Limits and Continuity

Def $f(x)$ has limit L as x approaches x_0 if

$$\boxed{\lim_{x \to x_0} f(x) = L}$$

[illegible]

means

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

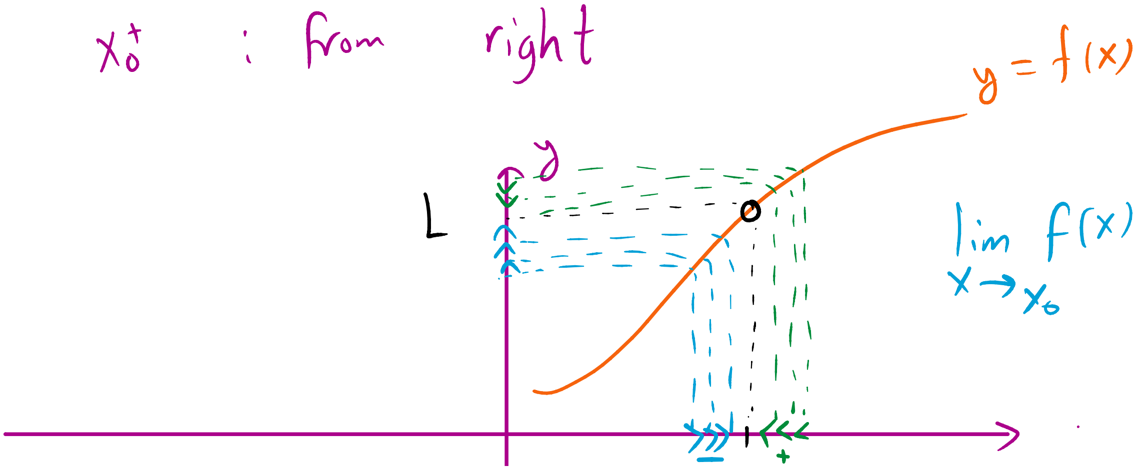
lim : limit

$x \to x_0$: x approaches x_0

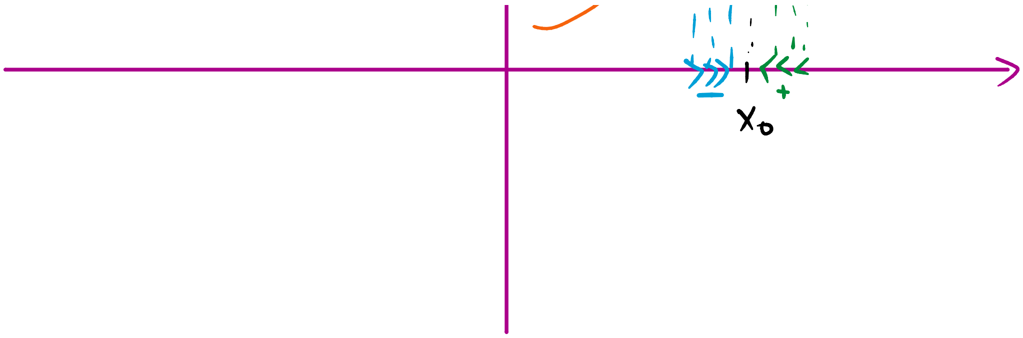
(this does not mean $x = x_0$)

x_0^- : from left

x_0^+ : from right



$$\lim_{x \to x_0} f(x) = L$$



Exp ① $\lim_{x \to 0} (x^2 - 3) = 0^2 - 3 = 0 - 3 = -3$ تعويض مباشر

$\frac{0}{0}$ ② $\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{(x-2)\cancel{(x+2)}}{\cancel{x+2}} = \lim_{x \to -2} (x-2)$

