# Moment - Area Theorems

The moment area theorems provide a way to find slopes and deflections without having to go through a full process of integration as described by double integration method.

There are two moment area theorems, one relates to the slope of the beam and the other relates to the deflection

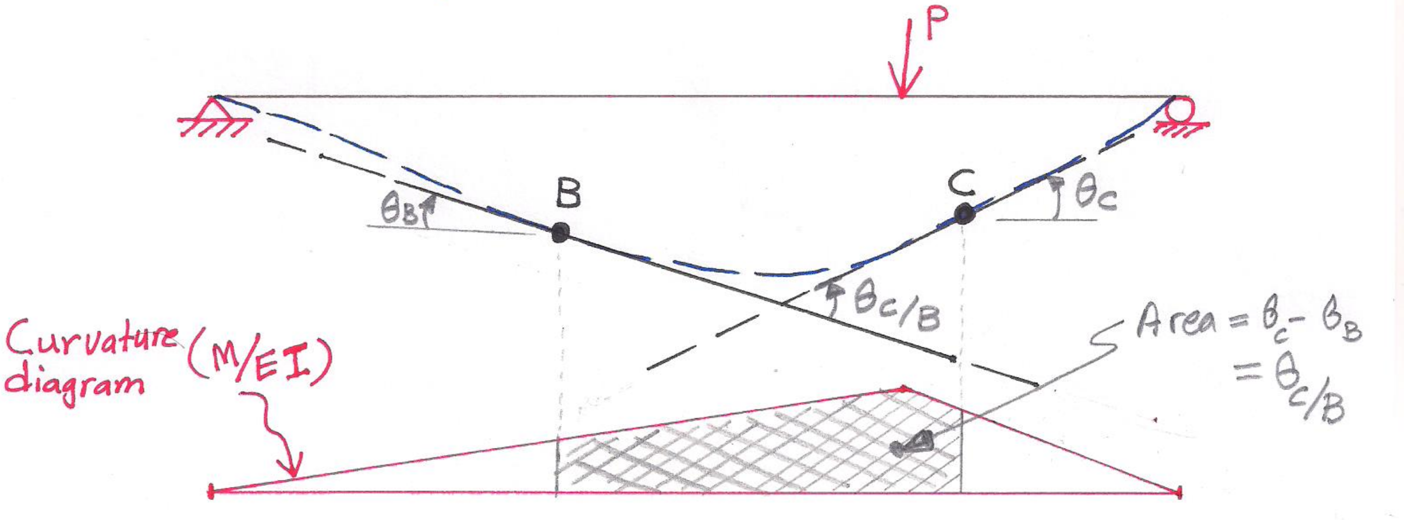
## First moment-area theorem

The change in the slope of a beam between two points is equal to the area under the curvature diagram between these two points.

$$\frac{d^2 v}{dx^2} = \frac{M}{EI} \rightarrow \frac{d\theta}{dx} = \frac{M}{EI} \rightarrow \int_{\theta_B}^{\theta_C} d\theta = \int_{x_B}^{x_C} \frac{M}{EI} dx$$

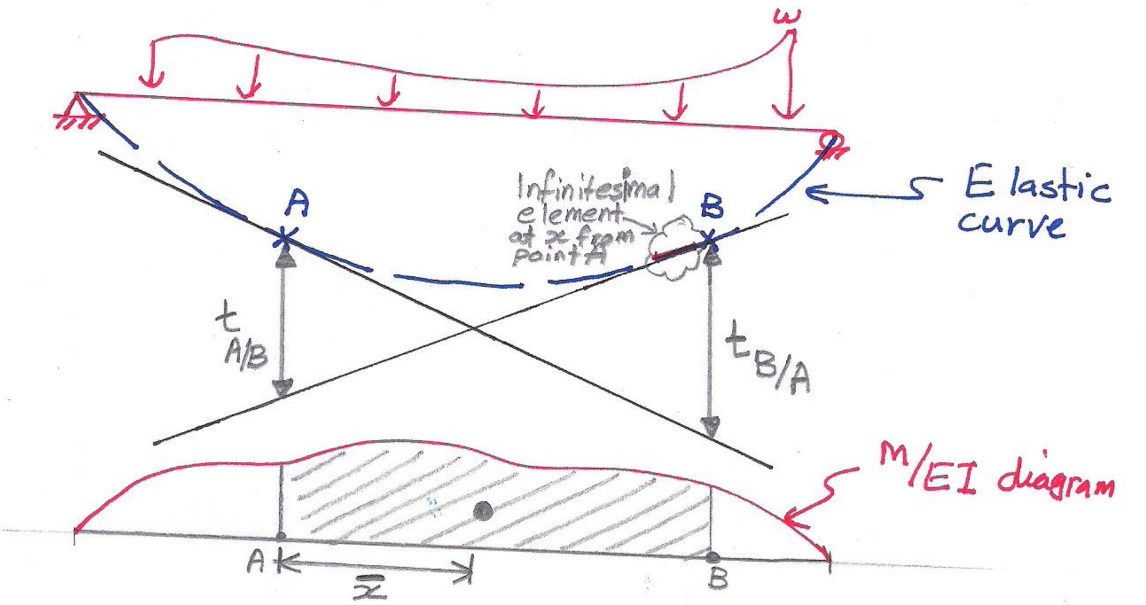
$$\theta_C - \theta_B = \int_{x_B}^{x_C} \frac{M}{EI} dx$$

