a,b]$, and y_0 is a value between $f(a)$ and $f(b)$, then there is $x_0 \in [a,b]$ such that $f(x_0) = y_0$



* Note. If $f(x)$ is cont. on $[a,b]$ and $f(a) * f(b) < 0$, the $f(x) = 0$ has at least one solution on $[a,b]$.



* Ex: $f(x) = x^3 + x^2 + 1$ on $[-2, 2]$

$f(-2) = -8 + 4 + 1 < 0$

$f(2) = 8 + 4 + 1 > 0$

$\Rightarrow f(-2) * f(2) < 0$

There is at least $c \in (-2, 2)$ such that $f(c) = 0$

* $R(x) = \frac{p(x)}{q(x)}$, $\lim_{x \to \pm\infty} R(x) = \pm\infty$

$\Rightarrow$ degree $p(x)$ < degree $q(x)$ $\Rightarrow 0$

degree $p(x)$ = degree $q(x)$ $\Rightarrow$ constant

degree $p(x)$ > degree $q(x)$ $\Rightarrow \pm\infty$

$$\lim_{\substack{x \to \infty \\ -\infty}} f(x) = 0 \quad , \quad \lim_{\substack{x \to a^+ \\ a^-}} f(x) = \mp \infty$$

isn't important

Ex: ① $\lim_{x \to \infty} \frac{2x+1}{x-1} = \boxed{2}$

② $\lim_{x \to \infty} \frac{x}{x^2+5x+10} = \boxed{\text{zero}}$

③ $\lim_{x \to \infty} \frac{x^3+1}{x^2+5} = \boxed{+\infty}$

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Rational function ⇒ Graph

$R(x) = \frac{p(x)}{q(x)}$, $q(x) = 0$

isn't important

* $\lim_{x \to a^+} R(x)$

* $\lim_{\substack{x \to \infty \\ -\infty}} R(x)$

Asymptotes Lines

$\boxed{1}$ → Def: the line $\boxed{y=b}$

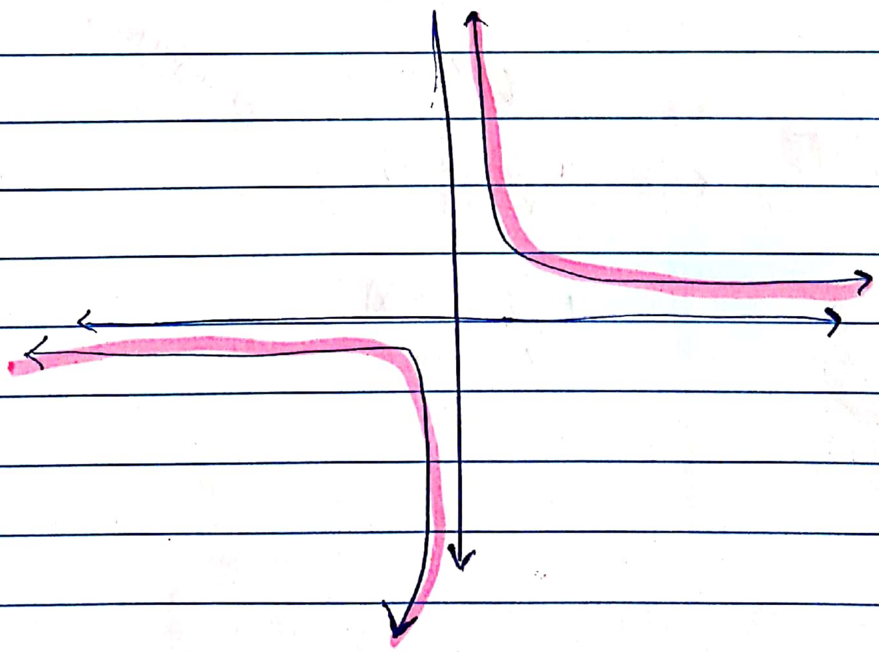
is a horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \boxed{\text{or}} \quad \lim_{x\to-\infty} f(x) = b$$

Ex: $f(x) = \frac{1}{x}$ $\Rightarrow$ $\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

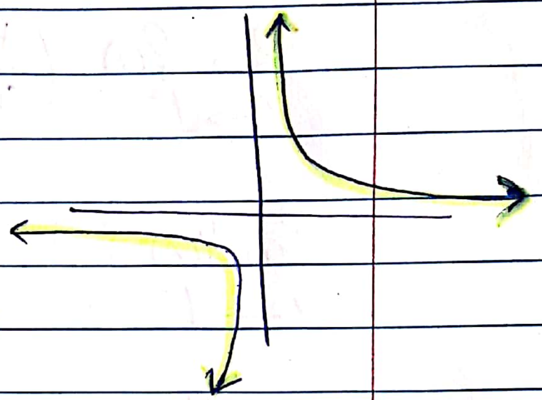
$\Rightarrow$ $\underset{\text{x-axis}}{\boxed{y=0}}$ is a horizontal asymptotes



$\boxed{2}$ **Def**: The line $\boxed{x=a}$, is a vertical asymptote for $y=f(x)$ if either

$\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

Ex: $f(x) = \frac{1}{x}$



$\lim_{x\to 0^+} \frac{1}{x} = \left(\frac{1}{0^+}\right) = +\infty$

$\lim_{x\to 0^-} \frac{1}{x} = \left(\frac{1}{0^-}\right) = -\infty$

$x = 0$ (y-axis) is a vertical asymptote

Ex: $f(x) = \frac{\sin x}{x}$

$\lim_{x\to\infty} \frac{\sin x}{x} = 0 \Rightarrow y=0$ H.Asy

Sandwich ~~theorem~~

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow$ No vertical asymptotes.