$= (-2) - 2$

$= -4$

$\lim_{x \to -2} \frac{2x}{1}$

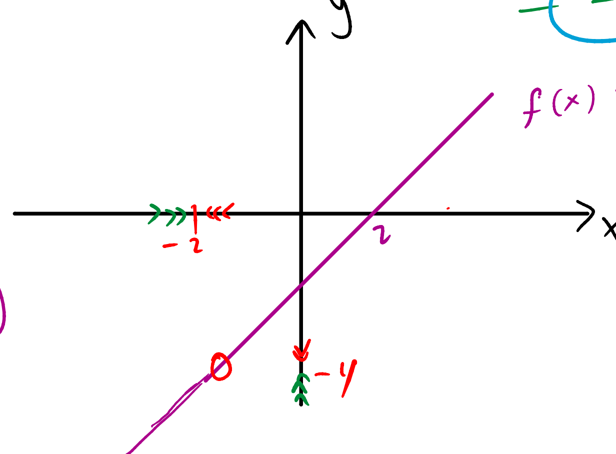
$2(-2)$

-4

$f(x)$

$f(-2)$ undefined

$f(x) = \frac{x^2 - 4}{x + 2}$



$f(-2)$ undefined

$\lim_{x \to -2} f(x) = -4$

f is not continuous at $x = -2$

③ $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{2x + 1}{2x - 1} = \frac{2 + 1}{2 - 1} = \frac{3}{1} = 3$ $\left(\frac{0}{0}\right)$

$(+2)(-1) = -2$

$+2 + (-1) = +1$

$\lim \frac{(x+2)\cancel{(x-1)}}{\ldots}$

$$\lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$\lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

1

(4) $$\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} \quad \boxed{\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}} \qquad \frac{0}{0}$$

$$= \lim_{x \to -1} \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \lim_{x \to -1} \frac{x^2+8-9}{(x+1)\left[\sqrt{x^2+8}+3\right]} = \lim_{x \to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \lim_{x \to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(\sqrt{x^2+8}+3)} = \frac{\boxed{-1}-1}{\sqrt{\boxed{-1}^2+8}+3}$$

$$= \frac{-2}{\sqrt{1+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3} = \frac{-2}{6}$$

$$= \frac{-1}{3}$$

a) $\frac{1}{3}$

a) $\frac{1}{3}$

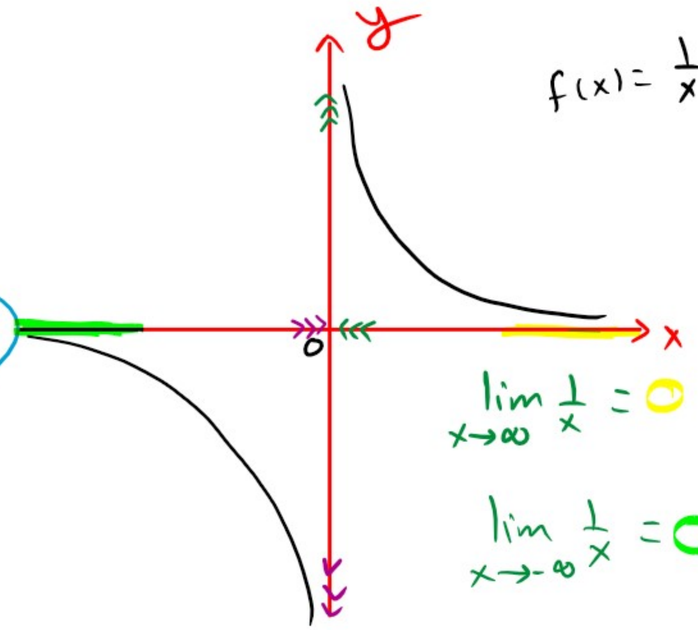
$\boxed{\text{b) } -\frac{1}{3}}$

c) 0

d) 3

(4) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\frac{1}{\text{small } +} = \infty$



$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\frac{1}{\text{small } -} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ DNE : Does not exist

Since $\lim_{x \to 0^+} \frac{1}{x} \neq \lim_{x \to 0^-} \frac{1}{x}$

$\infty \qquad -\infty$

(6) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin 2\theta}$

$\frac{1 - \cos 0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$ التعويض المباشر

$\sin 0 \quad 0 \quad 0-0$

$\theta \to 0 \quad \sin 2\theta$

$$\lim_{\theta \to 0} \frac{0 - -\sin\theta}{2\cos 2\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{2\cos 2\theta} = \frac{\sin 0}{2\cos 0} = \frac{0}{2(1)} = \frac{0}{2} = 0$$

⑦ $$\lim_{x \to \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-x}\right) \frac{\left(\sqrt{x^2+1} + \sqrt{x^2-x}\right)}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$\lim_{x \to \infty} \frac{(\cancel{x^2}+1) - (\cancel{x^2} - x)}{\sqrt{x^2+1} + \sqrt{x^2-x}} = \lim_{x \to \infty} \frac{\frac{1}{x}(1 + 1x)}{\frac{1}{x}\left(\sqrt{x^2+1} + \sqrt{x^2-x}\right)}$$