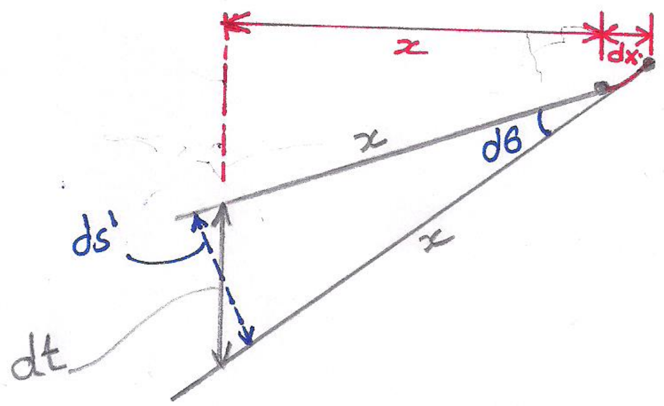
$$\theta_{C/B} = \int_{x_B}^{x_C} \frac{M}{EI} dx = \text{Area under curvature diagram between points B and C.}$$



# Second moment-area theorem

The vertical distance between the tangent at a point (A) on the elastic curve and the tangent extended from another point (B) equals the moment of the area under the curvature ($M/EI$) diagram between these two points (A and B). This moment is calculated about the point (A) where the vertical distance ($t_{A/B}$) is to be determined.




slope of elastic curve and its deflection are assumed to be very small

$ds' = dt$ $\quad$ tangent length $= x$

$$dt = x\, d\theta \quad \text{and} \quad d\theta = \frac{M}{EI}\, dx \Rightarrow \int_A^B dt = \int_{x_A}^{x_B} x \frac{M}{EI}\, dx$$

$\boxed{\bar{x}A = \int x\, dA}$

Area under $\frac{M}{EI}$ diagram

$$t_{A/B} = \bar{x} \int_{x_A}^{x_B} \frac{M}{EI}\, dx$$

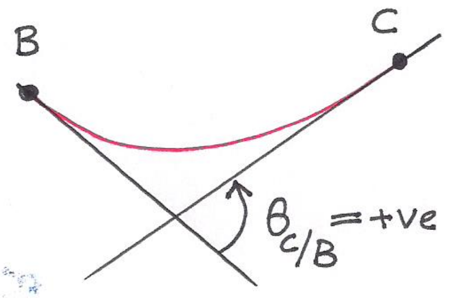
, $\bar{x}$ = is the distance from A to the centroid of the area under $\frac{M}{EI}$ diagram between A and B.

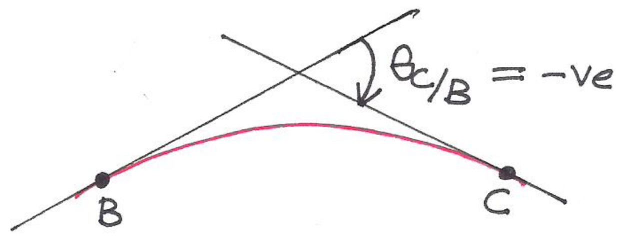
$t_{A/B} \neq t_{B/A}$

* Note that $t_{A/B} \neq t_{B/A}$

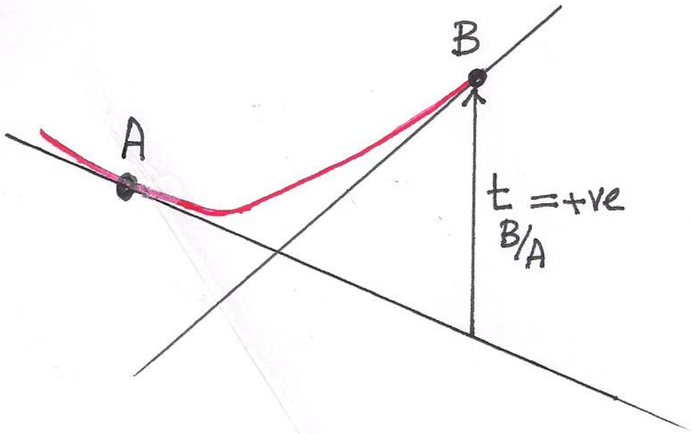
The moment of the area under the $\frac{M}{EI}$ diagram between A and B is calculated about point A to determine $t_{A/B}$, and it is calculated about point B to determine $t_{B/A}$.

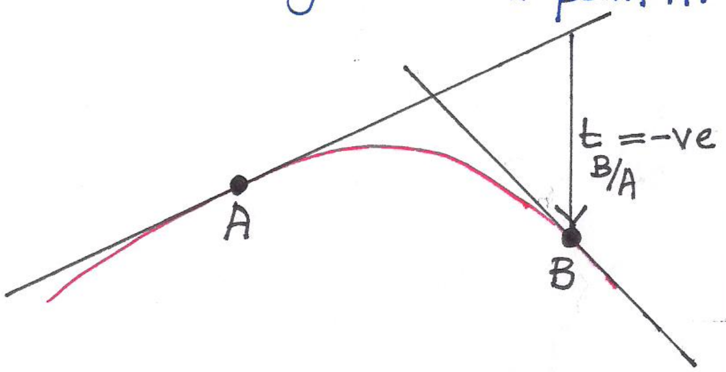
* A positive $\theta_{C/B}$ represents a counterclockwise rotation of tangent at C with respect to the tangent at A.



