6.4 Areas of Surfaces of Revolution

- If $f(x) \geq 0$ is continuously differentiable on $[a,b]$, then the area of the surface generated by revolving the graph of $y = f(x)$ about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

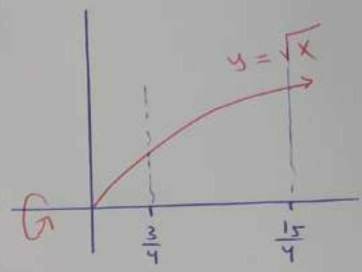
- If $g(\cancel{x}y) \geq 0$ is continuously differentiable on $[c,d]$, then the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, d\cancel{x}y$$

Question 14. Find the areas of the surface generated
Page 335 by revolving $y = \sqrt{x}$, $\frac{3}{4} \le x \le \frac{15}{4}$ about the x-axis.

$$S = \int_{3/4}^{15/4} 2\pi \, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{3/4}^{15/4} 2\pi \sqrt{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



$$y = \sqrt{x} \rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2\sqrt{x}}\right)^2 = \frac{1}{4x}$$

$$S = \int_{3/4}^{15/4} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx$$

$$S = \int_{3/4}^{15/4} 2\pi \sqrt{x} \sqrt{\frac{4x+1}{4x}} \, dx$$

$$S = \int_{3/4}^{15/4} \cancel{2}\pi \cancel{\sqrt{x}} \frac{\sqrt{4x+1}}{\cancel{2}\cancel{\sqrt{x}}} \, dx = \int_{3/4}^{15/4} \pi \sqrt{4x+1} \, dx$$

$$S = \pi \frac{(4x+1)^{3/2}}{\frac{3}{\cancel{2}} \cdot \cancel{4}2} \Bigg|_{3/4}^{15/4} = \frac{\pi}{6}\left[\left(4\left(\tfrac{15}{4}\right)+1\right)^{3/2} - \left(4\left(\tfrac{3}{4}\right)+1\right)^{3/2} \right]$$

$$= \frac{\pi}{6}\left[(16)^{3/2} - (4)^{3/2} \right] = \frac{\pi}{6}\left[64 - 8 \right]$$

$$= \boxed{\frac{28\pi}{3}}$$

15 Find the area of the surface generated by revolving the curve $y = \sqrt{2x - x^2}$, $0.5 \le x \le 1.5$ about the x-axis.

$$S = \int_{0.5}^{1.5} 2\pi \, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = \frac{1}{2}(2x - x^2)^{-\frac{1}{2}} (2 - 2x) = \frac{1 - x}{\sqrt{2x - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1 - x}{\sqrt{2x - x^2}}\right)^2 = 1 + \frac{(1 - x)^2}{2x - x^2}$$

$$= \frac{2x - x^2 + 1 - 2x + x^2}{2x - x^2} = \frac{1}{2x - x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{1}{2x - x^2}} = \frac{1}{\sqrt{2x - x^2}}$$

$$S = 2\pi \int_{0.5}^{1.5} \cancel{\sqrt{2x - x^2}} \cdot \frac{1}{\cancel{\sqrt{2x - x^2}}} \, dx$$

$$= 2\pi \times \Big|_{0.5}^{1.5} = 2\pi(1.5 - 0.5) = 2\pi(1)$$

$$= 2\pi$$

Question 18. Find the areas of the surface generated
page 335 by revolving the curve $x = \frac{1}{3} y^{3/2} - y^{1/2}$,

$1 \le y \le 3$, about y-axis

$$S = 2\pi \int_1^3 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = \frac{1}{3} y^{3/2} - y^{1/2}$$

$$\frac{dx}{dy} = \frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right) y^{1/2} - \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2} y^{1/2} - \frac{1}{2y^{1/2}}$$

$$= \frac{y-1}{2y^{1/2}}$$

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{y-1}{2y^{1/2}}\right)^2 = \frac{y^2 - 2y + 1}{4y}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2 - 2y + 1}{4y} = \frac{4y + y^2 - 2y + 1}{4y}$$

$$= \frac{y^2 + 2y + 1}{4y} = \frac{(y+1)^2}{4y}$$

$$S = 2\pi \int_1^3 \left(\frac{1}{3} y^{3/2} - y^{1/2}\right) \sqrt{\frac{(y+1)^2}{4y}}\, dy$$

$$= \cancel{2}\pi \int_1^3 \cancel{y^{1/2}}\left(\frac{1}{3}y - 1\right) \frac{(y+1)}{\cancel{2}\cancel{\sqrt{y}}}\, dy$$

$$= \pi \int_1^3 \frac{1}{3}y^2 - \frac{2}{3}y - 1\, dy$$

$$= \pi\left[\frac{1}{3}\left(\frac{y^3}{3}\right) - \frac{\cancel{2}}{3}\left(\frac{y^2}{\cancel{2}}\right) - y\right]_1^3 = \pi\left[\frac{y^3}{9} - \frac{y^2}{3} - y\right]_1^3$$