$\frac{\infty}{\infty}$

$1x$

$$\frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x \to \infty} \frac{\frac{1}{x} + 1}{\frac{\sqrt{x^2+1}}{x} + \frac{\sqrt{x^2-x}}{x}} = \lim_{x \to \infty} \frac{0+1}{\sqrt{\frac{x^2+1}{x^2}} + \sqrt{\frac{x^2-1}{x^2}}}$$

$\frac{1}{x} \to 0$

$$\lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0} + \sqrt{1+0}} = \frac{1}{1+1} = \frac{1}{2}$$

$\frac{1}{x^2} \to 0$

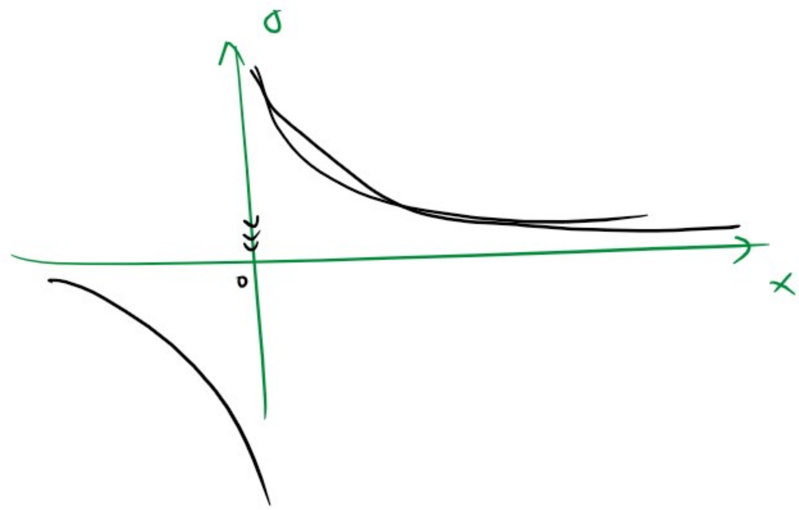
$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$\frac{\sqrt{x^2+1}}{x} = \sqrt{\frac{x^2+1}{x^2}}$$

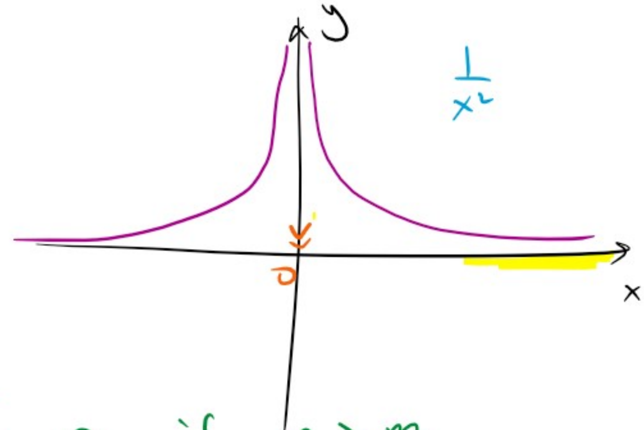
$$\sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} = \sqrt{\frac{5}{2^2}}$$

y



as $x \to \infty \Rightarrow \frac{1}{x} \to 0$



$\frac{1}{x^2}$

أكبر عدد من الدرجة m

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } n > m \\ \infty \text{ or } -\infty & \text{if } n < m \\ C & \text{if } n = m \end{cases}$$

