Lecture – 17

Linearization of Nonlinear Systems

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Problem statement

Problem: Given a nonlinear system

$$\dot{X} = f(X, U)$$

Derive an approximate linear system

$$\dot{X} = AX + BU$$

about an "Operating Point" (X_0, U_0)

Note: An operating point is a point through which the system trajectory passes.

Linearization:
Scalar homogeneous systems

Scalar system: $\dot{x} = f(x), \quad x \in R$

Operating point: x_0

Define: $x = x_0 + \Delta x$

Taylor series:

HOT

$$\dot{x}_0 + \Delta\dot{x} = f(x_0 + \Delta x) = f(x_0) + f'(x)\Big|_{x_0} \Delta x + \underbrace{\left\{ f''(x)_{x_0} \frac{(\Delta x)^2}{2!} + \cdots \right\}}_{\text{HOT}}$$

Neglecting HOT, $\dot{x}_0 + \Delta\dot{x} \approx f(x_0) + f'(x_0)\Delta x$

Linearization: Scalar homogeneous systems

x_0 satisfies the differential equation $\dot{x}_0 = f(x_0)$

This leads to $\Delta\dot{x} = \left[f'(x_0) \right] \Delta x = a \, \Delta x$

For convenience, redefine $\boxed{x \triangleq \Delta x}$

This leads to
$$\boxed{\begin{array}{l} \dot{x} = ax \\ \text{where } a = f'(x) \end{array}}$$

Example – 1

Linearize: $\dot{x} = x^2 - 1, \quad x(0) = \pm 1$

Solution: $a_1 = \left.\dfrac{df}{dx}\right|_{x_0=1} = 2x_0 \,|_{x_0=1} = 2$

$$a_2 = \frac{df}{dx}\,|_{x_0=-1} = 2x_0 \,|_{x_0=-1} = -2$$

The linearized system:

$$\dot{x} = 2x \qquad x_0 = 1$$

$$\dot{x} = -2x \qquad x_0 = -1$$

<u>Note:</u> As the reference point changes, the linearized approximation also changes!

Linearization: General homogeneous systems

Homogeneous System:

$$\dot{X} = f(X), \quad f \triangleq [f_1 \quad f_2 \quad \dots \quad f_n]^T, \quad X \triangleq [x_1 \quad x_2 \quad \cdots \quad x_n]^T$$

Taylor Series: $$f(X_0 + \Delta X) = f(X_0) + \left[\frac{\partial f}{\partial X}\right]_{X_0} \Delta X + \cancel{HOT}$$

$$\dot{X}_0 + \Delta\dot{X} \approx f(X_0) + \left[\frac{\partial f}{\partial X}\right]_{X_0} \Delta X$$

$$\Delta X \triangleq X$$

$$\dot{X} = A\, X$$

$$A = \left[\frac{\partial f}{\partial X}\right]_{X_0} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Example – 2: Van-der Pol's Oscillator (Limit cycle behaviour)

- Equation $\quad M\,\ddot{x} + 2c\left(x^2 - 1\right)\dot{x} + k\,x = 0 \qquad \{c, k > 0\}$

- State variables $\quad x_1 \triangleq x, \qquad x_2 \triangleq \dot{x}$

- State Space Equation

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} x_2 \\ -\dfrac{2c}{m}\left(x_1^2 - 1\right)x_2 - \dfrac{k}{m}x_1 \end{bmatrix}}_{F(X)} : \text{ Homogeneous nonlinear system}$$

Example – 2: Van-der Pol's Oscillator (Limit cycle behaviour)

- Operating Point: $\begin{bmatrix} x_{1_0} & x_{2_0} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

- Linearized State Space Equation

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \left. \begin{bmatrix} 0 & 1 \\ -\dfrac{2cx_2(2x_1)}{m} - \dfrac{k}{m}x_1 & -\dfrac{2c}{m}\left(x_1^2 - 1\right) \end{bmatrix} \right|_{X_0 = 0} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \dfrac{2c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X}$$

Linearization: General Systems

System having control input

$$\dot{X} = f\left(X, U\right), \qquad f, X \in \mathbb{R}^n, \quad U \in \mathbb{R}^m$$

