Iteration for NonLinear Systems:

1) Fixed point Iteration.

2) Gauss_Sidel Iteration

3) Newton's Method.

Fixed point Iteration:

Def: (p,q) is a fixed point of the system:

$$\begin{cases} x = g_1(x,y) \\ y = g_2(x,y) \end{cases} \quad \text{if} \quad \begin{cases} p = g_1(p,q) \\ q = g_2(p,q) \end{cases}$$

Similarly: (p,q,r) is a fixed point of the system

$$\begin{cases} x = g_1(x,y,z) \\ y = g_2(x,y,z) \\ z = g_3(x,y,z) \end{cases} \quad \text{if} \quad \begin{aligned} p &= g_1(p,q,r) \\ q &= g_2(p,q,r) \\ r &= g_3(p,q,r). \end{aligned}$$

For fixed point Iteration we start with (p_0, q_0) in 2D

for $\begin{cases} x = g_1(x,y) \\ y = g_2(x,y) \end{cases}$ then: $\begin{aligned} p_{n+1} &= g_1(p_n, q_n) \\ q_{n+1} &= g_2(p_n, q_n) \end{aligned}$

Exempl: $f_1(x,y) = x^2 - 2x - y + 0.5 = 0$

$f_2(x,y) = x^2 + 4y^2 - 4 = 0$

The solution is the point of Intersection.

which is around $(-0.2, 1)$ & $(1.9, 0.3)$.

We choose $g_1(x,y) = \dfrac{x^2 - y + 0.5}{2} = x$

$g_2(x,y) = \dfrac{-x^2 - 4y^2 + 8y + 4}{8} = y$ $[-8y]$

Assume $(p_0, q_0) = (0, 1)$ (Given).

then:

$p_1 = g_1(0,1) = \dfrac{0^2 - 1 + 0.5}{2}$

$= -0.25$

$q_1 = g_2(0,1) = \dfrac{-0^2 - 4(1)^2 + 8 + 4}{8}$

$= 1$

k	p_k	q_k
0	0	1
1	-0.25	1
2	-0.21875	0.9921875
3	-0.2221680	0.9939880
⋮	⋮	⋮
8	-0.2222145	0.9938084
9	-0.2222146	0.9938084

so The Iteration generates a sequence that Converges to the solution $(-0.2, 1)$.

If we start with $(p_0, q_0) = (2, 0)$, then:

The Iteration diverges.

k	p_k	q_k
0	2	0
1	2.25	0
2	2.78125	-0.1328125
⋮		
7	512263.3	-205477.82

Another g_1 & g_2 [adding $-2x$ for f_1 & $-11y$ for f_2]

$$g_1(x,y) = \frac{-x^2 + 4x + y - 0.5}{2} = x$$

$$g_2(x,y) = \frac{-x^2 - 4y^2 + 11y + 4}{11} = y.$$

start with $(2,0)$, then:

So the Iteration generates a sequence that converges to the solution $(1.9, 0.3)$

k	p_k	q_k
0	2	0
1	1.75	0
2	1.71875	0.0852273
⋮	⋮	⋮
12	1.900924	0.3112267
⋮	⋮	⋮
20	1.900677	0.3112196
⋮	⋮	⋮
24	1.900677	0.3112186

Thm: Fixed point Iteration for higher dimensions

Assume that $g_1(x,y)$, $g_2(x,y)$ and their partial derivatives are continuous on a Region that contains the fixed point (p,q).

If the starting point (p_0,q_0) is sufficiently close to the fixed point (p,q) and if

$$\left|\frac{\partial g_1}{\partial x}\right|_{(p,q)} + \left|\frac{\partial g_1}{\partial y}(p,q)\right| < 1$$

&

$$\left|\frac{\partial g_2}{\partial x}(p,q)\right| + \left|\frac{\partial g_2}{\partial y}(p,q)\right| < 1$$

Then the Iteration converges to the fixed point

(Similarly for 3D, 4D, ... we add $\left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| + \left|\frac{\partial g_3}{\partial z}\right| < 1$

Similarly for (3D): If (p_0, q_0, r_0) sufficiently close to (p, q, r) & if

$$\left|\frac{\partial g_1}{\partial x}(p,q,r)\right| + \left|\frac{\partial g_1}{\partial y}(p,q,r)\right| + \left|\frac{\partial g_1}{\partial z}(p,q,r)\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| + \left|\frac{\partial g_2}{\partial z}\right| < 1$$

$$\left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| + \left|\frac{\partial g_3}{\partial z}\right| < 1$$

Then the Iteration converges to the fixed point (p, q, r).

<u>Note</u>: If the conditions are not met, then the thm failed, but this does not mean always divergence).

$$x = \frac{x^2 - y + 0.5}{2}$$

$$y = \frac{-x^2 - 4y^2 + 8y + 4}{8}$$

<u>Example</u>: back to previous example! fixed point $(0.222215, 0.993803)$.

$$\frac{\partial g_1}{\partial x}(x,y) = x \quad , \quad \frac{\partial g_1}{\partial y}(x,y) = -\frac{1}{2}$$

$$\frac{\partial g_2}{\partial x}(x,y) = \frac{-x}{4} \quad , \quad \frac{\partial g_2}{\partial y}(x,y) = -y+1$$

Indeed for all (x,y) satisfying:

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |x| + \left|\frac{-1}{2}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = \left|\frac{-x}{4}\right| + |-y+1| < 1 \quad \cdots (***)$$

The fixed point Iteration converges to (p, q).

solving formula (***), we find that

$-0.5 < x < 0.5$ & $0.5 < y < 1.5$ change to this $0.125 < y < 1.875$

satisfying the partial derivatives..