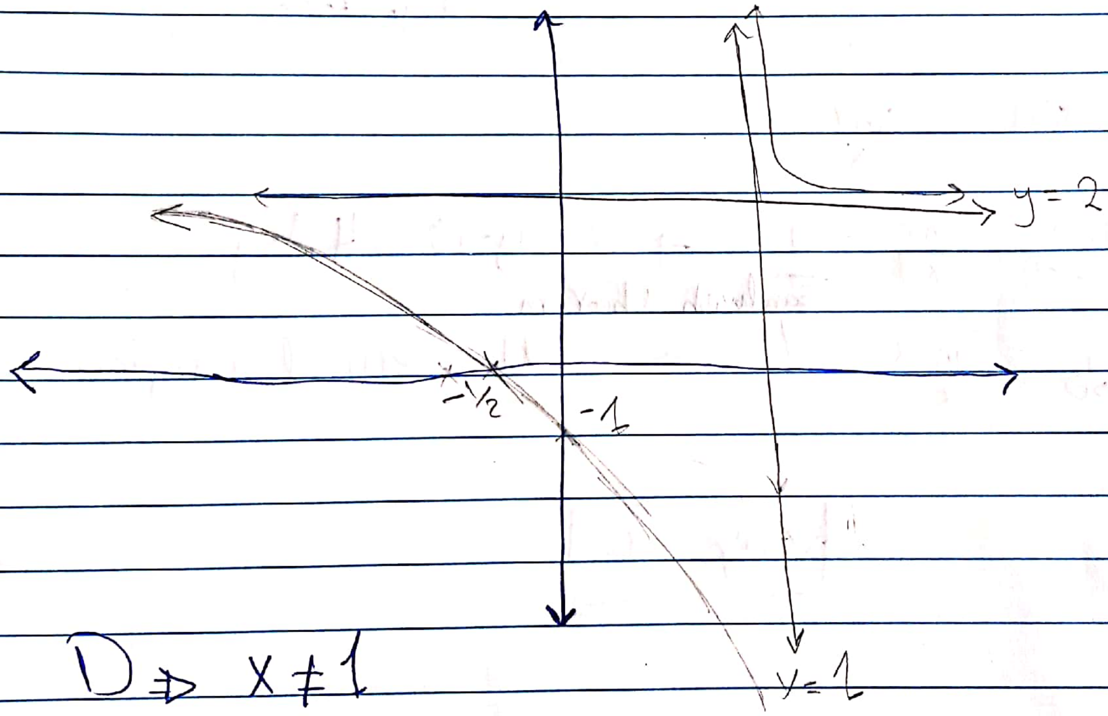
Ex: $f(x) = \dfrac{2x+1}{x-1}$

$\lim\limits_{x \to -\infty \atop \infty} f(x) = 2 \Rightarrow y = 2$ H. Asy

$\lim\limits_{x \to 1^+} f(x) \left(\dfrac{+}{0^-}\right) = +\infty$

$\lim\limits_{x \to 1^-} f(x) \left(\dfrac{+}{0^-}\right) = -\infty$

$\Rightarrow x = 1$ Vertical asy



D ⇒ $x \neq 1$

R ⇒ $y \neq 2$

Ex: $f(x) = \dfrac{x^2+x}{x^2-1}$ $\quad x \neq -1, 1$

$$\frac{x\cancel{(x+1)}}{\cancel{(x+1)}(x-1)} = \frac{x}{x-1}$$

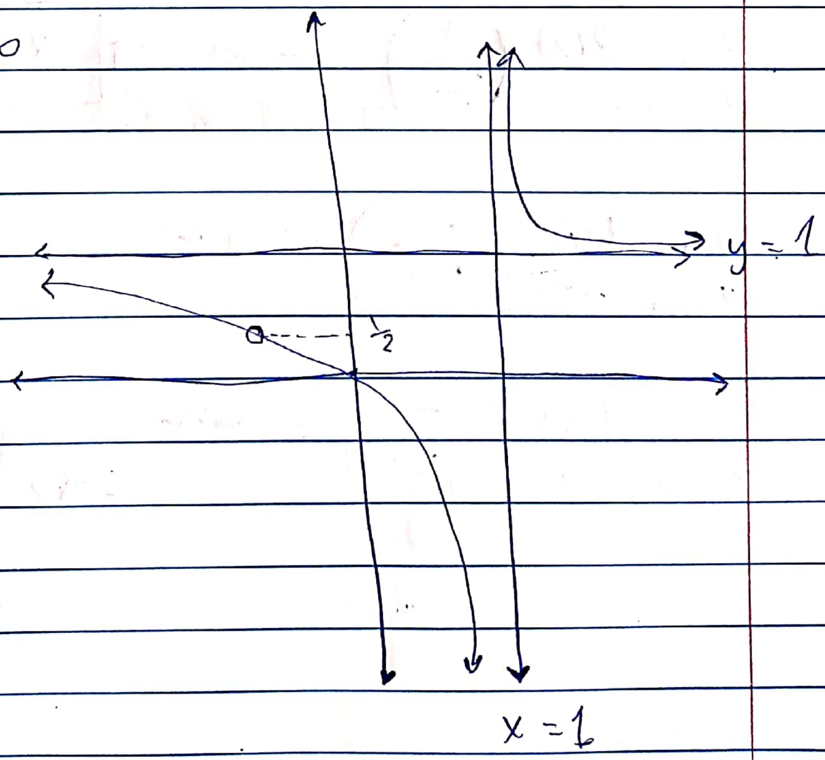
$$\lim_{x \to -1} f(x) = \frac{-1}{-2} = \frac{1}{2}$$

No vertical asy at $x = -1$

$$\lim_{x \to \mp\infty} f(x) = 1 \Rightarrow y = 1 \quad \text{H. Asy}$$

$$\lim_{x \to 1^+} f(x) = +\infty \Rightarrow x = 1 \quad \text{V. Asy}$$

$$\lim_{x \to 1^-} = -\infty$$



Ex: $f(x) = \frac{x}{x^2-4}$

$f(-x) = \frac{-x}{x^2-4} = -f(x)$ odd

y-int : $x=0 \Rightarrow y=0$

x-int : $y=0 \Rightarrow x=0$ $\Rightarrow (0,0)$

$\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = 0$

$\Rightarrow \boxed{y=0}$ H.A

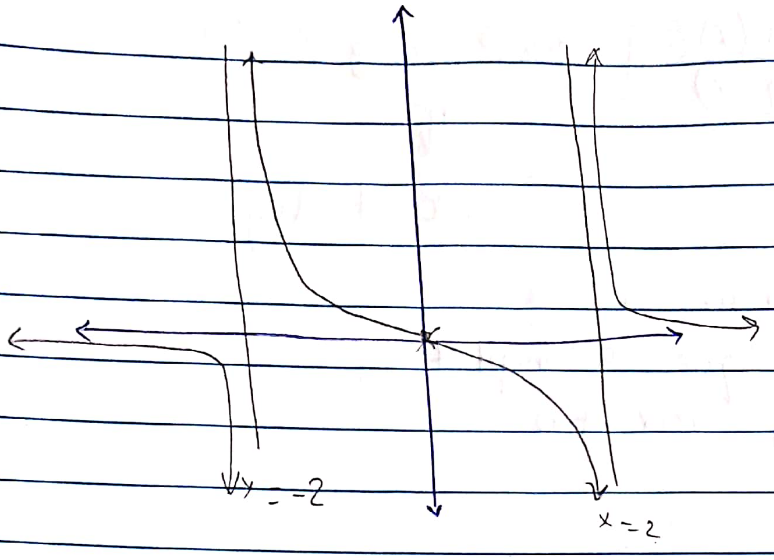
$\lim_{x \to 2^+} f(x) \left(\frac{+}{0^+}\right) = +\infty$

$\lim_{x \to 2^-} f(x) \left(\frac{+}{0^-}\right) = -\infty$ $\Rightarrow \boxed{x=2}$ V.A

$\lim_{x \to -2^+} f(x) \left(\frac{-}{0^-}\right) = +\infty$

$\lim_{x \to -2^-} f(x) \left(\frac{-}{0^+}\right) = -\infty$ $\Rightarrow \boxed{x=-2}$

V.A



Asymptotes: $R(x) = \dfrac{P(x)}{Q(x)}$

$\lim\limits_{x \to \infty} R(x) = b \Rightarrow y = b$ H.A

$\lim\limits_{\substack{x \to a^+ \\ a^-}} R(x) = \pm\infty \Rightarrow x = a$ V.A

* $R(x) = \dfrac{P(x)}{q(x)}$ $\boxed{\deg P = \deg (q+1)}$

$\Downarrow$

No H. Asy

= linear + ...

oblique asymptot

$y = mx + b$

Ex. $f(x) = \dfrac{x^2}{x-1}$, $x \neq 1$

$= \dfrac{x^2-1}{x-1} + \dfrac{1}{x-1}$

$= x+1 + \dfrac{1}{x-1}$

oblique asymptote

$f(x) = x+1 + \dfrac{1}{x-1}$

$$\boxed{x \to \begin{matrix}-\infty \\ +\infty\end{matrix} \Rightarrow f(x) \approx x+1}$$