$$=\pi\left(\frac{(3)^3}{9}-\frac{(3)^2}{3}-3-\left[\frac{(1)^3}{9}-\frac{(1)^2}{3}-1\right]\right)$$

$$=\pi\left(\frac{27}{9}-\frac{9}{3}-3-\left[\frac{1}{9}-\frac{3}{9}-\frac{9}{9}\right]\right)$$

$$=\pi\left(\cancel{3}-\cancel{3}-3-\left[-\frac{11}{9}\right]\right)$$

$$=\pi\left(-3+\frac{11}{9}\right)$$

$$=\pi\left(\frac{-27+11}{9}\right) = -\frac{16}{9}\pi$$

So the area of the surface $= \left|-\frac{16\pi}{9}\right|$

$$= \frac{16\pi}{9}$$

note that $S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ if

$x \geq 0$ but for $x = \frac{1}{3} y^{3/2} - y^{1/2}$, $1 \leq y \leq 3$

x is negative so the area $= \left|-\frac{16\pi}{9}\right| = \frac{16\pi}{9}$

Question 20. Find the area of the surface generated
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by revolving the curve $x = \sqrt{2y-1}$,

$\frac{5}{8} \le y \le 1$, about the y-axis.

$$S = \int_{5/8}^{1} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = (2y-1)^{1/2} \rightarrow \frac{dx}{dy} = \frac{1}{\cancel{2}}(2y-1)^{-1/2}(\cancel{2})$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{2y-1}}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{\sqrt{2y-1}}\right)^2 = 1 + \frac{1}{2y-1}$$

$$= \frac{2y - \cancel{1} + \cancel{1}}{2y-1} = \frac{2y}{2y-1}$$

$$S = \int_{5/8}^{1} 2\pi \sqrt{2y-1} \cdot \sqrt{\frac{2y}{2y-1}}\, dy$$

$$= 2\pi \int_{5/8}^{1} \cancel{\sqrt{2y-1}} \cdot \frac{\sqrt{2y}}{\cancel{\sqrt{2y-1}}}\, dy$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2y}\, dy = 2\pi\sqrt{2}\, \frac{y^{3/2}}{3/2} \Bigg|_{5/8}^{1}$$

$$= 2\pi\sqrt{2} \cdot \frac{2}{3}\left[(1)^{3/2} - \left(\frac{5}{8}\right)^{3/2}\right]$$

$$= \frac{4\pi\sqrt{2}}{3}\left[1 - \frac{5\sqrt{5}}{(2^3)^{3/2}}\right] = \frac{4\pi\sqrt{2}}{3}\left[1 - \frac{5\sqrt{5}}{16\sqrt{2}}\right]$$

$$= \frac{\cancel{4}\pi\cancel{\sqrt{2}}}{3}\left[\frac{16\sqrt{2} - 5\sqrt{5}}{\cancel{4}\cdot 16\cancel{\sqrt{2}}}\right] = \frac{\pi}{12}\left[16\sqrt{2} - 5\sqrt{5}\right]$$

Question 22. page 336 Find the area of the surface generated by revolving the curve

$y = \frac{1}{3}(x^2+2)^{3/2}$, $0 \le x \le \sqrt{2}$ about y-axis.

Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx and evaluate the integral $S = \int 2\pi\, S\, ds$

about y-axis

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right](dy)^2} = \sqrt{(dy)^2 + (dx)^2}$$

$$= \sqrt{(dx)^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right]} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

we will replace $\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ by $\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$y = \frac{1}{3}(x^2+2)^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{2}}\right)(x^2+2)^{1/2}\cdot(\cancel{2}x)$$

$$\frac{dy}{dx} = x\sqrt{x^2+2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x\sqrt{x^2+2}\right)^2 = 1 + x^2(x^2+2)$$

$$= 1 + x^4 + 2x^2 = (x^2+1)^2$$

$$S = \int_a^b 2\pi \, x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^{\sqrt{2}} 2\pi \, x \sqrt{(x^2+1)^2} \, dx$$

$$= 2\pi \int_0^{\sqrt{2}} x(x^2+1) \, dx$$

$$= 2\pi \int_0^{\sqrt{2}} x^3 + x \quad dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2} \right] \Big|_0^{\sqrt{2}}$$

$$= 2\pi \left[\frac{(\sqrt{2})^4}{4} + \frac{(\sqrt{2})^2}{2} - (0+0) \right]$$

$$= 2\pi \left[\frac{4}{4} + \frac{2}{2} \right]$$

$$= 2\pi (2) = 4\pi$$

Question 24. Write an integral for the area of

Page 336 the surface generated by revolving

the curve $y = \cos x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, about the x-axis

Don't evaluate the integral.

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi \, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = -\sin x \rightarrow \left(\frac{dy}{dx}\right)^2 = \sin^2 x$$

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi \cos x \sqrt{1 + \sin^2 x} \, dx$$