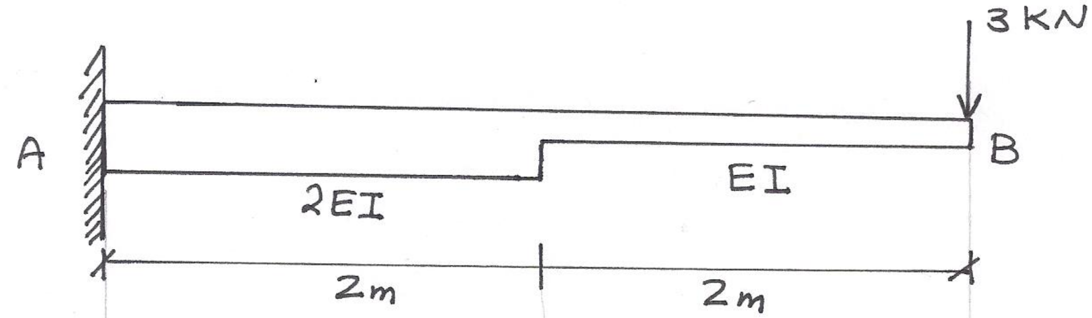

* A positive $t_{B/A}$ indicates that point B on the elastic curve lies above the extended tangent from point A.



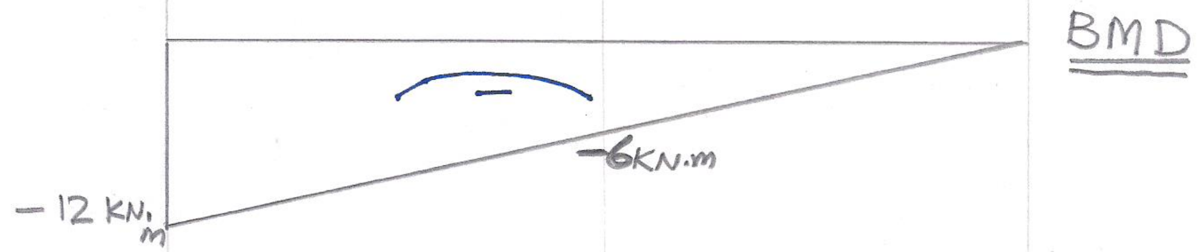



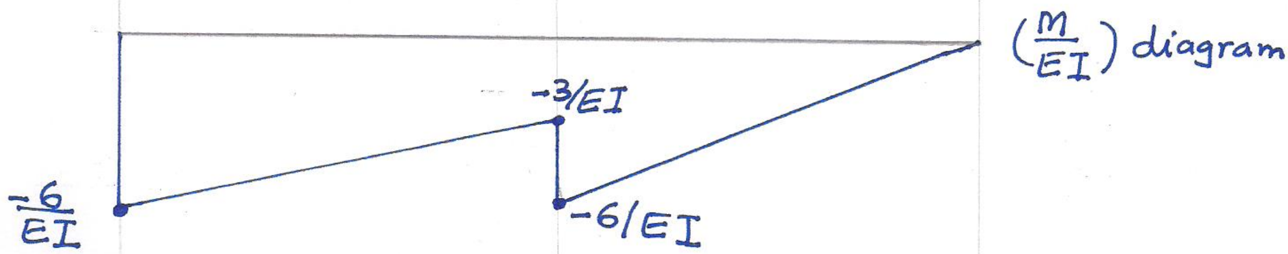
# Example

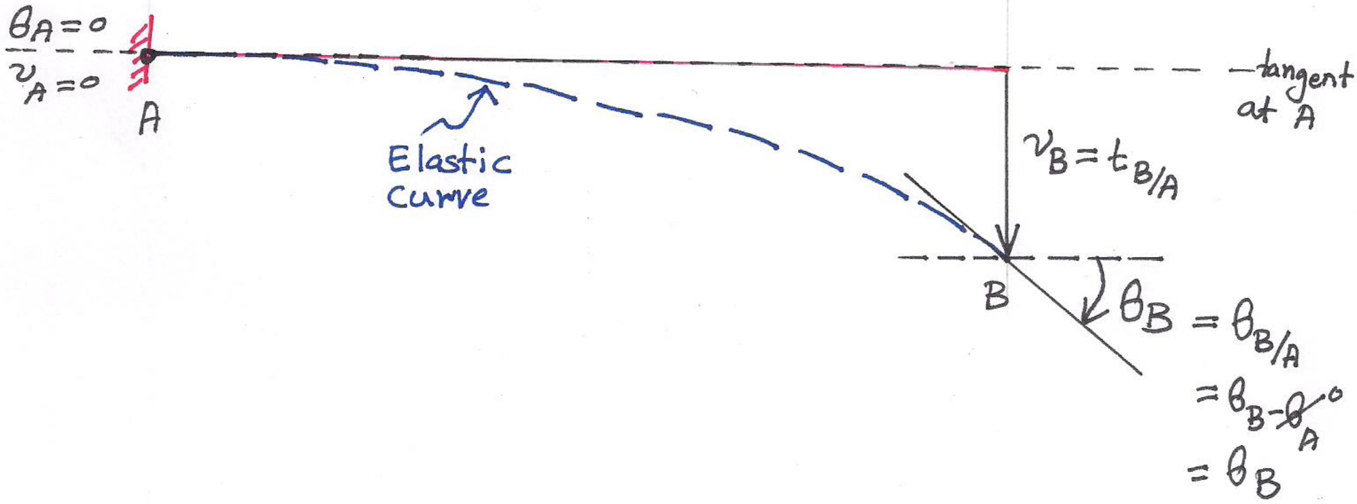
Beam AB is nonprismatic and subjected to the load shown. Determine slope and deflection at point B on the elastic curve.




