

## 14.5 Directional Derivatives and Gradient Vectors.

### Directional Derivatives in the plane.

Def: The derivative of  $f$  at  $P_0(x_0, y_0)$  in the

direction of the unit vector  $\vec{u} = u_1\vec{i} + u_2\vec{j}$  is

the number :

$$(D_{\vec{u}} f)_{P_0} = \left( \frac{df}{ds} \right)_{\vec{u}, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

"We read it: The derivative of  $f$  at  $P_0$  in the direction of  $\vec{u}$ ."

Note:  $f_x(x_0, y_0)$  is the directional derivative of  $f$

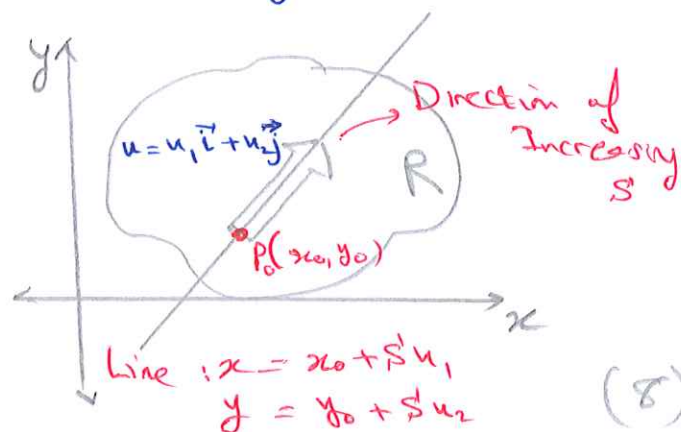
at  $P_0$  in the  $\vec{i}$  direction.

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$f_y(x_0, y_0)$  is the directional derivatives of  $f$  at  $P_0$

in the  $\vec{j}$  direction.



Example: Using the definition, find the derivative of

$f(x,y) = x^2 + xy$  at  $P_0(1,2)$  in the Direction

of the unit vector  $\vec{u} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$

$$\text{so } (D_{\vec{u}} f)_{P_0(1,2)} = \lim_{s \rightarrow 0} \frac{f\left(1 + \frac{1}{\sqrt{2}}s, 2 + \frac{1}{\sqrt{2}}s\right) - f(1,2)}{s}$$

$$= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{1}{\sqrt{2}}s\right)^2 + \left(1 + \frac{1}{\sqrt{2}}s\right)\left(2 + \frac{1}{\sqrt{2}}s\right) - [1+2]}{s}$$

$$= \lim_{s \rightarrow 0} \frac{\left(1 + \sqrt{2}s + \frac{1}{2}s^2\right) + \left(2 + \frac{3}{\sqrt{2}}s + \frac{1}{2}s^2\right) - 3}{s}$$

$$= \lim_{s \rightarrow 0} \frac{\frac{5}{\sqrt{2}}s + s^2}{s}$$

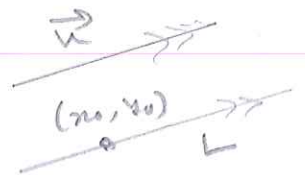
$$= \lim_{s \rightarrow 0} \frac{5}{\sqrt{2}} + s = \frac{5}{\sqrt{2}}$$

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←  
(1-e) The rate of change of  $f(x,y) = x^2 + xy$  at  $P_0(1,2)$  in the Direction of  $\vec{u}$  is  $\frac{\sqrt{5}}{2}$  (9)

## Calculation of Gradients:



Consider the line  $L: x = x_0 + u_1 s, y = y_0 + u_2 s$

through  $P_0(x_0, y_0)$ , parametrized with the arc length parameter  $s$  increasing in the direction of  $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ .

