Physics 310
Notes on Coordinate Systems and Unit Vectors

A general system of coordinates uses a set of parameters to define a vector. For example, x, y and z are the parameters that define a vector $\boldsymbol{r}$ in Cartesian coordinates:

$$\boldsymbol{r} = \hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y + \hat{\boldsymbol{k}}z \tag{1}$$

Similarly a vector in cylindrical polar coordinates is described in terms of the parameters r, θ and z since a vector $\boldsymbol{r}$ can be written as $\boldsymbol{r} = r\hat{\boldsymbol{r}} + z\hat{\boldsymbol{k}}$. The dependence on θ is not obvious here, but the unit vector $\hat{\boldsymbol{r}}$ is actually a function of the polar angle, θ. If you want, you can make this dependence explicit by writing

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}(\theta) + \hat{\boldsymbol{k}}z \tag{2}$$

Finally, a vector in spherical coordinates is described in terms of the parameters r, the polar angle θ and the azimuthal angle ϕ as follows:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}(\theta, \phi) \tag{3}$$

where the dependence of the unit vector $\hat{\boldsymbol{r}}$ on the parameters θ and ϕ has been made explicit.

It can be very useful to express the unit vectors in these various coordinate systems in terms of their components in a Cartesian coordinate system. For example, in cylindrical polar coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{4}$$

while in spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned} \tag{5}$$

Using these representations, we can construct the components of all unit vectors in these coordinate systems and in this way define explicitly the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, *etc.*

If a vector, $\boldsymbol{r}$ depends on a parameters u, then a vector that points in the "direction" of increasing u is defined by

$$\boldsymbol{e}_u = \frac{\partial \boldsymbol{r}}{\partial u}. \tag{6}$$

This vector is not necessarily normalized to have unit length, but from it we can always construct the unit vector

$$\hat{\boldsymbol{e}}_u = \frac{\boldsymbol{e}_u}{|\boldsymbol{e}_u|} \tag{7}$$

We will apply this definition to the Cartesian, cylindrical and spherical coordinate systems to illustrate the construction of their unit vectors.

The case of Cartesian coordinates is almost trivial:

$$\begin{aligned}
\boldsymbol{e}_x &= \frac{\partial \boldsymbol{r}}{\partial x} = \hat{\boldsymbol{\imath}} && (8)\\
\boldsymbol{e}_y &= \frac{\partial \boldsymbol{r}}{\partial y} = \hat{\boldsymbol{\jmath}} && (9)\\
\boldsymbol{e}_z &= \frac{\partial \boldsymbol{r}}{\partial z} = \hat{\boldsymbol{k}}. && (10)
\end{aligned}$$

It also turns out that each of these vectors is already normalized to have unit length.

In the case of cylindrical polar coordinates, using Equations 2 and 4,

$$\begin{aligned}
\boldsymbol{e}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = \hat{\boldsymbol{r}}(\theta) \\
&= \hat{\boldsymbol{\imath}} \cos\theta + \hat{\boldsymbol{\jmath}} \sin\theta, && (11)\\
\boldsymbol{e}_\theta &= \frac{\partial \boldsymbol{r}}{\partial \theta} = r\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} \\
&= -\hat{\boldsymbol{\imath}} r \sin\theta + \hat{\boldsymbol{\jmath}} r \cos\theta, && (12)\\
\boldsymbol{e}_z &= \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{k} && (13)
\end{aligned}$$

The unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are the constructed using Equation 7 as follows:

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \frac{\boldsymbol{e}_r}{\sqrt{\cos^2\theta + \sin^2\theta}} = \boldsymbol{e}_r && (14)\\
\hat{\boldsymbol{\theta}} &= \frac{\boldsymbol{e}_\theta}{r\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{\boldsymbol{e}_\theta}{r} && (15)
\end{aligned}$$

so it turns out that $\boldsymbol{e}_r$ was already normalized to unit length.