## Solution



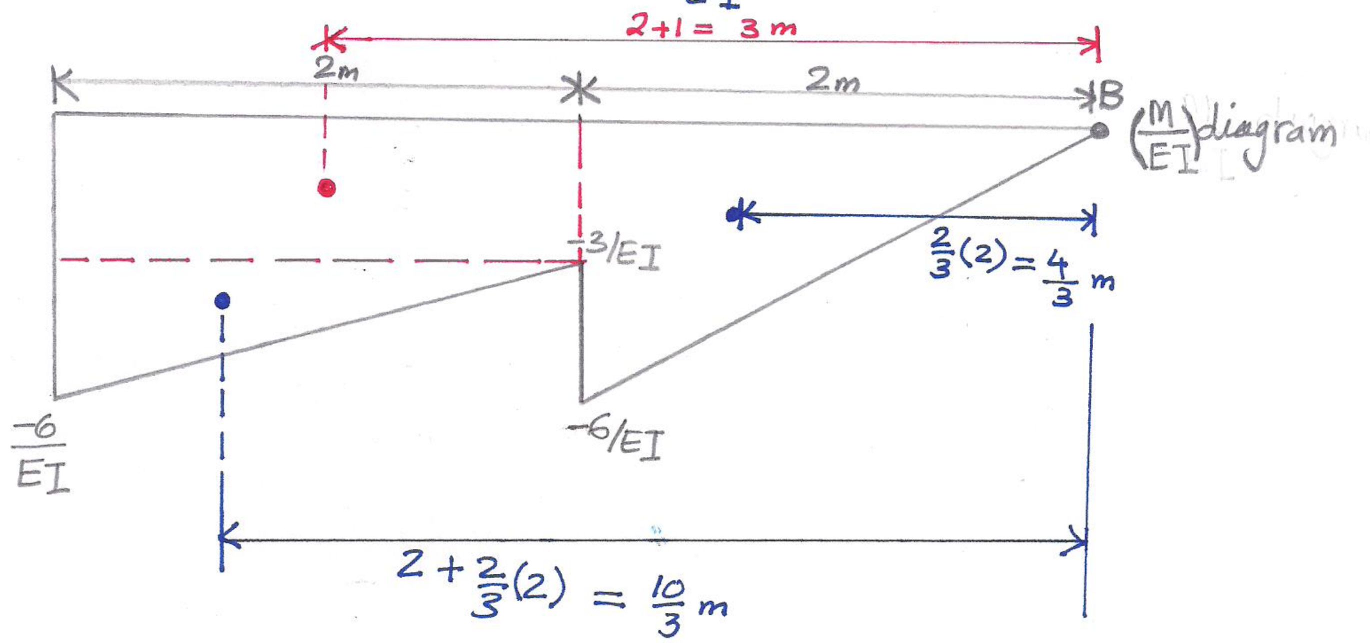



$\theta_{B/A}$ = Area under $\left(\frac{M}{EI}\right)$ diagram from "A" to "B"

$$\theta_B = \frac{1}{2}\left(\frac{-6}{EI} + \frac{-3}{EI}\right)(2) + \frac{1}{2}\left(\frac{-6}{EI}\right)(2) = \frac{-15}{EI}\ \text{rad}$$

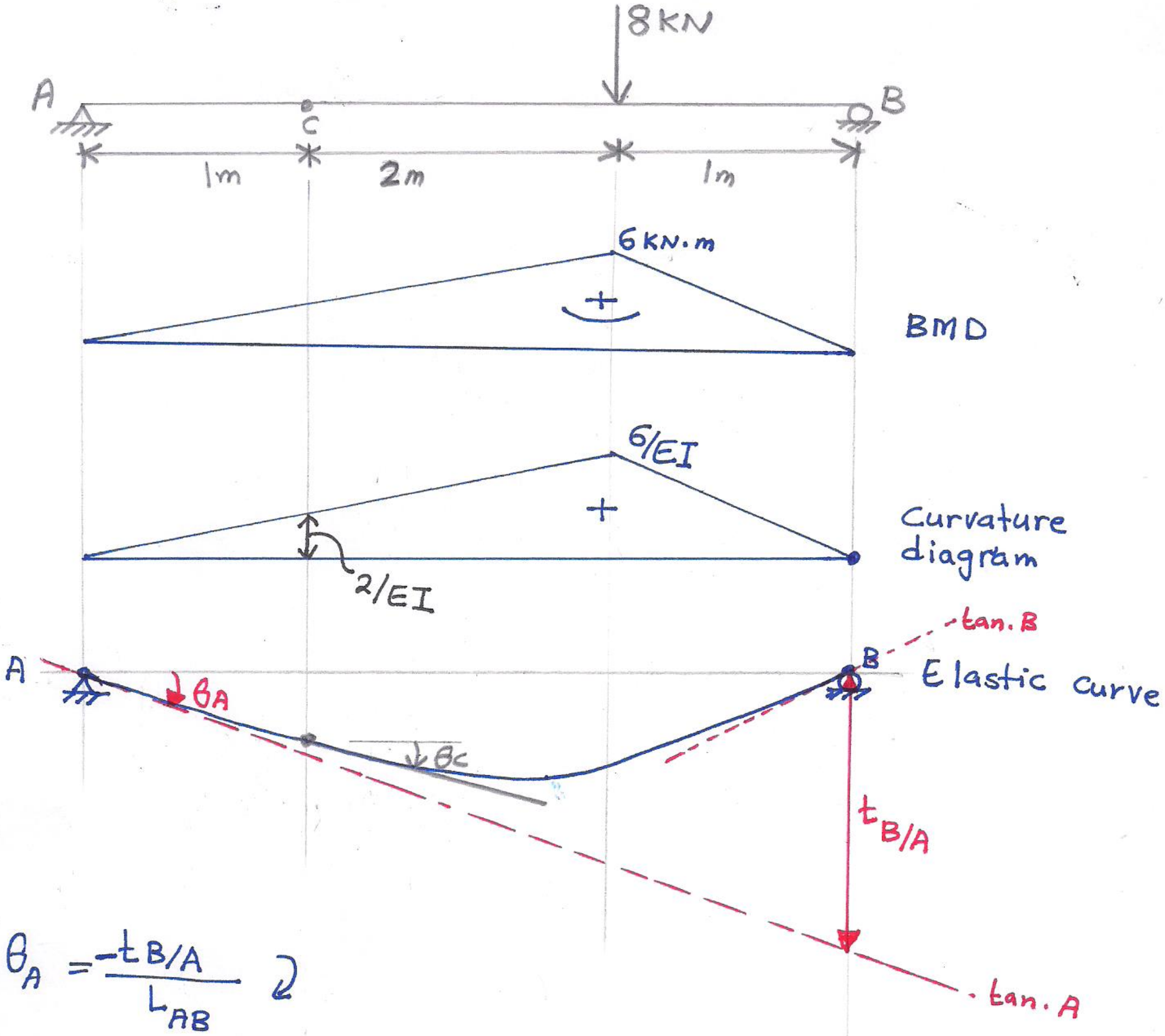
$t_{B/A}$ = $\sum x.$(Area under $\frac{M}{EI}$ diagram)