أكبر عدد من الدرجة n

$$\lim_{x\to\infty} \frac{4x^{2}+1}{3-5x^{2}} = \frac{4}{-5}$$

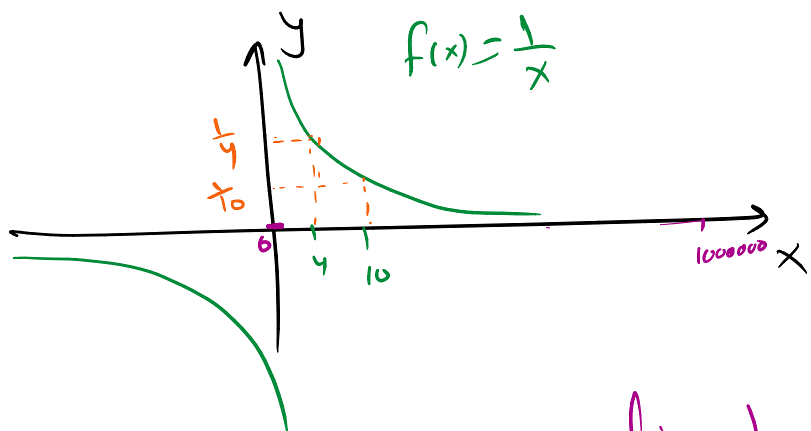
$$\lim_{x\to\infty} \frac{\frac{1}{x^2}(4x^2+1)}{\frac{1}{x^2}(3-5x^2)} = \lim_{x\to\infty} \frac{4+\frac{1}{x^2}}{\frac{3}{x^2}-5} = \frac{4}{-5}$$

$\frac{1}{x^2} \to 0$, $\frac{3}{x^2} \to 0$

$$\lim \frac{2x^{3}}{\ldots} = \infty$$

$$\lim_{x \to \infty} \frac{2x^2}{1+x^1} = \infty$$

$$\lim_{x \to \infty} \frac{-2x^3}{1+x} = -\infty$$

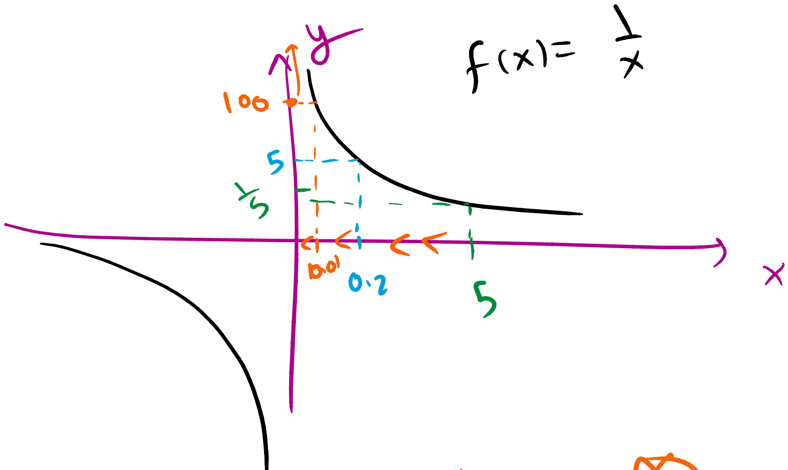


$f(4) = \frac{1}{4}$

$f(10) = \frac{1}{10}$

$f(1000\,000)$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$



$f(5) = \frac{1}{5}$

$f(0.2) = \frac{1}{0.2} = 5$

$f(0.01) = \frac{1}{0.01} = 100$

$f(0.000001) = 1000000$

$$\lim_{x \to 0} \frac{1}{x} = \infty$$

__Th__ (Sandwich Th)

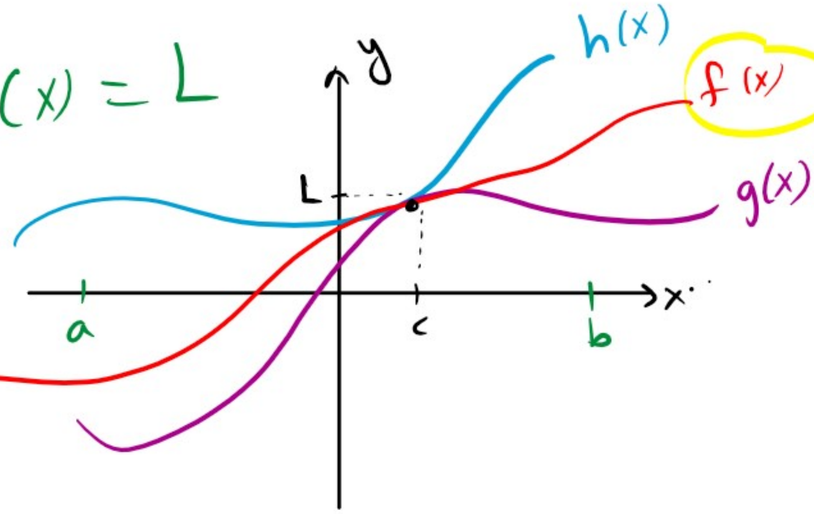
$[a, b]$

Th (Sandwich)