$x \to 1 \Rightarrow f(x) \approx \dfrac{1}{x-1}$

$\boxed{y = x+1}$ oblique

$\Rightarrow \dfrac{x^3}{x^2+1}$

No H. Asy

No V. Asy

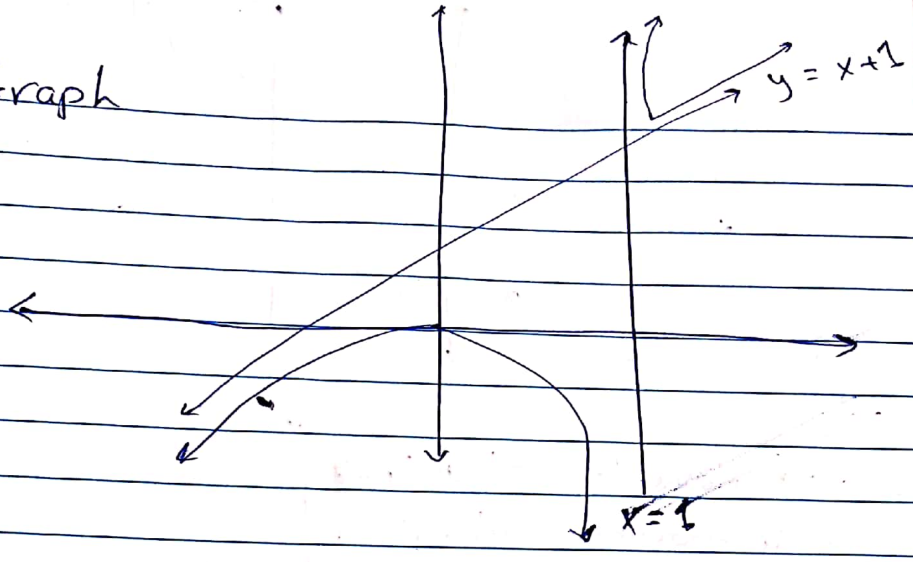
$\lim\limits_{x \to 1^+} f(x) = \left(\dfrac{+}{0^+}\right) = +\infty$

$\lim\limits_{x \to 1^-} f(x) = \left(\dfrac{+}{0^-}\right) = -\infty$

$\boxed{y = 1}$ vertical

$x = 0 \Rightarrow f(x) = 0$

Graph



Ex: $f(x) = \dfrac{x^3+1}{x^2-1}$, $x \neq +1, -1$ → may be (V.As) ; - No (H.Asy) ; - oblique

degree x^3+1 > degree x^2-1

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3+1 \\ & -x^3 \pm x \\ \hline & x+1 \end{array} \Longrightarrow \quad f(x) = x + \frac{x+1}{x^2-1} = x + \frac{1}{x-1}$$

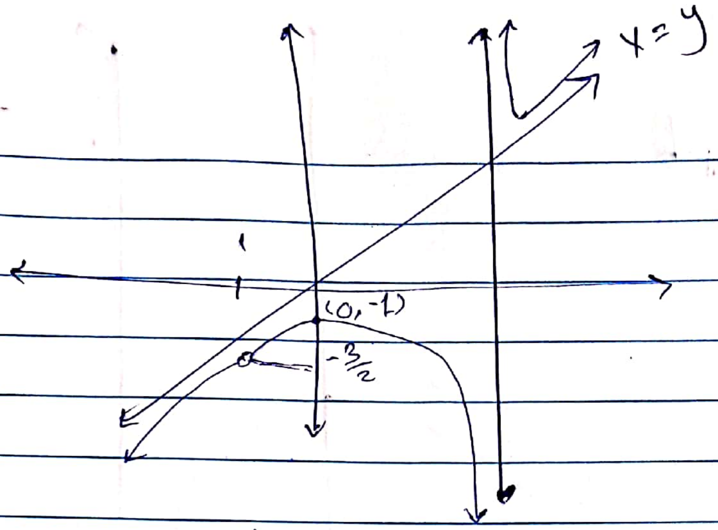
$\Rightarrow \lim\limits_{x \to -1} f(x) = -\dfrac{3}{2} \Rightarrow$ $\boxed{\text{No. V.Asy}}$

$\Rightarrow y = x \Rightarrow$ $\boxed{\text{oblique Asy}}$

$\Rightarrow \lim\limits_{x \to 1^+} f(x) = +\infty$

$\lim\limits_{x \to 1^-} f(x) = -\infty$

$x = 1$ V.Asy

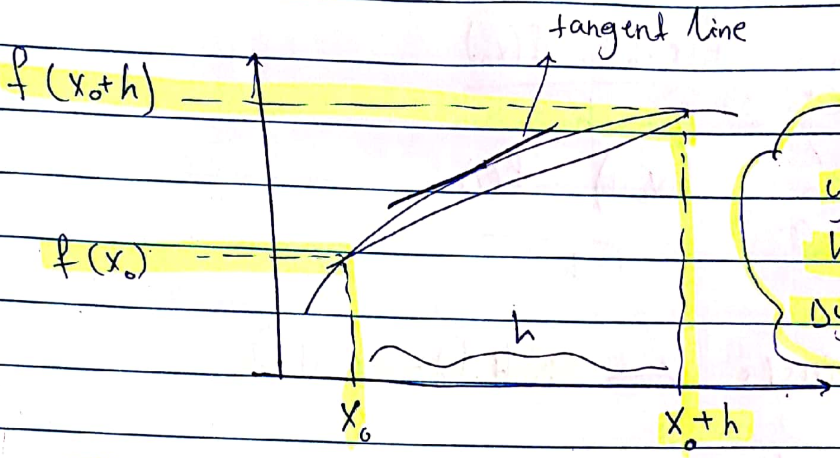


x = -1 ⇒ removable dicontinuity

x = 1 ⇒ irremovable dicontinuity

$$f(x) = \begin{cases} f(x) & , x \neq -1 \\ -\frac{3}{2} & , x = -1 \end{cases}$$

3.1 → Differentiation



$y = f(x)$

$h = \Delta x$

$\Delta y = f(x_0+h) - f(x_0)$

average rate of change ↙ $\frac{\Delta y}{\Delta x}$ = slop of the secant line

3.1 → **Def**: Given a function $f(x)$ the first derivative of $f(x)$, with respect to x at $x = x_0$, dented by $f(x)$ is given by the folowing Limit

$$f(x) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided that the limit exists

$$f(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$f(x)$ is a differentiable function at $x = x_0$

If $y = f(x)$, then $f(x)$ is diff of $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists for all x in the domain of $f(x)$.

$f'(x) = \frac{dy}{dx}$

R.H.D $\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$

L.H.D $\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$

$f'(x_0)$ exists $\iff$ **R.H.D = L.H.D**

Ex: $f(x) = |x|$

$$f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

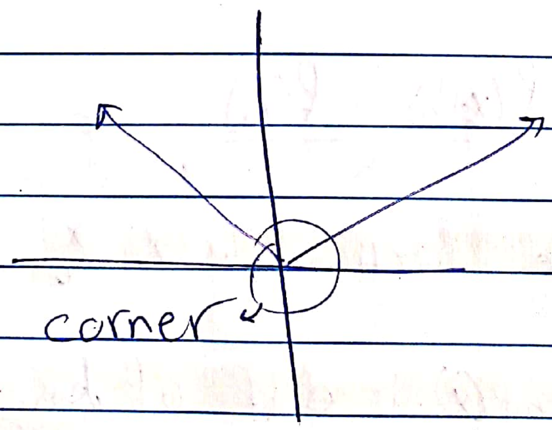
$$L.H.D = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

$R.H.D \neq L.H.D$

$1 \neq -1$

So $f(x)$ is not diff

at $(x = 0)$



Derivatives

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x\to 0} \frac{f(x) - f(a)}{x-a}$$

$f'(x) = f'(a) \equiv$ slope of the tangent to the curve $f(x)$ at $x=a$

$f(x) = |x| \Rightarrow$ not diff at $x=0$ cont

Note: If $f(x)$ is diff at $x=c$, then $f(x)$ is cont at $x=c$

diff $\rightarrow$ Cont

cont $\not\rightarrow$ diff

3.2 $\rightarrow$ **Rules of differentiarable:-**

$\boxed{1}$ $y = f(x) = x^n$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$f'(x) = n x^{n-1}$$

$\boxed{2}$ If f, g are differentiable then

$$(f \pm g)' = f' \pm g'$$

$\boxed{3}$ $(f.g)' = f'g + g'f$

$\boxed{4}$ $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$\boxed{5}$ $(f \circ g)(x) = f(g(x))$

The Chain Rule: If $y = f(u)$ and $u = g(x)$
$y = f(g(x)) = (f \circ g)(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= f'(u) \cdot g'(x)$$
$$= f'(g(x)) \cdot g'(x)$$

3.3 → Trigonometric functions:

$\boxed{6}$ $\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx} \sin u(x) = \cos(u(x)) . u'(x)$ → * to all trigonometric function.