$$t_{B/A} = \frac{1}{2}(2)\left(\frac{-3}{EI}\right)\left(\frac{10}{3}\right) + (2)\left(\frac{-3}{EI}\right)(3) + \frac{1}{2}(2)\left(\frac{-6}{EI}\right)\left(\frac{4}{3}\right) = \frac{-36}{EI}\ (m)$$

## Example

Determine (a) the slope at point C, (b) location and magnitude of maximum deflection. EI = constant.




$$\theta_A = \frac{-t_{B/A}}{L_{AB}}$$

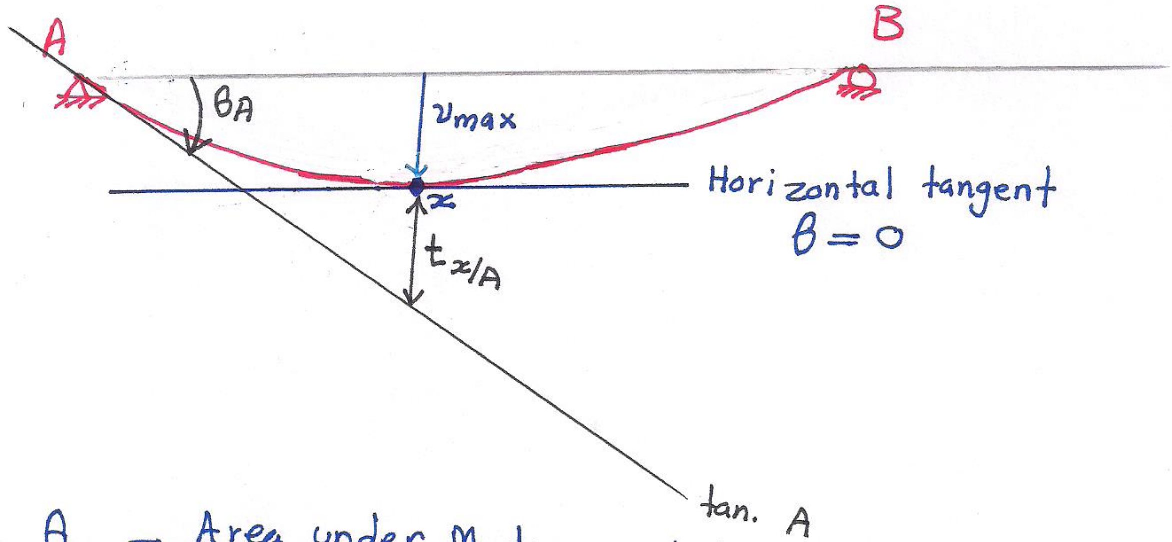
$$t_{B/A} = \frac{1}{2}(6/EI)(1)\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{6}{EI}\right)(3)(2) = +\frac{20}{EI}$$

$$\theta_A = -\frac{20/EI}{4} = \frac{-5}{EI} \text{ rad}$$

$\theta_{C/A}$ = Area between A and C under $\frac{M}{EI}$ diagram

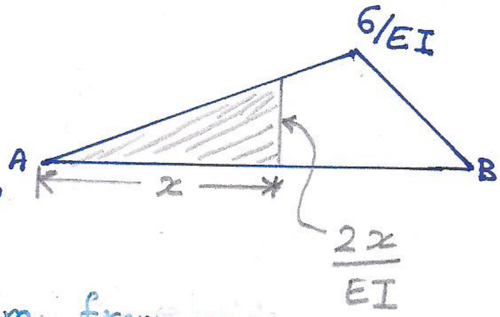
$$\theta_C - \left(\frac{-5}{EI}\right) = +\frac{1}{2}(1)\left(\frac{2}{EI}\right) \rightarrow \theta_C = -\frac{4}{EI} \text{ rad.}$$

## Maximum deflection




$\theta_x - \theta_A$ = Area under $\frac{M}{EI}$ diagram between A and $x$.

$$\theta_x - \left(\frac{-5}{EI}\right) = \frac{1}{2}(x)\left(\frac{2x}{EI}\right)$$




$\theta_x = 0$ : $\quad \frac{5}{EI} = \frac{x^2}{EI} \rightarrow x = \sqrt{5}\ m < 3\ m$

OK.