Assume $g(x) \le f(x) \le h(x) \quad \forall x \in [a,b]$

If $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$

Then $\lim\limits_{x \to c} f(x) = L$



Exp

Assume $5 - x^3 \le f(x) \le 5 + 3x^{13}$

$5 - x^3 \to 5$, $\quad 5 + 3x^{13} \to 5$

Find $\lim\limits_{x \to 0} f(x)$

$\lim\limits_{x \to 0} (5 + 3x^{13}) = 5 + 0 = 5$

$\lim\limits_{x \to 0} (5 - x^3) = 5$

Hence, $\lim\limits_{x \to 0} f(x) = 5$ by Sandwich th.

Exp $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$

$\dfrac{-1}{x} \le \left(\dfrac{\sin x}{x}\right) \le \left(\dfrac{1}{x}\right)$

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{-1}{x} = 0 \qquad \lim_{x\to\infty} \frac{1}{x} = 0$$

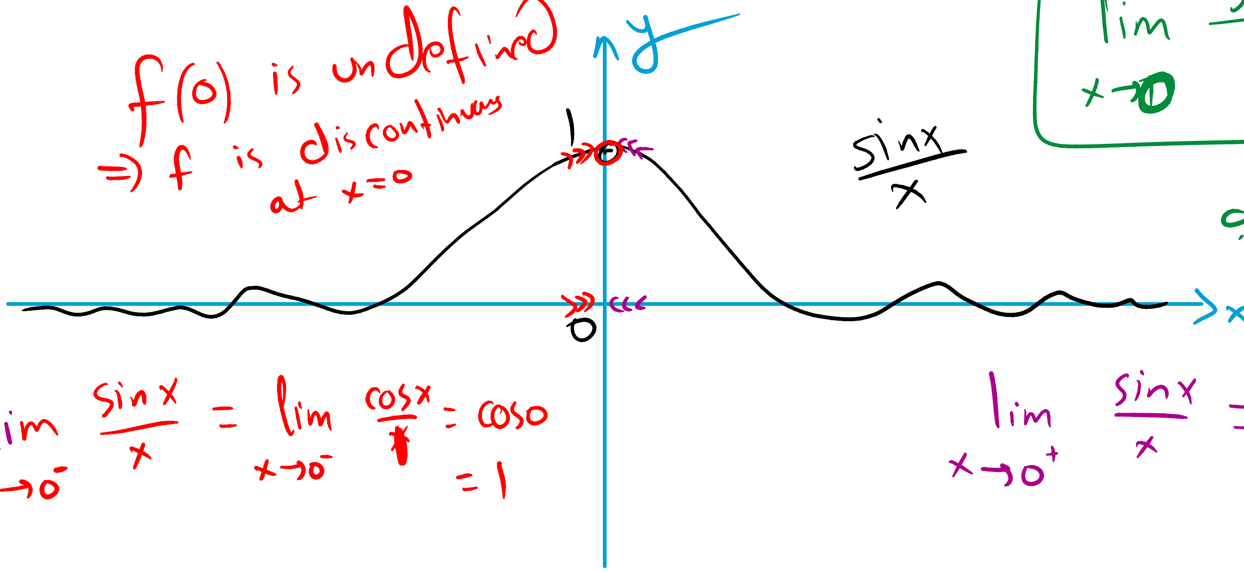
$$\lim_{x\to\infty} \frac{\sin x}{x} = 0 \quad \text{by } \underline{\underline{\text{Sandwich Th}}}$$

$\underline{\underline{\text{Exp}}}$ Draw $f(x) = \frac{\sin x}{x}$ $\quad D = \mathbb{R} \setminus \{0\}$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