Then, Using chain Rule:

$$\left( \frac{df}{ds} \right)_{\vec{u}, P_0} = \left( \frac{\partial f}{\partial x} \right)_{P_0} \cdot \frac{dx}{ds} + \left( \frac{\partial f}{\partial y} \right)_{P_0} \cdot \frac{dy}{ds}$$

$$= \left( \frac{\partial f}{\partial x} \right)_{P_0} \cdot u_1 + \left( \frac{\partial f}{\partial y} \right)_{P_0} \cdot u_2$$

$$= \left[ \left( \frac{\partial f}{\partial x} \right)_{P_0} \vec{i} + \left( \frac{\partial f}{\partial y} \right)_{P_0} \vec{j} \right] \cdot \left[ u_1 \vec{i} + u_2 \vec{j} \right]$$

Gradient of  $f$   
at  $P_0$

Direction  
 $\vec{u}$

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$$\left( \frac{df}{ds} \right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

$\nabla f$  is real "grad  $f$ " or "gradient of  $f$ " or " $\nabla f$ "

Def: The gradient vector (gradient) of  $f(x, y)$

at a point  $P_0(x_0, y_0)$  is the vector

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

Theorem: The Directional derivative is a dot product

$$\left( \frac{df}{ds} \right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

Example: Find the derivative of  $f(x, y) = xe^y + \cos(xy)$

at the point  $(2, 0)$  in the direction of  $\vec{v} = 3\vec{i} - 4\vec{j}$

Sol:  $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$  ,  $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$ .

Now,  $f_x|_{(2,0)} = \left[ e^y + -\sin(xy)[y] \right]_{(2,0)} = 1 - 0 = \boxed{1}$

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$$f_y|_{(2,0)} = \left[ xe^y - x \sin(xy) \right]_{(2,0)} = 2 - 0 = \boxed{2}$$

$$\therefore \nabla f|_{(2,0)} = \vec{i} + 2\vec{j}$$

$$\therefore (D_{\vec{u}} f)_{P_0} = (\nabla f)_{(2,0)} \cdot \vec{u} = \frac{3}{5}(1) - \left(\frac{4}{5}\right)(2) = \frac{-5}{5} = \boxed{-1}$$

(11)



Recall:  $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$   $\theta$  between  $\vec{u}$  &  $\vec{v}$

$$\Rightarrow (D_{\vec{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u} = |\nabla f|_{P_0} |\vec{u}| \cos \theta$$

$\downarrow$   
unit vector

$$= |\nabla f| \cos \theta$$

So, we have the following properties:

- 1) The function  $f$  increases most rapidly when  $\cos \theta = 1$ , or when  $\theta = 0$  and  $\vec{u}$  is the direction of  $\nabla f$ . That is, at each point  $P$  in its domain,  $f$  increases most rapidly in the direction of the gradient vector  $\nabla f$  at  $P$ . In this case

$$D_{\vec{u}} f = |\nabla f| \cos(0) = |\nabla f|$$

- 2) Similarly,  $f$  decreases most rapidly in the direction of  $-\nabla f$ . In this case:

$$D_{\vec{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$$

- 3) Any direction  $\vec{u}$  orthogonal to  $\nabla f \neq 0$  is a direction of zero change in  $f$ , because  $\theta = \frac{\pi}{2}$

$$\Rightarrow D_{\vec{u}} f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = 0$$

Example: Find the Directions in which

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$$

(a) increases most rapidly at the point (1, 1).

(b) decrease most rapidly at the point (1, 1).

(c) what are the directions of zero change in  $f$  at (1, 1)?

So 1:

$$f_x \Big|_{(1,1)} = x = 1, \quad f_y \Big|_{(1,1)} = y = 1$$

$$\therefore \nabla f = \vec{i} + \vec{j}, \quad |\nabla f| = \sqrt{2}.$$

$$(a) \quad \vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}.$$

$$\text{Note: } (D_{\vec{u}} f)_{(1,1)} = |\nabla f| = \sqrt{2}.$$

$$(b) \quad \vec{u} = \frac{-\nabla f}{|\nabla f|} = -\frac{1}{\sqrt{2}} \vec{i} - \frac{1}{\sqrt{2}} \vec{j}$$

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$$(c) \quad (D_{\vec{u}} f)_{(1,1)} = (\nabla f)_{(1,1)} \cdot \vec{u} = 0.$$

$$\Rightarrow (\vec{i} + \vec{j}) \cdot (u_1 \vec{i} + u_2 \vec{j}) = 0$$

$$\Rightarrow u_1 + u_2 = 0 \quad \text{and we know } \underbrace{u_1^2 + u_2^2 = 1}_{\text{unit vector}}$$

$$\Rightarrow u_1 = -u_2$$

$$\Rightarrow u_1^2 + u_1^2 = 1 \Rightarrow 2u_1^2 = 1 \Rightarrow u_1 = \pm \frac{1}{\sqrt{2}}$$

$$\text{If } u_1 = \frac{1}{\sqrt{2}}, \text{ then } u_2 = -\frac{1}{\sqrt{2}} \Rightarrow \vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

$$\text{If } u_1 = -\frac{1}{\sqrt{2}}, \text{ then } u_2 = \frac{1}{\sqrt{2}} \Rightarrow \vec{v} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

$$(\text{check: } (\nabla f)|_{(1,1)} \cdot \vec{u} = 0 \quad \& \quad (\nabla f)|_{(1,1)} \cdot \vec{v} = 0)$$