Reference point: $\left(X_0, U_0\right)$

Taylor series expansion:

$$f\left(X_0 + \Delta X,\ U_0 + \Delta U\right)$$

$$= f\left(X_0, U_0\right) + \left[\frac{\partial f}{\partial X}\right]_{(X_0, U_0)} \Delta X + \left[\frac{\partial f}{\partial U}\right]_{(X_0, U_0)} \Delta U + HOT$$

Linearization

$$\cancel{\dot{X}_0} + \Delta\dot{X} \approx \cancel{f(X_0, U_0)} + \left[\frac{\partial f}{\partial X}\right]_{(X_0, U_0)} \Delta X + \left[\frac{\partial f}{\partial U}\right]_{(X_0, U_0)} \Delta U$$

$$\Delta\dot{X} = A\,\Delta X + B\,\Delta U$$

Re-define: $\quad \Delta X \triangleq X, \quad \Delta U \triangleq U$

This leads to $\quad \dot{X} = AX + BU$

$$A_{n\times n} = \left[\frac{\partial f}{\partial X}\right]_{(X_0, U_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(X_0, U_0)} \qquad B_{n\times m} = \left[\frac{\partial f}{\partial U}\right]_{(X_0, U_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{(X_0, U_0)}$$

Example – 3: Spinning Body Dynamics (Satellite dynamics)

Dynamics:

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 + \left(\frac{1}{I_1}\right)\tau_1$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 + \left(\frac{1}{I_2}\right)\tau_2$$

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 + \left(\frac{1}{I_3}\right)\tau_3$$

I_1, I_2, I_3 : MI about principal axes

$\omega_1, \omega_2, \omega_3$: Angular velocities about principal axes

τ_1, τ_2, τ_3: Torques about principal axes

Example – 3: Spinning Body Dynamics (Satellite dynamics)

- Operating Point: $\begin{bmatrix} \omega_{1_0} & \omega_{2_0} & \omega_{3_0} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$

  $\begin{bmatrix} \tau_{1_0} & \tau_{2_0} & \tau_{3_0} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$

- Linearized State Space Equation (Double Integrator)

$$\underbrace{\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} (1/I_1) & 0 & 0 \\ 0 & (1/I_2) & 0 \\ 0 & 0 & (1/I_3) \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}}_{U}$$

Example – 4: Airplane Dynamics, Six Degree-of-Freedom Nonlinear Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$\dot{U} = VR - WQ - g\sin\Theta + \left(F_{A_X} + F_{T_X}\right)/m$$

$$\dot{V} = WP - UR + g\sin\Phi\cos\Theta + \left(F_{A_Y} + F_{T_Y}\right)/m$$

$$\dot{W} = UQ - VP + g\cos\Phi\cos\Theta + \left(F_{A_Z} + F_{T_Z}\right)/m$$

$$\dot{P} = c_1 QR + c_2 PQ + c_3\left(L_A + L_T\right) + c_4\left(N_A + N_T\right)$$

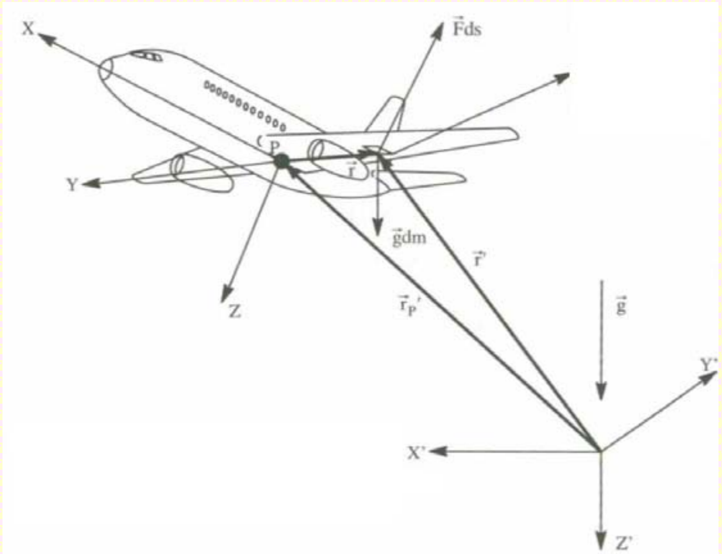
$$\dot{Q} = c_5 PR - c_6\left(P^2 - R^2\right) + c_7\left(M_A + M_T\right)$$

