Exercises.

Q35: Find the mean and variance of the beta distribution

$$\rightsquigarrow E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 x \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{\beta(\alpha,\beta)} \int_0^1 x^{(\alpha+1)-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{\beta(\alpha,\beta)} \beta(\alpha+1, \beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta)\cdot(\alpha+\beta)} \frac{\Gamma(\alpha)\cdot\alpha}{\Gamma(\alpha)}$$

$$E(X) = \frac{\alpha}{\alpha+\beta}$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

Q36: Determine the constant c in each of the following so that each f(x) is a beta p.d.f.

a. $f(x) = \begin{cases} cx(1-x)^3 , & 0<x<1 \\ 0 , & \text{elsewhere} \end{cases}$

$$g(x) = \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{c} x^{\alpha-1} (1-x)^{\beta-1}$$

$\Rightarrow \alpha = 2 , \quad \beta = 4$.

$$\Rightarrow C = \frac{\Gamma(2+4)}{\Gamma(2)\Gamma(4)} = \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}$$

we know $\Gamma(\alpha) = (\alpha-1)!$ since $\alpha = 2, 3, \dots$.

$$\Rightarrow C = \frac{5!}{1!\ 3!} = 20 .$$

b. $f(x) = \begin{cases} cx^4 (1-x)^5 , & 0<x<1 \\ 0 , & \text{elsewhere} \end{cases}$

$\alpha = 5, \quad \beta = 6$

$$\Rightarrow C = \frac{\Gamma(5+6)}{\Gamma(5)\Gamma(6)} = \frac{10!}{4!\ 5!} = 1260 .$$

c. $f(x) = \begin{cases} cx^2(1-x)^8 & , \; 0<x<1 \\ 0 & , \; \text{elsewhere} \end{cases}$

$\alpha = 3 \quad , \quad \beta = 9$

$\rightarrow c = \dfrac{\Gamma(3+9)}{\Gamma(3)\,\Gamma(9)} = \dfrac{11!}{2!\;8!} = 495$.

Q37: Determine the constant c so that $f(x) = \begin{cases} cx(3-x)^4 & , \; 0<x<3 \\ 0 & , \; \text{elsewhere} \end{cases}$ is a p.d.f. p-dist مش نفس ←

$\rightarrow \displaystyle\int_0^3 cx(3-x)^4\,dx = 1$

$c\displaystyle\int_0^3 x(3-x)^4\,dx = 1$ By part

$x \quad\quad (3-x)^4$

$1 \quad\quad \dfrac{-(3-x)^5}{5}$

$c\left[-\dfrac{1}{5}x(3-x)^5 - \displaystyle\int_0^3 -\dfrac{1}{5}(3-x)^5\,dx\right] = 1$

$c\left[-\dfrac{1}{5}x(3-x)^5 + \dfrac{1}{5}\left(\dfrac{-(3-x)^6}{6}\right)\right] = 1$

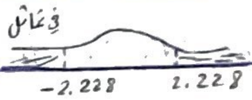
$c\left[-\dfrac{1}{5}x(3-x)^5 - \dfrac{1}{30}(3-x)^6 \Big|_0^3\right] = 1$

$c\left[0 - \left(0 - \dfrac{1}{30}(3)^6\right)\right] = 1$

$c\;\dfrac{729}{30} = 1$

$c\;\dfrac{243}{10} = 1 \rightarrow \boxed{c = \dfrac{10}{243}}$

Q46: let T have a t-distribution with 10 degree of freedom. Find $Pr(|T| > 2.228)$. use table.



$$Pr(|T| > 2.228) = 2\, Pr(T > 2.228)$$
$$= 2(1 - Pr(T < 2.228))$$
$$= 2(1 - 0.975)$$
$$= 2(0.025)$$
$$= 0.05$$

Q42: let F have an F-distribution with parameters r_1 and r_2. Prove that $\frac{1}{F}$ has an F distributions r_2 and r_1.

We know F-distribution is a Ratio of two chi-squar distributed R.V.

say X and Y each devided by there degree of freedom.

X have r_1 d.f, y have r_2 d.f and both indep.

So the F dist defined by $F = \frac{X/r_1}{y/r_2}$

Now, take inverse of $F \rightarrow \frac{1}{F}$

$$\frac{1}{F} = \frac{y/r_2}{X/r_1}$$

So $\frac{1}{F}$ follows an F distribution itself but with switched parameters

$\Rightarrow \frac{1}{F}$ follows an F-dist. with parameters r_2 (d.f of y) and r_1 (d.f of X)

Therefor $\frac{1}{F}$ has an F-dist. with parameter r_2, r_1.

Q43: if F has an F distribution with parameters $r_1 = 5$, $r_2 = 10$
Find a and b so that $P_r(F \leq a) = 0.05$ and $P_r(F \leq b) = 0.95$ and accordingly $P_r(a < F < b) = 0.9$.

$\rightarrow$ $P_r(F \leq a) = P_r\left(\frac{1}{F} \geq \frac{1}{a}\right) = 1 - P_r\left(\frac{1}{F} \leq \frac{1}{a}\right)$

$$0.05 = 1 - P_r\left(\frac{1}{F} \leq \frac{1}{a}\right)$$

$$0.95 = P_r\left(\frac{1}{F} \leq \frac{1}{a}\right)$$

$\Rightarrow$ $\frac{1}{a} = 4.74$ $\rightarrow$ $a = \frac{1}{4.74}$ $\underbrace{r_1 = 10 , r_2 = 5}_{\frac{1}{F} \text{ عكسنا لأن}}$

$\rightarrow$ $P_r(F \leq b) = 0.95$ with $r_1 = 5$, $r_2 = 10$

By table $\Rightarrow$ $b = 3.33$.

Q44: let $T = \frac{W}{\sqrt{V/r}}$, where the indep. variables W and V are resp. Normal with mean 0 and variance 1 and chi square with r d.f .
show that T^2 has an F-distribution with parameters $r_1 = 1$, $r_2 = r$.

$W \sim N(0,1)$

$V \sim \chi^2(r)$

$\rightarrow$ $T^2 = \frac{W^2}{V/r}$ But $W^2 \sim \chi^2(1)$

$\Rightarrow$ $T^2 = \frac{W^2/1}{V/r}$ So T^2 follows an F dist. with $r_1 = 1$ and $r_2 = r$

$\downarrow$

يعني صار T^2 عبارة عن قسمة chi-square chi.

والبسط df=1 والمقام r

وهي عبارة عن F-dist .