### Gradients and Tangents to Level Curves:

If  $f(x, y)$  has a constant value  $C$  along a smooth

curve  $\vec{r} = g(t)\vec{i} + h(t)\vec{j}$ , (making the

curve a level curve of  $f$ ), then:

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$$f(g(t), h(t)) = C.$$

$$\frac{df}{dt}(g(t), h(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot g'(t) + \frac{\partial f}{\partial y} \cdot h'(t) = 0$$

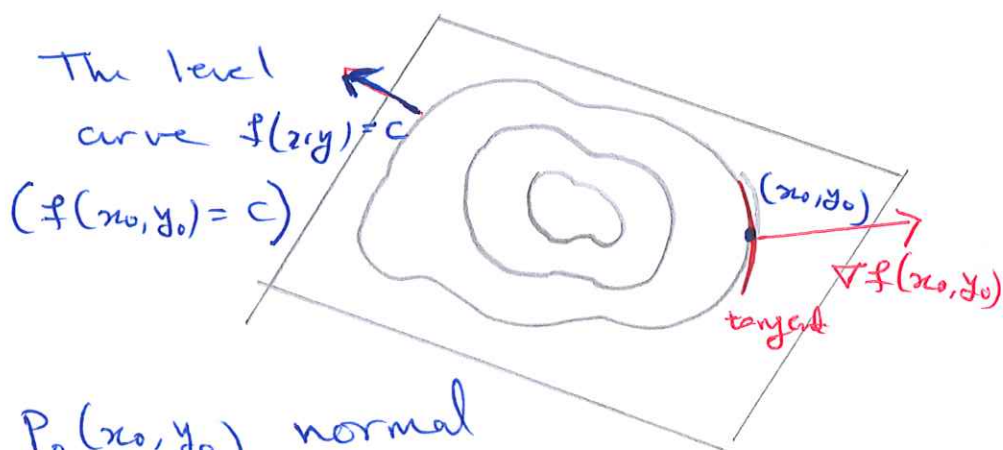
(14)

$$\Rightarrow \left( \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \cdot \left( g'(t) \vec{i} + h'(t) \vec{j} \right) = 0$$

$$\nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

which means that  $\nabla f$  is normal to the tangent vector  $\frac{d\vec{r}}{dt}$ , so  $\nabla f$  is normal to the Curve.

Result: At every point  $(x_0, y_0)$  in the domain of a differentiable function  $f(x, y)$ , the gradient of  $f$  is Normal to the Level curve through  $(x_0, y_0)$ .



The line through  $P_0(x_0, y_0)$  normal

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to the vector  $\vec{N} = A\vec{i} + B\vec{j}$

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has the equation  $A(x - x_0) + B(y - y_0) = 0$

$$\text{If } \vec{N} = (\nabla f)_{(x_0, y_0)} = f_x(x_0, y_0) \vec{i} + f_y(x_0, y_0) \vec{j}$$

then the equation is the tangent line given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \quad (15)$$



Equation of the Tangent Line:

$$y - y_0 = m (x - x_0) \quad , \quad m = \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$$

$$\Rightarrow y - y_0 = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} (x - x_0)$$

$$\Rightarrow y - y_0 = \frac{-f_x(x_0, y_0)}{f_y(x_0, y_0)} (x - x_0)$$

$$\Rightarrow (y - y_0) f_y(x_0, y_0) = -f_x(x_0, y_0) (x - x_0)$$

$$\Rightarrow f_x(x_0, y_0) x + f_y(x_0, y_0) y = f_x(x_0, y_0) x_0 + f_y(x_0, y_0) y_0$$