$\boxed{7}$ $\frac{d}{dx}(\cos(x)) = -\sin x$

$\boxed{8}$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\boxed{9}$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\boxed{10}$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\boxed{11}$ $\frac{d}{dx} \cot x = -\csc^2 x$

Ex. $\boxed{1}$ $y = (x^2+3)^{\frac{7}{3}}$
$y' = \frac{7}{3}(x^2+3)^{\frac{4}{3}} * 2x = \frac{14}{3}(x^2+3)^{4/3}$

$\boxed{2}$ $y = \sec\sqrt{x}$
$y' = \sec\sqrt{x} \tan\sqrt{x} * \frac{1}{2\sqrt{x}}$

$\boxed{3}$ $y = \tan x^2 * \cos(2x+3)$

$$\left(\sec^2 x^2 * 2x\,(\cos(2x+3))\right) + \left(\tan x^2 * -\sin(2x+3) * 2\right)$$

$\boxed{4}$ $y = \sin^3(2x+4)$

$y' = 3\sin^2(2x+4) * \cos(2x+4) * 2$

3.4 → **Implicit Differentiation**

$y = f(x) \Rightarrow \frac{dy}{dx}$

y ⇓ Dep, x ⇓ Indep

$\boxed{1}$ $x^3 + xy^2 + 10y = 0$

$3x^2 + x\,2yy' + y^2 + x\,10y' + 10y' = 0$

$3x^2 + y^2 + \cancel{10y} + y'(2xy + 10x) = 0$

$y'(2xy + 10x) = -3x^2 - y^2 \cancel{-10y}$

$$y' = \frac{-3x^2 - y^2 \cancel{-10y}}{2xy + 10x}$$

$y', y'', y''', y^{(4)}$

$\boxed{2}$ $xy = \cos(xy)$

$xy' + y = -\sin(xy) * (xy' + y)$

$xy' + y = -[\sin(xy) * xy' + y\sin xy]$

$(x + x\sin xy)y' = -[y\sin xy + y]$

Ex: If $x^{2/3} + y^{2/3} = 1$ find y'' ?

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \Rightarrow y' = -\left(\frac{y}{x}\right)^{1/3}$

$$y'' = -\frac{1}{3}\left[\frac{xy' - y}{x^2}\right]^{-2/3}$$

$$= -\frac{1}{3}\left[\frac{-x\left(\frac{y}{x}\right)^{1/3} - y}{x^2}\right]^{-2/3}$$

Exp : Draw this function $f(x) = x^3 - 3x^2 + 1$

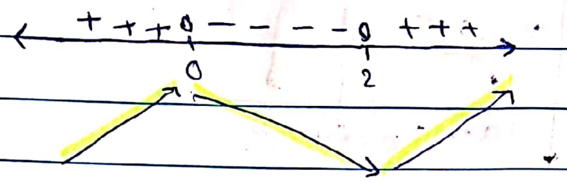
[1] $D(f) = \mathbb{R}$

[2] ↑↓ : $f'(x) = 3x^2 - 6x$

$0 = 3x(x-2)$

(critical point) ⇒ $x = 0$, $x = 2 \in D(f)$

$f'(x)$



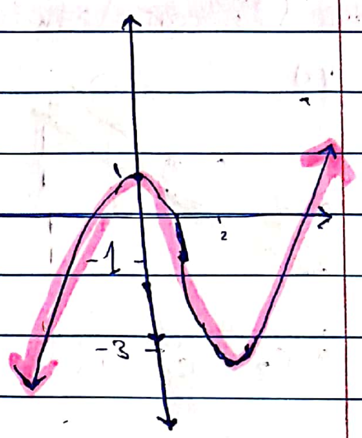
$f(x)$ behave

* f is ↑ on $(-\infty, 0] \cup [2, \infty)$
* f is ↓ on $[0, 2]$

[3] key point $(0,1)$

$f(0) = 1$ is local max

$f(2) = -3$ is local min



[4] inflection point

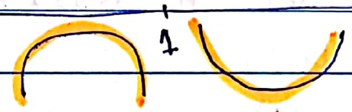
$f''(x) = 6x - 6$

$0 = 6(x-1)$

$x = 1$

$f''(x)$ − − − − 0 + + + +

1



$\Rightarrow$ 3.5 Linearization and differentials

Linearization:

$y - y_1 = m(x - x_1)$

$y = y_1 + m(x - x_1)$

$x_1 = a \Rightarrow L(x) = f(a) + f'(a)(x - a)$

Exp: Find the linearzation? ~~and differential~~

$f(x) = \tan x$ at $x = \frac{\pi}{4}$?

$f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1 \Rightarrow (\frac{\pi}{4}, 1)$

$f'(x) = \sec^2 x = \sec^2 \frac{\pi}{4} = 2 = f'(\frac{\pi}{4})$

$x = \frac{\pi}{4}$, $y = 1$, $m = 2$

$y - 1 = 2(x - \frac{\pi}{4})$

$y = 1 + 2(x + \frac{\pi}{4}) = L(x)$

Differentials:-

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) \Rightarrow \text{derivative.}$$

$$dy = f'(x)\,dx \Rightarrow \text{differentials.}$$

exp: $y = x^2 \Rightarrow y' = 2x$ dervative

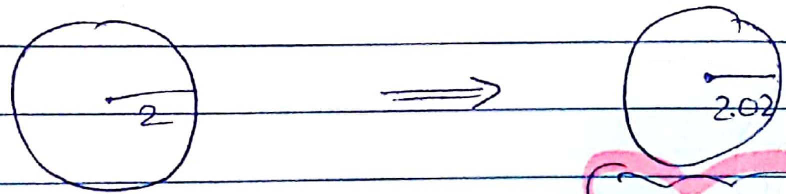
$dy = 2x\,dx$ differentials of y.

exp: Radius of a circale increased from 2 to 2.02.

(a) Estimate the resulting Change in area

(b) express the estimate as a perecentage of the circle's original areas

$A = \pi(2.02)^2$



$r = 2.02 = 2 + 0.02$
$= r + dr$

(a) $A(r) = \pi r^2$

$dA = 2\pi r\,dr = 2\pi(2)(0.02) = 0.08\pi$

$A(2) = \pi(2)^2 = 4\pi$

estimate area $= A + dA$

$= 4\pi + 0.08\pi = 4.08\pi$

(b) True area $= \pi(2.02)^2 = (4.0804)\pi$

error $= |\text{True} - \text{estimate}|$

$= |4.0804\pi - 4.08\pi|$

$= 0.0004\pi$

Applications of derivatives

4.1 → Increasing and decreasing functions:

Def: Let $f(x)$ be a function defined on an Interval I. Then

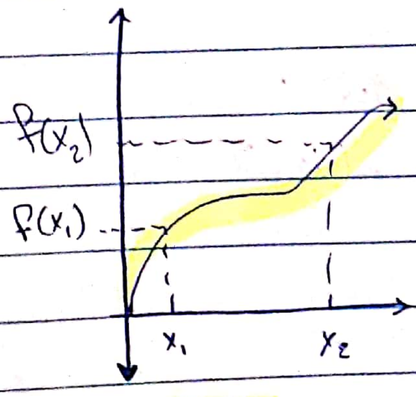
(a) f is increasing on I if whenever $x_2 > x_1$ then $f(x_2) > f(x_1)$, for all x_1, x_2 in I.