السبب

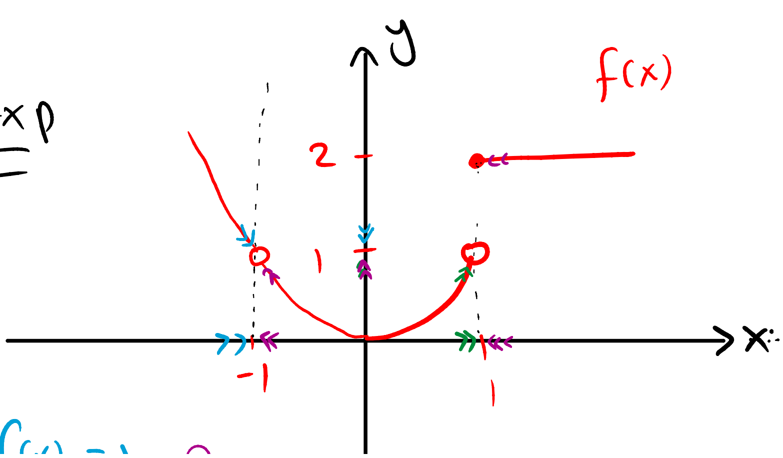
$f(0)$ is undefined
$\Rightarrow f$ is discontinuous at $x=0$



$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\cos x}{1} = \cos 0 = 1$$

$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = \cos 0 = 1$$

$\underline{\underline{\text{Exp}}}$



$$\lim_{x\to 1^+} f(x) = 2$$

$$\lim_{x\to 1^-} f(x) = 1$$

$\lim f(x)$ DNE

$\lim_{x \to -1^-} f(x) = 1$

$\lim_{x \to -1^+} f(x) = 1$

$\Rightarrow \lim_{x \to -1} f(x) = 1$

$\lim_{x \to 1} f(x)$ DNE

$f(1) = 2$

$f(-1) =$ undefined

Exp $f(x) = \begin{cases} x^2 + a^2 & \text{if } x \geq \frac{1}{2} \\ a & \text{if } x < \frac{1}{2} \end{cases}$

$\frac{1}{2}$

Find a so that $\lim_{x \to \frac{1}{2}} f(x)$ exists

$\lim_{x \to \frac{1}{2}^+} f(x) = \left(\frac{1}{2}\right)^2 + a^2 = \lim_{x \to \frac{1}{2}^-} f(x) = a$

$\frac{1}{4} + a^2 = a$

$a^2 - a + \frac{1}{4} = 0$

$(a - \frac{1}{2})(a - \frac{1}{2}) = 0$

$(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4}$

$-\frac{1}{2} + -\frac{1}{2} = -1$

$a = \frac{1}{2}$

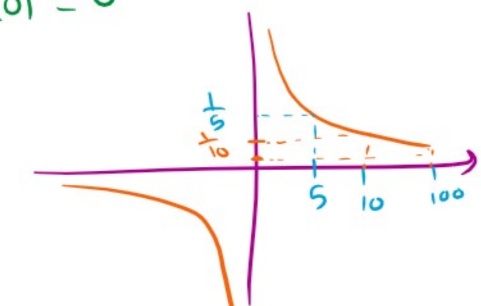
$\lim \frac{1}{x} = 0$

$\lim \frac{1}{x} = 0$??

$$\lim_{x\to\infty} \frac{-1}{x} = 0 \quad ??$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$-1 \lim_{x\to\infty} \frac{1}{x} = (-1)(0) = 0$$



Exp Assume $\sin x + \cos x \le f(x) \le \cot x + 1$

Find $\lim_{x\to\frac{\pi}{2}} f(x) = 1$ by Sandwich Th

a) 0

b) $\frac{\pi}{2}$

c) π

d) DNE

e) ∞

f) $-\infty$

$\boxed{\text{h) } 1}$

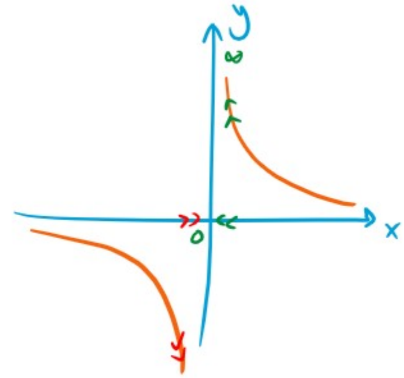
$$\lim_{x\to\frac{\pi}{2}} (\sin x + \cos x) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$$

$$\lim_{x\to\frac{\pi}{2}} (\cot x + 1) = \cot\frac{\pi}{2} + 1 = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} + 1 = 0 + 1 = 1$$

Exp $f(x) = \frac{1}{x}$

$$\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{\text{small}+} = \infty$$

$$\lim \frac{1}{x} = \frac{1}{\text{small}-} = -\infty$$



$$\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\text{small}^-} = -\infty$$