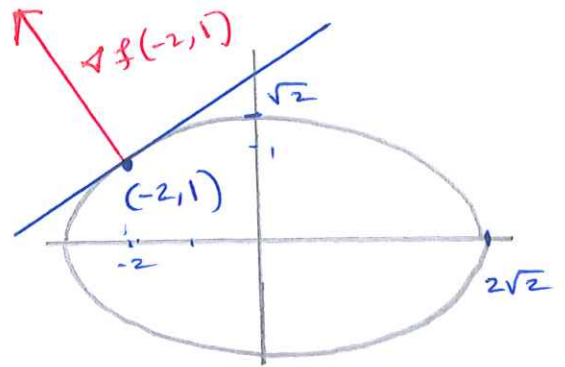
Example: Find an equation for the tangent to the

ellipse :  $\frac{x^2}{4} + y^2 = 2$  at  $(-2, 1)$

so 1:  $f(x, y) = \frac{x^2}{4} + y^2 = 2$

∴ Ellipse is a level of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$



$$f_x \Big|_{(-2, 1)} = \frac{x}{2} \Big|_{(-2, 1)} = -1, \quad f_y \Big|_{(-2, 1)} = 2y \Big|_{(-2, 1)} = 2$$

$$\therefore \vec{N} = (\nabla f)_{(x_0, y_0)} = -\vec{i} + 2\vec{j}.$$

∴ The tangent line is

$$f_x(-2, 1)(x - (-2)) + f_y(-2, 1)(y - 1) = 0$$

$$\Rightarrow (-1)(x + 2) + 2(y - 1) = 0$$

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$$\Rightarrow x - 2y = -4$$

$$\Rightarrow y = \frac{x + 4}{2}$$

## Algebra Rules for Gradients:

$$1) \quad \nabla (f + g) = \nabla f + \nabla g$$

$$2) \quad \nabla (f - g) = \nabla f - \nabla g$$

$$3) \quad \nabla (kf) = k \nabla f$$

$$4) \quad \nabla (fg) = f \nabla g + g \nabla f.$$

$$5) \quad \nabla \left( \frac{f}{g} \right) = \frac{(g \nabla f - f \nabla g)}{g^2}$$

Example: If  $f(x, y) = x - y$ ,  $g(x, y) = 3y$

$$\text{then } \nabla (fg) = \nabla ((x - y) \cdot (3y)) = \nabla (3xy - 3y^2)$$

$$= 3y \vec{i} - (6y - 3x) \vec{j}$$

$$= 3y \vec{i} + 3x \vec{j} - 6y \vec{j}$$

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$$\text{Now : } f \nabla g + g \nabla f = (x - y)(3 \vec{j}) + (3y)(\vec{i} - \vec{j})$$

$$= 3x \vec{j} - 3y \vec{j} + 3y \vec{i} - 3y \vec{j}$$

$$= 3y \vec{i} + 3x \vec{j} - 6y \vec{j}$$

## Functions of three Variables:

For a differentiable function  $f(x, y, z)$  and a unit vector  $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$  in space

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

and  $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3$ .

Example: (a) Find the derivative of  $f(x, y, z) = x^3 - xy^2 - z$  at  $P_0(1, 1, 0)$  in the direction of  $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ .

(b) In what directions does  $f$  change most rapidly at  $P_0$ , and what are the rates of change in these directions.

Sol:  $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7} \vec{i} - \frac{3}{7} \vec{j} + \frac{6}{7} \vec{k}, |\vec{v}| = 7$

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$$\Rightarrow f_x(1, 1, 0) = (3x^2 - y^2) \big|_{(1, 1, 0)} = 2$$

$$f_y(1, 1, 0) = -2xy \big|_{(1, 1, 0)} = -2$$

$$f_z(1, 1, 0) = -1$$



$$\therefore \nabla f|_{P_0} = 2\vec{i} - 2\vec{j} - \vec{k}$$

$$\begin{aligned} \therefore (D_{\vec{u}} f)|_{P_0} &= \nabla f|_{P_0} \cdot \vec{u} = (2\vec{i} - 2\vec{j} - \vec{k}) \cdot \left(\frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}\right) \\ &= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7} \end{aligned}$$

(b)  $f$  increases most rapidly in the direction of  $\nabla f$ .

$$\therefore \text{rate of change is } |\nabla f| = \sqrt{9} = 3$$

$f$  decreases most rapidly in the direction of  $-\nabla f$ .

$$\therefore \text{rate of change} = -|\nabla f| = -3.$$