(b) f is decreasing on I if whenever $x_2 > x_1$ then $f(x_2) < f(x_1)$, for all x_1, x_2 in I.

Theorem:- Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) then:-

(a) If $f'(x) > 0$, for all $x \in (a,b)$ then f is increasing on (a,b)

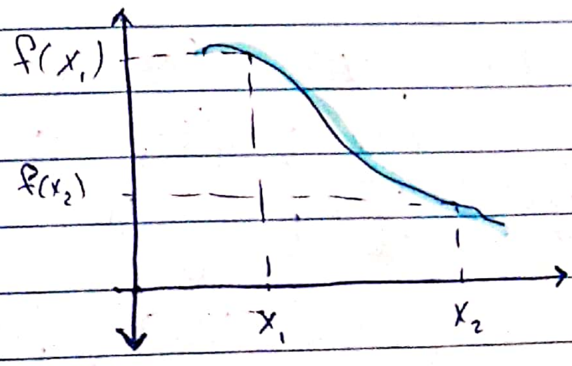
(b) If $f'(x) < 0$, for all $x \in (a,b)$ then f is decreasing on (a,b)



$x_2 > x_1$

$f(x_2) > f(x_1)$

increasing



$x_2 > x_1$

$f(x_2) < f(x_1)$

decreasing

4.2 → Extrem Values

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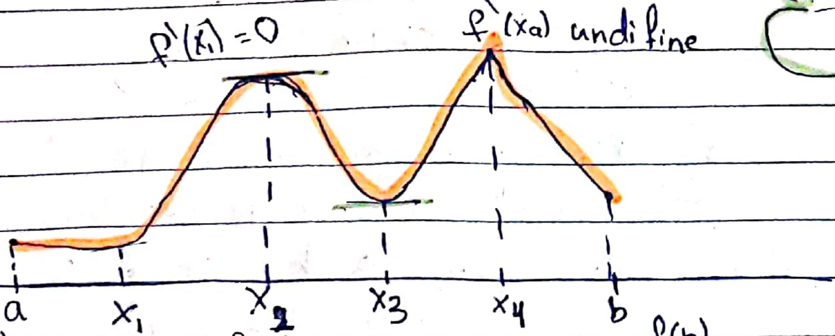
لمحور السينات
slop = 0



f(a) is "Local" max

f(x1) is Abs min "Local" min

f(x2) is "Local" max

f(x3) "Local" min

f(x4) Abs Max "Local" max

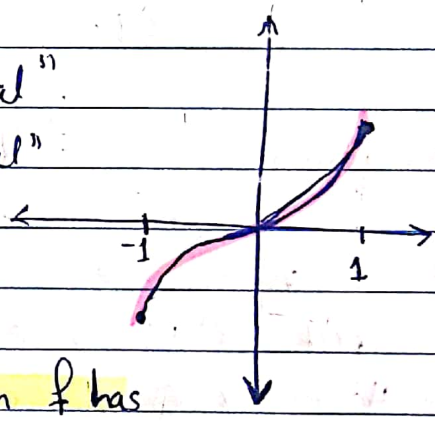
f(b) is "Local" min

« figur 1 »

Exp: $f(x) = x^3$ on $[-1, 1]$, find the extrem value?

$f(1) = 1^3 = 1$ is Aps Max "local"

$f(-1) = -1$ is Aps Min "local"

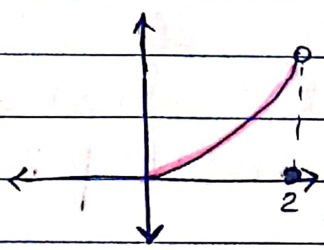


Th (Extrem Value theorem)

⇒ If f cont. on [a, b] then f has Abs Max and Abs Min.

Exp: Consider this function.

$$f(x) = \begin{cases} x^2 & 0 \le x < 2 \\ 0 & x = 2 \end{cases}$$



find Extrem Values?

$f(0) = 0 = 0$ is Abs Min

$f(2) = 0$ is Abs Min

NO Abs Max.

* f(x) is discont. at x = 2

Def. An * interior point موجودة في المجال c $(c \in D(f))$ is called critical point if ① $f'(c)=0$ or ② $f'(c)$ undefined.

Remark: Note the Extrem Values occur at either critical points or End points for a function defined on $[a,b]$ see fig 1.

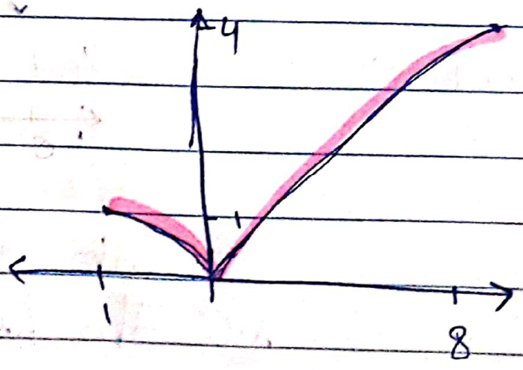
Exp: Find extrem values for $f(x) = x^{2/3}$ on $[-1,8]$

End points: $f(-1) = (-1)^{\frac{2}{3}} = \sqrt[3]{(-1)^2} = \boxed{1}$
$f(8) = (8)^{2/3} = \sqrt[3]{(8)^2} = \boxed{4}$

Critical points $f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$

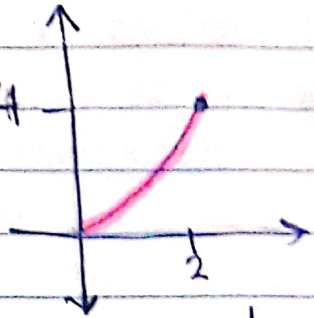
f' never zero.
but f' is undefined at $x=0 \in [-1,8]$
Henec $x=0$ is a critical point
$f(0) = \boxed{0}$

So $f(0)=0$ is Abs Min
and $f(8)=4$ is Abs Max
*(f) is Cont. on $[-1,8]$ | f 1 is local Max

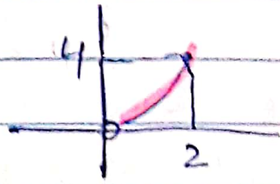


Exp: $f(x) = x^2$ on ① $[0,2]$

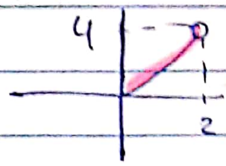
$f(0)=0$ is Abs-Min
$f(2)=4$ is Abs-Max



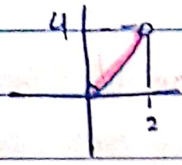
② on $(0,2]$ $f(2)=4$ is Abs-Max



③ on $[0,2)$ $f(0)=0$ is Abs-Min

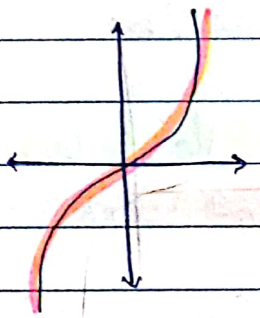


④ on $(0,2)$ No extrem Values.