$$\dot{R} = c_8 PQ - c_2 QR + c_4\left(L_A + L_T\right) + c_9\left(N_A + N_T\right)$$

$$\dot{\Phi} = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta$$

$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi$$

$$\dot{\Psi} = \left(Q\sin\Phi + R\cos\Phi\right)\sec\Theta$$



$$\begin{bmatrix}\dot{X}'\\ \dot{Y}'\\ \dot{Z}'\end{bmatrix} = \begin{bmatrix}\cos\Psi & -\sin\Psi & 0\\ \sin\Psi & \cos\Psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\Theta & 0 & \sin\Theta\\ 0 & 1 & 0\\ -\sin\Theta & 0 & \cos\Theta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\Phi & -\sin\Phi\\ 0 & \sin\Phi & \cos\Phi\end{bmatrix}\begin{bmatrix}U\\ V\\ W\end{bmatrix}$$

$\left[\text{Note:} \quad \dot{h} = -\dot{Z}'\right]$

Linearization using Small Perturbation Theory

Perturbation in the variables:

$$U = U_0 + \Delta U \qquad V = V_0 + \Delta V \qquad W = W_0 + \Delta W$$

$$P = P_0 + \Delta P \qquad Q = Q_0 + \Delta Q \qquad R = R_0 + \Delta R$$

$$X = X_0 + \Delta X \qquad Y = Y_0 + \Delta Y \qquad Z = Z_0 + \Delta Z$$

$$X_T = X_{T_0} + \Delta X_T \quad Y_T = Y_{T_0} + \Delta Y_T \qquad Z_T = Z_{T_0} + \Delta Z$$

$$M = M_0 + \Delta M \quad N = N_0 + \Delta N \qquad L = L_0 + \Delta L$$

$$\Phi = \Phi_0 + \Delta\phi \qquad \Theta = \Theta_0 + \Delta\theta \qquad \Psi = \Psi_0 + \Delta\psi$$

$$\delta_A = \delta_{A_0} + \Delta\delta_A \quad \delta_E = \delta_{E_0} + \Delta\delta_E \quad \delta_R = \delta_{R_0} + \Delta\delta_R$$

Trim Condition for Straight and Level Flight

- Assume: $V_0 = P_0 = Q_0 = R_0 = \Phi_0 = \underbrace{Y_{T_0} = Z_{T_0} = 0}_{\text{Typically True } \forall t}$
- Select: $X_{T_0},\, z_{I_0} \ (i.e.\ h_0)$
- Enforce: $\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = \dot{\Phi} = \dot{\Theta} = \dot{z}_I = 0$
- Solve for: $U_0, W_0, X_0, Y_0, Z_0, L_0, M_0, N_0, \Theta_0$
- Verify: $Y_0 = L_0 = M_0 = N_0 = 0$

Linearization using Small Perturbation Theory

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

$$\Delta X = \frac{\partial X}{\partial U}\Delta U + \frac{\partial X}{\partial W}\Delta W + \frac{\partial X}{\partial \delta_E}\Delta\delta_E + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

$$\Delta Y = \frac{\partial Y}{\partial V}\Delta V + \frac{\partial Y}{\partial P}\Delta P + \frac{\partial Y}{\partial R}\Delta R + \frac{\partial Y}{\partial \delta_R}\Delta\delta_R$$

$$\Delta Z = \frac{\partial Z}{\partial U}\Delta U + \frac{\partial Z}{\partial W}\Delta W + \frac{\partial Z}{\partial \dot{W}}\Delta\dot{W} + \frac{\partial Z}{\partial Q}\Delta Q + \frac{\partial Z}{\partial \delta_E}\Delta\delta_E + \frac{\partial Z}{\partial \delta_T}\Delta\delta_T$$