For the last example, in spherical coordinates, using Equations 3 and 5,

$$\begin{aligned}
\boldsymbol{e}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = \hat{\boldsymbol{r}}(\theta, \phi) \\
&= \hat{\boldsymbol{\imath}} \sin\theta\cos\phi + \hat{\boldsymbol{\jmath}} \sin\theta\sin\phi + \hat{\boldsymbol{k}} \cos\theta, && (16)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{e}_\phi &= \frac{\partial \boldsymbol{r}}{\partial \phi} = r\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} \\
&= -\hat{\boldsymbol{\imath}} r \sin\theta \sin\phi + \hat{\boldsymbol{\jmath}} r \sin\theta\cos\phi, \qquad (17)\\
\boldsymbol{e}_\theta &= \frac{\partial \boldsymbol{r}}{\partial \theta} = r\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} \\
&= \hat{\boldsymbol{\imath}} r \cos\theta\cos\phi + \hat{\boldsymbol{\jmath}} r\cos\theta\sin\phi - \hat{\boldsymbol{k}} r \sin\theta \qquad (18)
\end{aligned}$$

The unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ are the constructed using Equation 7 as follows:

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \frac{\boldsymbol{e}_r}{\sqrt{\sin^2\theta(\sin^2\phi+\cos^2\phi)+\cos^2\theta}} = \frac{\boldsymbol{e}_r}{\sqrt{\sin^2\theta+\cos^2\theta}} = \boldsymbol{e}_r \qquad (19)\\
\hat{\boldsymbol{\phi}} &= \frac{\boldsymbol{e}_\phi}{r\sin\theta\sqrt{\sin^2\phi+\cos^2\phi}} = \frac{\boldsymbol{e}_\phi}{r\sin\theta} \qquad (20)\\
\hat{\boldsymbol{\theta}} &= \frac{\boldsymbol{e}_\theta}{r\sqrt{\sin^2\theta(\sin^2\phi+\cos^2\phi)+\cos^2\theta}} = \frac{\boldsymbol{e}_\theta}{r\sqrt{\sin^2\theta+\cos^2\theta}} = \frac{\boldsymbol{e}_\theta}{r} \qquad (21)
\end{aligned}$$

so it turns out that $\boldsymbol{e}_r$ was already normalized to unit length. From Equation 20 you can see that the direction of $\hat{\boldsymbol{\phi}}$ becomes completely undefined when $\theta = 0$ or $\theta = \pi$.

We usually express time derivatives of the unit vectors in a particular coordinate system in terms of the unit vectors themselves. Since all unit vectors in a Cartesian coordinate system are constant, their time derivatives vanish, but in the case of polar and spherical coordinates they do not.

In polar coordinates,

$$\begin{aligned}
\frac{d\hat{\boldsymbol{r}}}{dt} &= (-\hat{\boldsymbol{\imath}}\sin\theta + \hat{\boldsymbol{\jmath}}\cos\theta)\frac{d\theta}{dt} = \hat{\boldsymbol{\theta}}\dot{\theta} \qquad (22)\\
\frac{d\hat{\boldsymbol{\theta}}}{dt} &= (-\hat{\boldsymbol{\imath}}\cos\theta - \hat{\boldsymbol{\jmath}}\sin\theta)\frac{d\theta}{dt} = -\hat{\boldsymbol{r}}\dot{\theta} \qquad (23)
\end{aligned}$$

In spherical coordinates,

$$\begin{aligned}
\frac{d\hat{\boldsymbol{r}}}{dt} &= \frac{d\hat{\boldsymbol{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{\boldsymbol{r}}}{d\phi}\frac{d\phi}{dt} \\
&= (\hat{\boldsymbol{\imath}}\cos\theta\cos\phi + \hat{\boldsymbol{\jmath}}\cos\theta\sin\phi - \hat{\boldsymbol{k}}\sin\theta)\frac{d\theta}{dt} + (-\hat{\boldsymbol{\imath}}\sin\theta\sin\phi + \hat{\boldsymbol{\jmath}}\sin\theta\cos\phi)\frac{d\phi}{dt} \\
&= \hat{\boldsymbol{\theta}}\dot{\theta} + \hat{\boldsymbol{\phi}}\sin\theta\dot{\phi} \qquad (24)\\
\frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\hat{\boldsymbol{r}}\dot{\phi}\sin\theta - \hat{\boldsymbol{\phi}}\dot{\phi}\cos\theta \qquad (25)\\
\frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\hat{\boldsymbol{r}}\dot{\theta} + \hat{\boldsymbol{\phi}}\dot{\phi}\cos\theta \qquad (26)
\end{aligned}$$