The maximum deflection is at $x = \sqrt{5}$ m from point A.

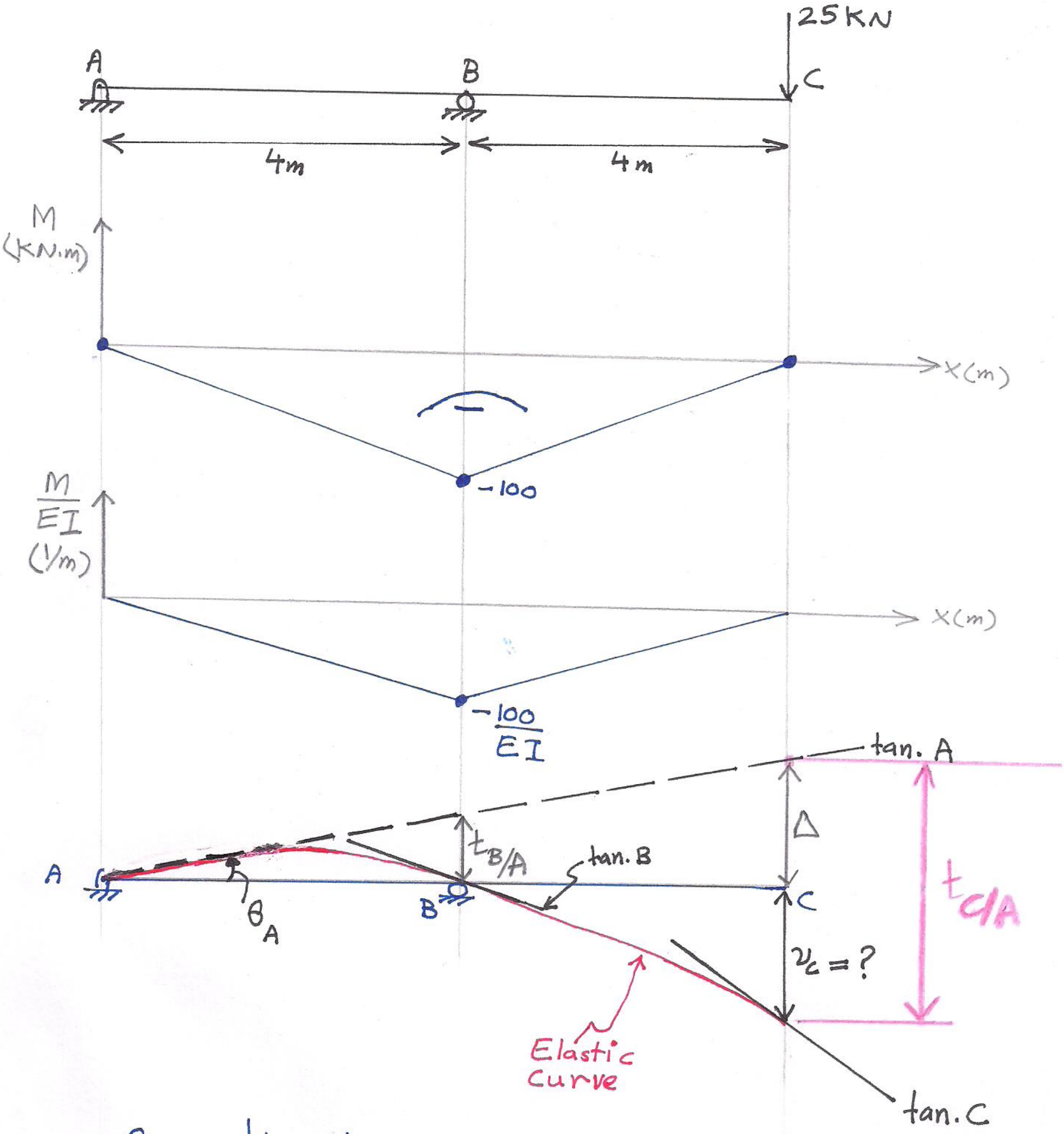
$$|\theta_A| = \frac{|v_{max} + t_{x/A}|}{x}$$

$$t_{x/A} = \frac{1}{2}(\sqrt{5})\left(\frac{2\sqrt{5}}{EI}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{5\sqrt{5}}{3EI}$$

$$\theta_A = \frac{5}{EI} = \frac{v_{max} + \frac{5\sqrt{5}}{3EI}}{\sqrt{5}} \longrightarrow v_{max} = \frac{7.453}{EI}\ m\ (\downarrow)$$

## Example

Determine the displacement at point C for the steel overhanging beam shown. Use $E_{st} = 200\ GPa$, $I = 50 \times 10^6\ mm^4$.