$$\Delta L = \frac{\partial L}{\partial V}\Delta V + \frac{\partial L}{\partial P}\Delta P + \frac{\partial L}{\partial R}\Delta R + \frac{\partial L}{\partial \delta_R}\Delta\delta_R + \frac{\partial L}{\partial \delta_A}\Delta\delta_A$$

$$\Delta M = \frac{\partial M}{\partial U}\Delta U + \frac{\partial M}{\partial W}\Delta W + \frac{\partial M}{\partial \dot{W}}\Delta\dot{W} + \frac{\partial M}{\partial Q}\Delta Q + \frac{\partial M}{\partial \delta_E}\Delta\delta_E + \frac{\partial M}{\partial \delta_T}\Delta\delta_T$$

$$\Delta N = \frac{\partial N}{\partial V}\Delta V + \frac{\partial N}{\partial P}\Delta P + \frac{\partial N}{\partial R}\Delta R + \frac{\partial N}{\partial \delta_R}\Delta\delta_R + \frac{\partial N}{\partial \delta_A}\Delta\delta_A$$

State Variable Representation of Longitudinal Dynamics

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

State space form:

$$\dot{X} = AX + BU_c$$

$$A = \begin{bmatrix} X_U & X_W & 0 & -g \\ Z_U & Z_W & U_0 & 0 \\ M_U + M_{\dot{W}} Z_U & M_W + M_{\dot{W}} Z_W & M_Q + M_{\dot{W}} U_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} \Delta U \\ \Delta W \\ \Delta Q \\ \Delta \theta \end{bmatrix}$$

$$B = \begin{bmatrix} X_{\delta_E} & X_{\delta_T} \\ Z_{\delta_E} & Z_{\delta_T} \\ M_{\delta_E} + M_{\dot{W}} Z_{\delta_E} & M_{\delta T} + M_{\dot{W}} Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \qquad U_c = \begin{bmatrix} \Delta \delta_E \\ \Delta \delta_T \end{bmatrix}$$

$$X_U = \frac{1}{m}\left(\frac{\partial X}{\partial U}\right), \quad X_W = \frac{1}{m}\left(\frac{\partial X}{\partial W}\right) \quad etc.$$

State Variable Representation of Lateral Dynamics

State space form: $\dot{X} = AX + BU_c$

$$A = \begin{bmatrix} Y_V & Y_P & -(U_0 - Y_R) & g\cos\theta_0 \\ L_V{}^* + \dfrac{I_{XZ}}{I_X} N_V{}^* & L_P{}^* + \dfrac{I_{XZ}}{I_X} N_P{}^* & L_R{}^* + \dfrac{I_{XZ}}{I_X} N_R{}^* & 0 \\ N_V{}^* + \dfrac{I_{XZ}}{I_Z} L_V{}^* & N_P{}^* + \dfrac{I_{XZ}}{I_Z} L_P{}^* & N_R{}^* + \dfrac{I_{XZ}}{I_Z} L_R{}^* & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} \Delta V \\ \Delta P \\ \Delta R \\ \Delta \phi \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & Y_{\delta_R} \\ L_{\delta_A}{}^* + \dfrac{I_{XZ}}{I_X} N_{\delta_A} & L_{\delta_R}{}^* + \dfrac{I_{XZ}}{I_X} N_{\delta_R} \\ N_{\delta_A}{}^* + \dfrac{I_{XZ}}{I_Z} L_{\delta_A}{}^* & N_{\delta_R}{}^* + \dfrac{I_{XZ}}{I_Z} L_{\delta_R}{}^* \\ 0 & 0 \end{bmatrix} \qquad U_c = \begin{bmatrix} \Delta\delta_A \\ \Delta\delta_R \end{bmatrix}$$

Linearization: Points to remember

- Linearized system is always a **local approximation** about the operating point
- As the operating point changes, the linearized model changes (for the same nonlinear system)
- The usual objective of control design using the linearized dynamics is "deviation minimization" (i.e. regulation)
- Control design based on linearized dynamics always relies on the philosophy of "gain scheduling" (i.e. gain interpolation)

Thanks for the Attention...!