The if $f(x)$ is diff at c and $f(x)$ has extrem value at $x=0$, then $f'(c)=0$

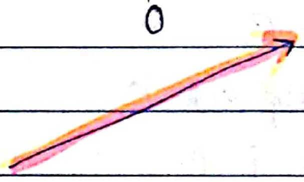
The converse is not true



Exm:
$f(x) = x^3$ Cont. on $\mathbb{R}$
$f'(x) = 3x^2 = 0$
$x = 0$
$f(0) = 0$

$f'(x)$ + + + + + | + + + + +
0

$f(x)$

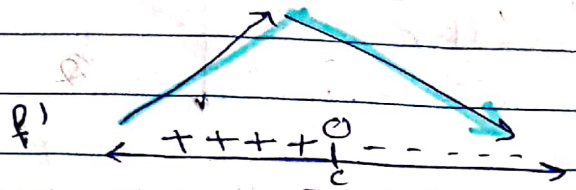


but $f(0)$ is neither local Max nor local Min.
$\Rightarrow x=0$ is Critical point
$0 \in D(f)$

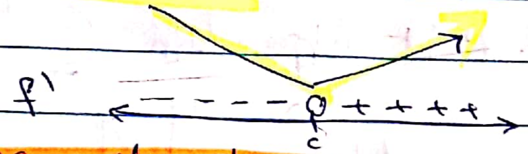
The (1^{st}) Derivative Test for Extrem Values

Assume $f'(c) = 0$ where c is interiar point

- If f' changes signs from $+$ to $-$ around c then $f(c)$ is local Max



- If f' changes signs from $-$ to $+$ around c then $f(c)$ is local Min.



- If f' does not change signs around c then f has no extrem values

Th (2^{nd} Derivative test)

Assume $f'(c) = 0$ where c is interior point

- If $f''(c) < 0$, then $f(c)$ is local Max
- If $f''(c) > 0$, then $f(c)$ is local Min
- If $f''(c) = 0$ the test doesn't work

Remark * An interior point c $(c \in D(f))$

where ① f has tangent at c.

and ② f changes concavity at c. تقعر

is called inflection point.

نقطة انعطاف

* To find the inflections we start with we start with $f''(x) = 0$ and find x.

Exp : Draw this function $f(x) = x^3 - 3x^2 + 1$

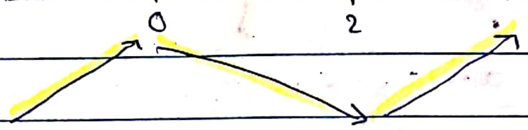
1. $D(f) = \mathbb{R}$

2. ↑↓ : $f'(x) = 3x^2 - 6x$

$0 = 3x(x-2)$

(critical point) ⇒ $x = 0$, $x = 2$ $\in D(f)$

$f'(x)$: + + + 0 − − − − 0 + + + (0 ... 2)

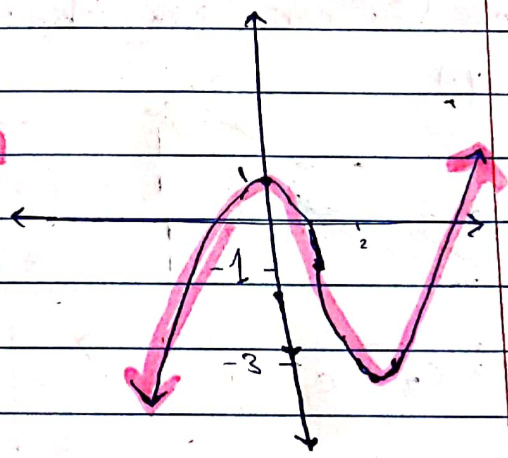
f(x) behave


* f is ↑ on $(-\infty, 0] \cup [2, \infty)$
* f is ↓ on $[0, 2]$

3. key point $(0,1)$

$f(0) = 1$ is **local max**

$f(2) = -3$ is **local min**



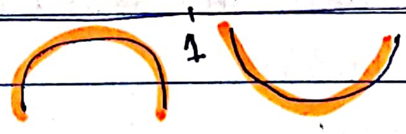
4. inflection point

$f''(x) = 6x - 6$

$0 = 6(x-1)$

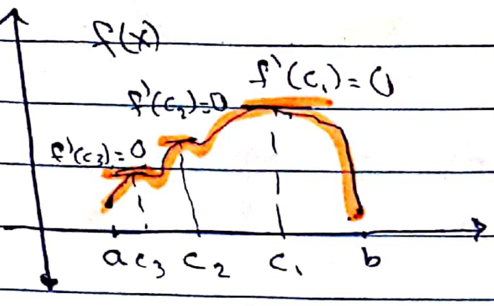
$x = 1$

$f''(x)$: − − − − 0 + + + + (1)



Ex Th (Roll's Th) : Assume f is cont on $[a,b]$ and f is diff on (a,b) with $f(a) = f(b)$

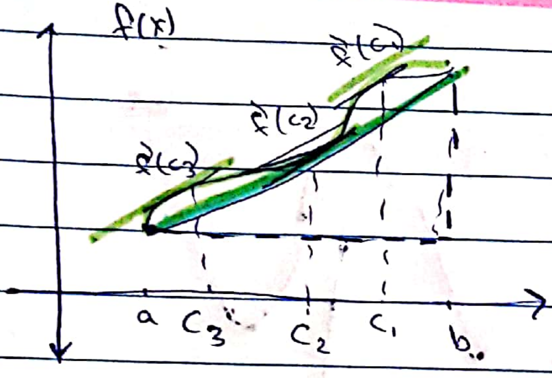
* Then $\exists$ at least one point $c \in (a,b)$ s.t $f'(c) = 0$.



Th (Mean Value Th) : Assume f is Cont On $[a,b]$ and f is diff on (a,b)

Then, $\exists$ at least one Point $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$f'(c_1)$ = Slop of secant

$$f'(c_2) = \frac{f(b) - f(a)}{b - a}$$

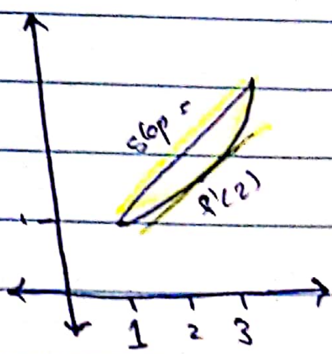
Ex Find c that satifics the MVT for $f(x) = x^2$ on $[1,3]$

- f cont on $\mathbb{R} \Rightarrow f$ Cont on $[1,3]$
- $f' = 2x$ cont on $\mathbb{R} \Rightarrow f'$ cont on $(1,3)$
- Henec $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$2c = \frac{f(3) - f(1)}{3 - 1}$$

$$2c = \frac{9 - 1}{2}$$

$$c = 2$$



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Exp: Find a, b, c so that f satisfies the MVT on $[0,2]$.

$$f(x) = \begin{cases} 3 & , x = 0 \\ -x^2+3x+a & , 0<x<1 \\ cx+b & , 1 \le x \le 2 \end{cases}$$

✓ f is cont. on $[0,2]$, f is cont at $x=0$

$f(0) = \lim_{x \to 0^+} f(x) \Rightarrow \boxed{3 = a}$

✓ f is cont at $x=1 \Rightarrow \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

$c+b = -1+3+3$

$\boxed{c+b = 5}$ — ①

✓ f is diff on $[0,2]$

$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$

$c = -2(1) + 3 \Rightarrow \boxed{c = 1}$

$\Rightarrow \boxed{b = 4}$

Exp: Given $y = f(x) = x\sqrt{8-x^2}$

$y' = \dfrac{8-2x^2}{\sqrt{8-x^2}}$

$y'' = \dfrac{2x^3 - 24x}{\sqrt{(8-x^2)^3}}$

find ① D_f ② $R(f)$ ③ ↑↓ ④ critical point ⑤ Extrem values ⑥ inflection point ⑦ Concave (up + down) ⑧ graph f