$$\theta_A = \frac{|t_{B/A}|}{L_{AB}} = \frac{\Delta}{L_{AC}} \rightarrow \Delta = \left(\frac{L_{AC}}{L_{AB}}\right)|t_{B/A}|$$

$$v_c = |t_{C/A}| - \Delta$$

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$$t_{B/A} = \frac{1}{2}(4)\left(\frac{-100}{EI}\right)\left(\frac{1}{3}\times 4\right) = \frac{-800}{3EI}$$

$$\Delta = \left(\frac{8}{4}\right)\left(\frac{800}{3EI}\right) = \frac{1600}{3EI}$$

$$t_{C/A} = \frac{1}{2}(8)\left(\frac{-100}{EI}\right)(4) = \frac{-1600}{EI}$$

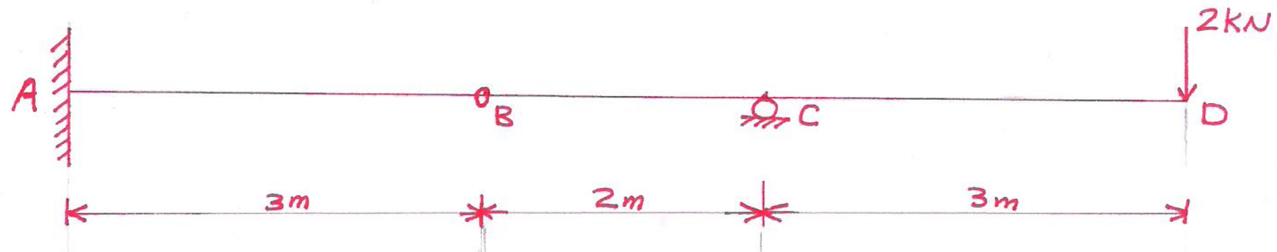
$$v_c = \frac{1600}{EI} - \frac{1600}{3EI} = \frac{3200\ kN.m^3}{3EI} \quad (m)$$

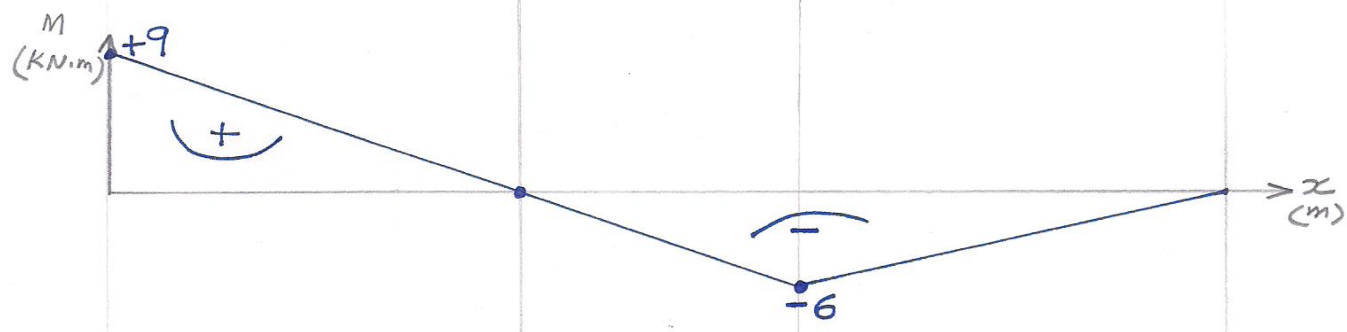
$$v_c = \frac{3200\ kN.m^3}{3\left(200\times 10^6 \frac{kN}{m^2}\right)\left(50\times 10^6 \times 10^{-12} m^4\right)} = 0.1067\ m\ (\downarrow)$$