① $8 - x^2 \geq 0$

$-2\sqrt{2} \leq x \leq 2\sqrt{2}$

$D = [-2\sqrt{2}, 2\sqrt{2}]$

② wait for ⑧

③ $y' = 0 \Leftrightarrow 8-2x^2 = 0$

④ $\sqrt{x^2} = \sqrt{4}$

$x = 2$ or $x = -2$, $\in D(f)$

critical points

$(2, 4)$, $(-2, -4)$

③ y' is undefined: $8 - x^2 = 0$

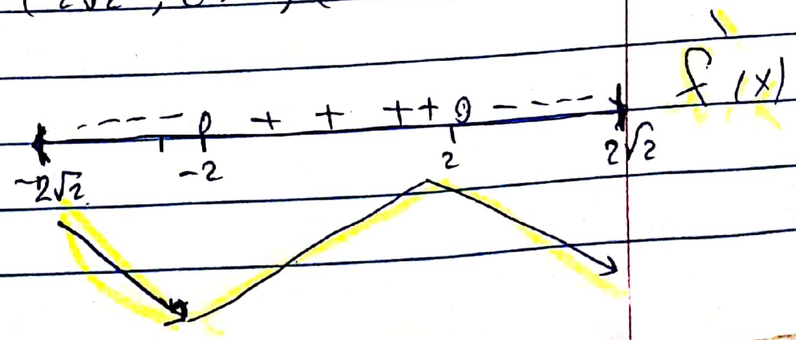
$x^2 = 8$

$x = 2\sqrt{2}, -2\sqrt{2} \in D(f)$

$(2\sqrt{2}, 0)$, $(-2\sqrt{2}, 0)$

• (f) is decreasing on $[-2\sqrt{2}, -2] \cup [2, 2\sqrt{2}]$

• (f) is increasing on $[-2, 2]$



5. Extrem Value ⇒ $(-2, -4)$ is local min (Abs Min).

$(2\sqrt{2}, 0)$ is Local min.

$(2, 4)$ is local max (Abs Max).

$(-2\sqrt{2}, 0)$ is local max.

6. inflection point $(0, 0)$

7. $y'' = 0 \Rightarrow 2x^3 - 24x = 0$

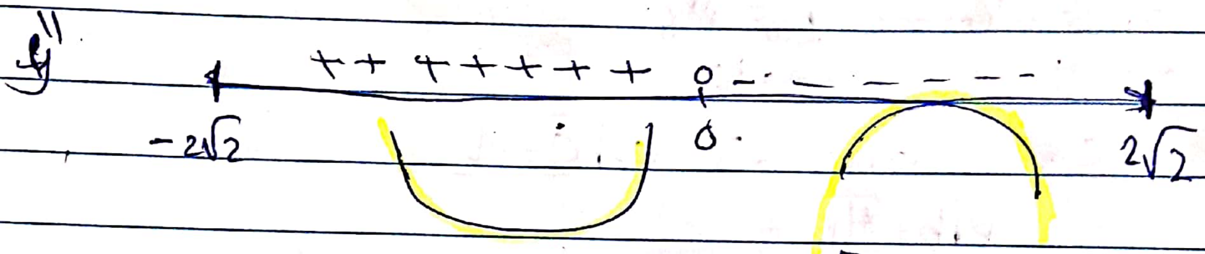
$2x\,[x^2 - 12] = 0$

$x = 0 \in D(f)$

or $x^2 - 12 = 0$

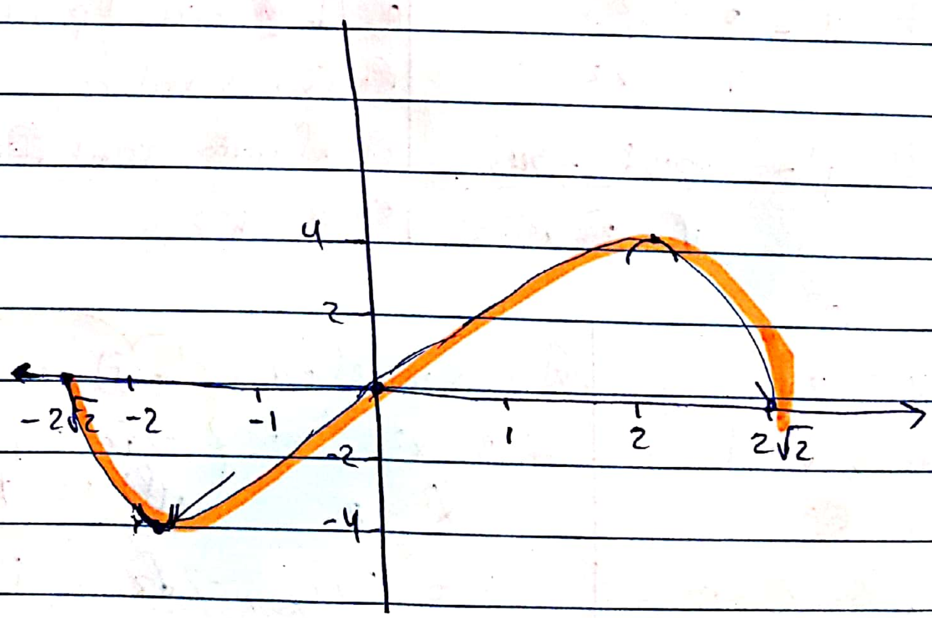
$x^2 = 12$

$|x| = \sqrt{12} = 2\sqrt{3}, -2\sqrt{3} \notin D(f)$



- f is concave up on $[-2\sqrt{2}, 0]$
- f is concave down on $[0, 2\sqrt{2}]$

8.



Integration

Def: The function $F(x)$ is called anti-derivative for the function $f(x)$ on interval I
if $F'(x) = f(x) \quad \forall x \in I$

Exp Give 4 antiderivatives for $f(x) = 2x$,
$\boxed{1}$ $F(x) = x^2+5 \Rightarrow$ ① x^2+4 ③ x^2+9
② $x^2+\sqrt{3}$ ④ x^2-5

$F(x) = x^2+c$

Def ⇒
The set of all anti-derivatives for $f(x)$ is called Indefinite integral
تكامل غير محدود

$\int 2x\,dx = x^2+c \longrightarrow \int f(x)dx = F(x)$
$f(x) = F'(x)$

$\boxed{1}$ $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c \quad , n \neq -1$

$\boxed{2}$ $\int \cos x\,dx = \sin x + c$

$\boxed{3}$ $\int \sin x\,dx = -\cos x + c$

$\boxed{4}$ $\int \sec^2 x\,dx = \tan x + c$

$\boxed{5}$ $\int \csc^2 x\,dx = -\cot x + c$

$\boxed{6}$ $\int \sec x \tan x\,dx = \sec x + c$

$\boxed{7}$ $\int \csc x \cot x\,dx = -\csc x + c$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Exp: $\int \sin^2 x \, dx$

$$= \int \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{2} x - \frac{1}{2\times 2} \sin 2x + c = \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

Exp: $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx$

$$= \frac{x}{2} + \frac{1}{4} \sin 2x + C$$

Exp: $\int \frac{\csc\theta}{\csc\theta - \sin\theta} \cdot d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} \cdot d\theta$

$$= \int \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}} \, d\theta = \int \frac{1}{\cos^2\theta} \, d\theta = \int \sec^2\theta \, d\theta = \tan\theta + c$$

Exp: $\int \frac{\sec z \tan z \, dz}{\sqrt{\sec z}}$

$u = \sec z$

$du = \sec z \tan z \, dz$

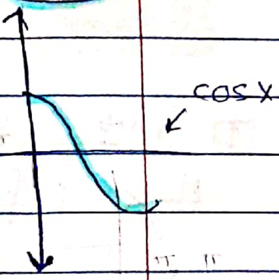
$$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2} \, du = 2\sqrt{u} + c$$
$$= 2\sqrt{\sec z \tan z} + c$$