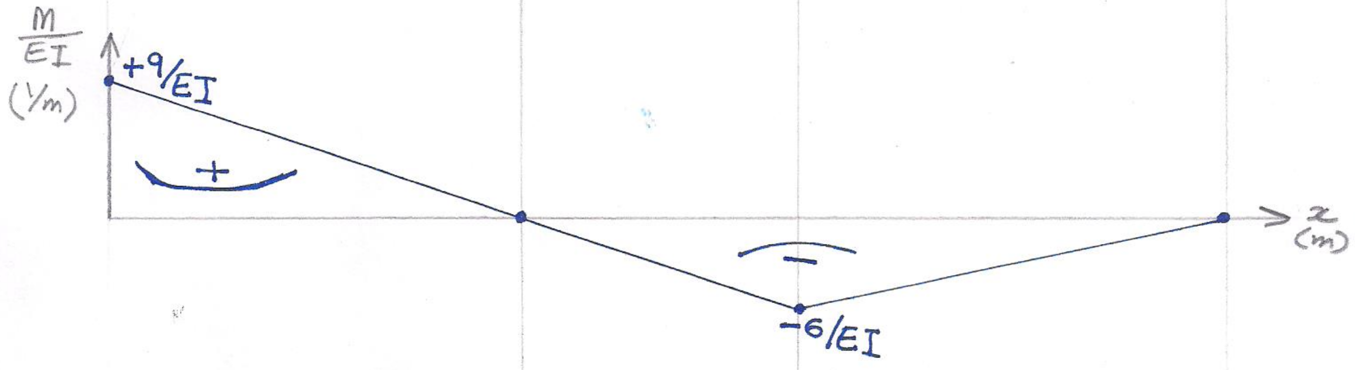
$$= 106.7\ mm\ (\downarrow)$$

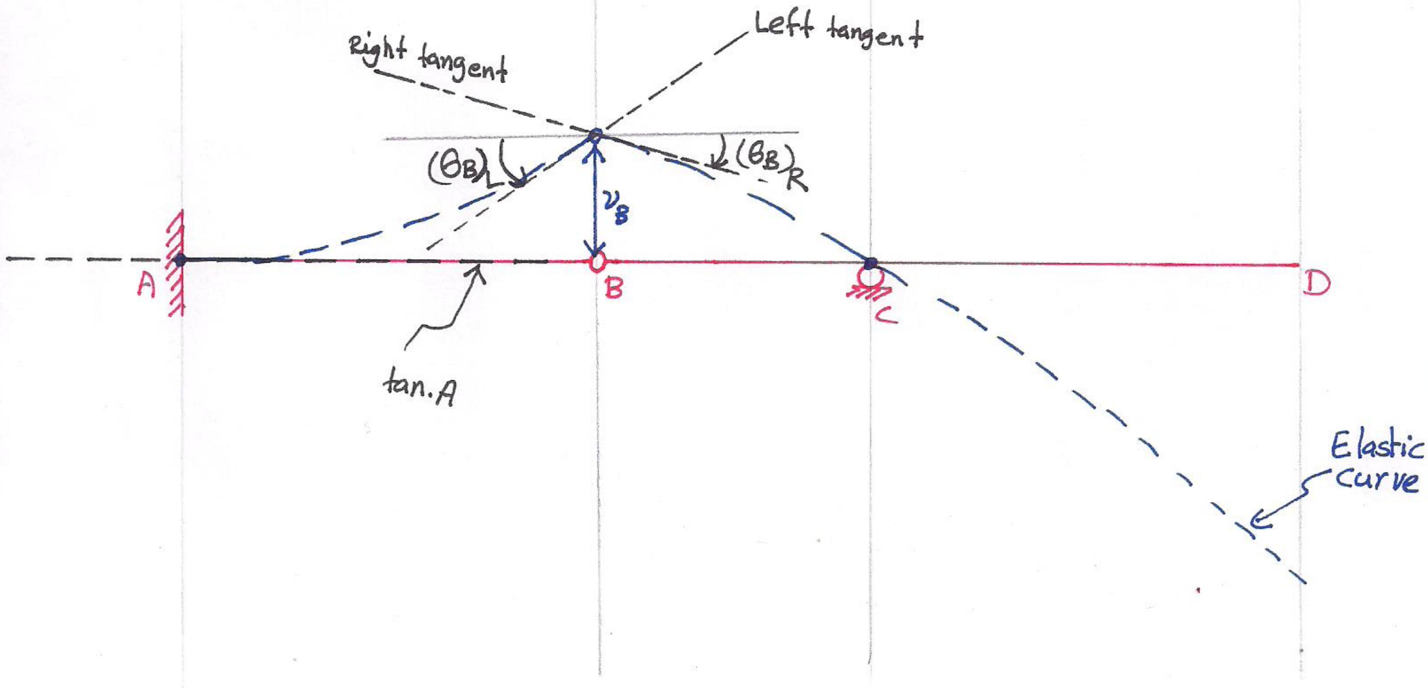
# Example

Beam ABCD shown has a fixed support at A, an internal hinge at B, a roller support at C, and a free end at D. Determine the slope and deflection of point B. EI = constant.



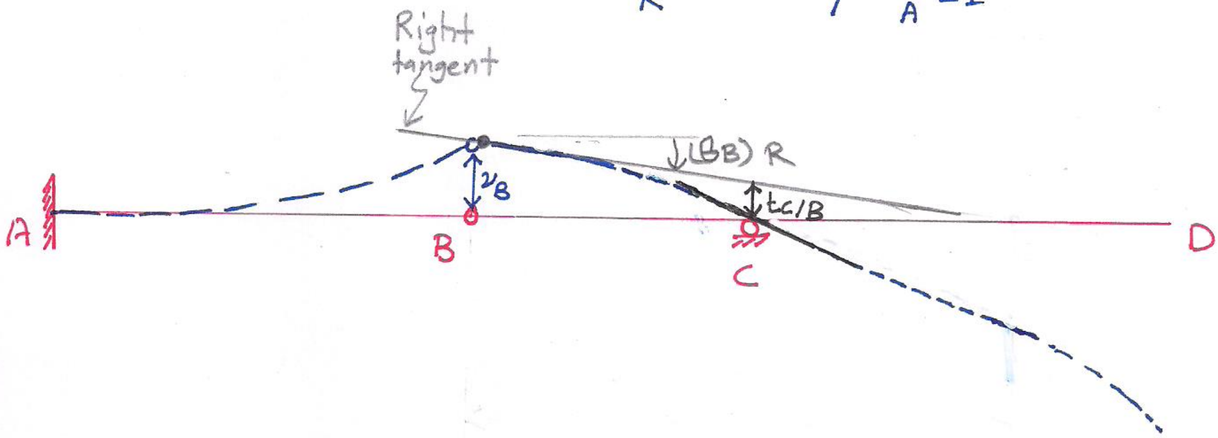



$(\theta_B)_L - \theta_A$ = Area under $\frac{M}{EI}$ diagram between A and B.

$$(\theta_B)_L - 0 = \frac{1}{2}(3)\left(\frac{+9}{EI}\right) \rightarrow (\theta_B)_L = \frac{+27}{2EI} \text{ rad} \circlearrowleft$$

$$v_B = t_{B/A} = \frac{1}{2}(3)\left(\frac{+9}{EI}\right)\left(\frac{2}{3} * 3\right) = \frac{+27}{EI} \text{ m } (\uparrow)$$

For $(\theta_B)_R$, moment-area theorms cannot be used between A and the point just right of point B, because of the discontinuity at B "internal hinge". $(\theta_B)_R - \theta_A \neq \int_A^B \frac{M}{EI}dx$




$$t_{C/B} = \frac{1}{2}(2)\left(\frac{-6}{EI}\right)\left(\frac{1}{3} * 2\right) = \frac{-4}{EI} \ (\downarrow)$$

$$(\theta_B)_R = \frac{|v_B| - |t_{C/B}|}{L_{BC}} = \frac{\frac{27}{EI} - \frac{4}{EI}}{2} = \frac{-23}{2EI} \circlearrowright \text{ radian}$$