Exp: $\displaystyle\int_0^{\pi} (\cos x + |\cos x|)\, dx$ absolut value

النتيجة يجب أن تكون موجبة لوجود القيمة المطلقة

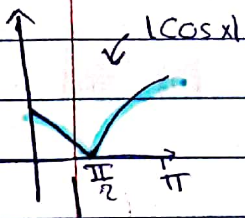
$$\int_0^{\pi} \cos x + \int_0^{\pi} |\cos x|\, dx$$

$$\sin x \Big|_0^{\pi} \Rightarrow \underbrace{0}_{} + \int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x\, dx$$



$$\sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\pi}$$

$$1 - (0-1) = \boxed{2}$$

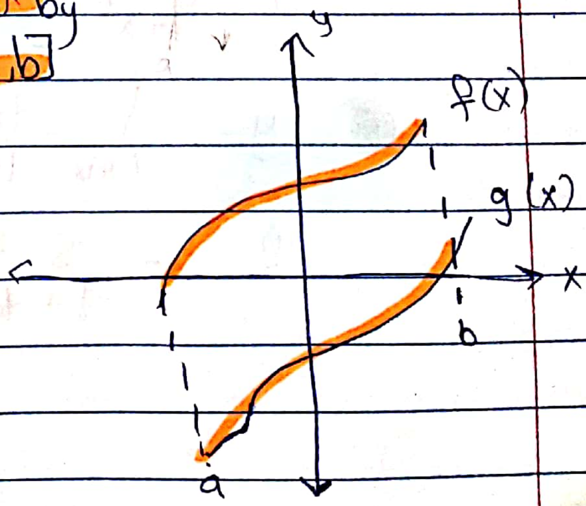


Remark: If $f(x) \geq 0 \;\; \forall x \in [a,b]$, then

$\int_a^b f(x)\, dx$ represents the area bounded between the x axis and the curve of $f(x)$ on $[a,b]$

Remark If $f(x) \geq g(x)$ $\forall x \in [a,b]$

then: $\int_a^b (f(x) - g(x))\, dx$ represents the area bounded by the 2 curve on $[a,b]$



Th: fundemental theorem of Calculus:

Assume $f(x)$ is Cont on $[a,b]$ and

1. If $F(x)$ is anti derivative for $f(x)$
then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

∴ definite integral ⟹ gives real number.
تكامل محدود

2. If $F(x) = \int_a^x f(t)\,dt$, then $F(x)$ is Cont on $[a,b]$
and $F(x)$ is diff on (a,b)
with $F'(x) = f(x)$

Exp: Find y' if (1) $y = \int_0^x \cos t\,dt$

(1) $y' = \cos x$

(2) $y = \int_0^2 \cos t\,dt \rightarrow y' = 0$

(3) $y = \int_0^{x^2} \frac{dt}{1+t^2}$

$y' = \frac{2x}{1+x^4}$ ⟹

عند اشتقاق الاقتران
المكامل ، نزيل إشارة التكامل
ونضع بدل dt مشتقة المتغير

(4) $y = \int_{\tan x}^{0} \frac{dz}{1+z^2}$

$y' = -\frac{\sec^2 x}{1+\tan^2 x} = -\frac{\sec^2 x}{\sec^2 x} = -1$

Find the area between the curve $y = 3x\sqrt{x^2+1}$ and x-axis on [0,1]

$$A = \int_0^1 3x\sqrt{x^2+1}\,.dx$$

$U = x^2+1$ $\left\{ \begin{array}{l} x=1 \to 2 \\ x=2 \to 2 \end{array} \right.$

$du = 2x$

$$\frac{3}{2}\int_0^1 \sqrt{U} = \frac{\cancel{3}}{2}\,\frac{U^{\frac{3}{2}}}{\cancel{\frac{3}{2}}} =$$

$$= \sqrt{U^3}\,\Big|_1^2 = \sqrt{(x^2+1)^3}\,\Big|_1^2$$

Exp: Find area enclosed by $f(x) = 2-x^2$ and $g(x) = -x$.

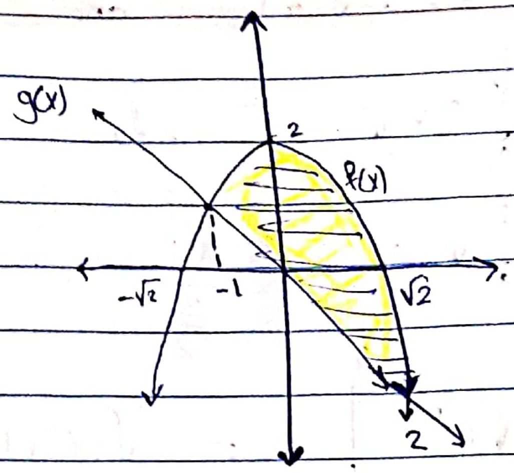
$f(x) = g(x)$

$2-x^2 = -x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1 , 2$



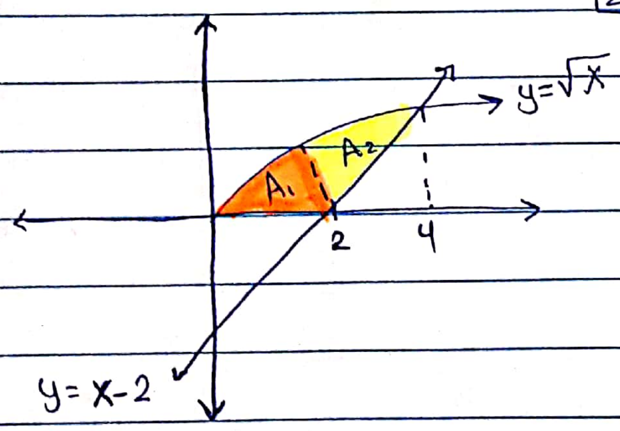
$$\int_{-1}^{2} \left(f(x) - g(x)\right).dx$$

$$\int_{-1}^{2} \left(2 - x^2 + x\right).dx = 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2} = \frac{9}{2}$$

Exp: Find the area bounded by x-axis, $y = \sqrt{x}$, $y = x-2$ using $\boxed{1}$ Integral w.r.t x

$\boxed{2}$ Integral w.r.t y

w.r.t: with respect to



$(\sqrt{x} = x-2)^2$

$x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 4 , 1$
✓ ✗

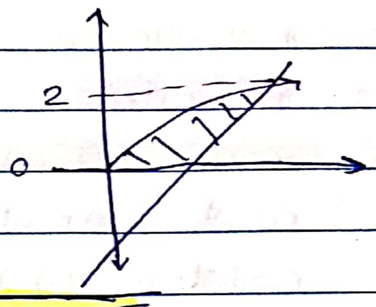
$$\int_{0}^{2} \sqrt{x}\ dx + \int_{2}^{4} \sqrt{x} - (x-2).dx = \frac{10}{3}$$

$\boxed{2}$ $A = \int_0^2 (y+2 - y^2) \, dy$

① $y = x - 2 \Rightarrow x = y+2$

② $y = \sqrt{x} \Rightarrow y^2 = x$

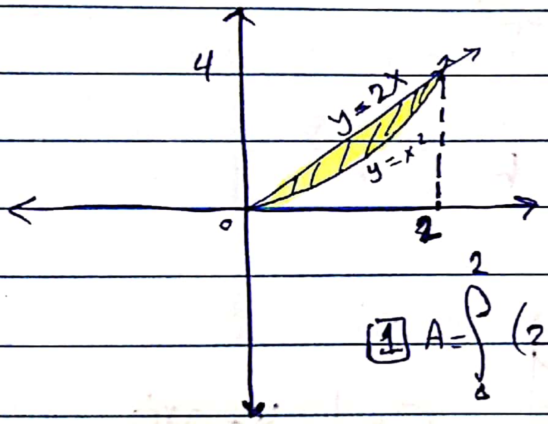
$= \frac{10}{3}$



Exp: Find the area bounded by $y = x^2$, $y = 2x$

$\boxed{1}$ Integral w.r.t x

$\boxed{2}$ Integral w.r.t y



$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

$\boxed{1}$ $A = \int_0^2 (2x - x^2) \, dx = \frac{4}{3}$

$\boxed{2}$ $A = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right) dy = \frac{4}{3}$