<u>Note:</u> Near the other fixed point $(1.90068, 0.31122)$ the partial derivates don't meet with the thm then the convergence is not guaranted.

3.6 Iteration method for Linear system:

1) Jacobi Iteration.

Example:

$$4x - y + z = 7$$
$$4x - 8y + z = -21 \quad , \text{ exact Solution } (2,4,3)$$
$$-2x + y + 5z = 15$$

$$x = \frac{7 + y - z}{4} \qquad\qquad p_{k+1} = \frac{7 + q_k - r_k}{4} \quad \left|\tfrac{1}{4}\right| + \left|\tfrac{1}{4}\right| < 1$$

$$y = \frac{-21 - 4x - z}{8} \implies \qquad q_{k+1} = \frac{-21 - 4p_k - r_k}{-8} \quad \left|\tfrac{4}{8}\right| + \left|\tfrac{1}{8}\right| < 1$$

$$z = \frac{15 + 2x - y}{5} \qquad\qquad r_{k+1} = \frac{15 + 2p_k - q_k}{5} \quad \left|\tfrac{2}{5}\right| + \left|\tfrac{1}{5}\right| < 1$$

After 19 Iterations starting with (1,2,2) it will converge

* Gauss - Seidel Iteration:

Given $x = g_1(x, y)$
$y = g_2(x, y)$ starting with (p_0, q_0)

then: $p_1 = g_1(p_0, q_0)$

$q_1 = g_2(p_1, q_0)$

In General: $p_{n+1} = g_1(p_n, q_n)$

$q_{n+1} = g_2(p_{n+1}, q_n)$.

similarly: If $x = g_1(x, y, z)$

$y = g_2(x, y, z)$ starting (p_0, q_0, r_0)

$z = g_3(x, y, z)$

then: $p_{n+1} = g_1(p_n, q_n, r_n)$

$q_{n+1} = g_2(p_{n+1}, q_n, r_n)$ for Linear system

$r_{n+1} = g_3(p_{n+1}, q_{n+1}, r_n)$.

Def: A matrix $A_{n\times n}$ is strictly diagonally dominant iff $|a_{ii}| > \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}|$, $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -1 \\ 4 & 2 & 8 \end{bmatrix}$

Thm: If $A_{n\times n}$ is strictly diagonally dominant then (Jacobi & Gauss Seidel) to solve $\boxed{Ax = b}$ is converge.

ERI

Newton's Method for Nonlinear systems.

Def: If f_1, f_2 are two ~~continuous~~ functions in two variables x, y, Then the Jacobian matrix is given by:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

this method is always converge

write $f_1(x,y) = 0$
$f_2(x,y) = 0$ $\Longrightarrow$ Starting with $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$

(s.t)
& $f_1(p,q) = 0$
$f_2(p,q) = 0$ close to $\begin{pmatrix} p \\ q \end{pmatrix}$.

cost

then use taylor expansion in Two dimensions at $\underline{(p_0, q_0)}$

$$f_1(x,y) \cong f_1(p_0, q_0) + \frac{\partial f_1}{\partial x}\Big|_{(p_0, q_0)} (x - p_0) + \frac{\partial f_1}{\partial y}\Big|_{(p_0, q_0)} (y - q_0)$$

$$f_2(x,y) = f_2(p_0, q_0) + \frac{\partial f_2}{\partial x}\Big|_{(p_0, q_0)} (x - p_0) + \frac{\partial f_2}{\partial y}\Big|_{(p_0, q_0)} (y - q_0)$$

Substitute $\underline{\underline{(p, q)}}$

$$0 = f_1(p_0, q_0) + \frac{\partial f_1}{\partial x}\Big|_{(p_0, q_0)} (p - p_0) + \frac{\partial f_1}{\partial y}\Big|_{(p_0, q_0)} (q - q_0)$$

$$0 = f_2(p_0, q_0) + \frac{\partial f_2}{\partial x}\Big|_{(p_0, q_0)} (p - p_0) + \frac{\partial f_2}{\partial y}\Big|_{(p_0, q_0)} (q - q_0)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} p-p_0 \\ q-q_0 \end{bmatrix}$$

$$-\begin{bmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{bmatrix} = J\Big|_{(p_0,q_0)} \begin{bmatrix} p-p_0 \\ q-q_0 \end{bmatrix},$$

$\Delta p = p - p_0$

$\Delta q = q - q_0$

Then Cramers

call $p = p_1$

& $q = q_1$

This method is called Direct Method

$$\Rightarrow \quad -J^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} p-p_0 \\ q-q_0 \end{bmatrix}$$

$$-J^{-1} \begin{bmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} - \begin{bmatrix} p_0 \\ q_0 \end{bmatrix}$$

$$\begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - J^{-1} \begin{bmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$$

This method is called Inverse method.

<u>In General:</u> Inverse method:

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - J^{-1}\Big|_{(p_n,q_n)} \begin{bmatrix} f_1(p_n,q_n) \\ f_2(p_n,q_n) \end{bmatrix}$$

Direct Method:

$$-\begin{bmatrix} f_1(p_n,q_n) \\ f_2(p_n,q_n) \end{bmatrix} = J\Big|_{(p_n,q_n)} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix}.$$

Example: Solve the following using Newton Method:

<u>Inverse method:</u>

$$x^2 - 2x - y = -0.5 \longrightarrow x^2 - 2x - y + 0.5 = f_1(x,y)$$

$$x^2 + 4y^2 = 4 \longrightarrow x^2 + 4y^2 - 4 = f_2(x,y)$$

Let $(P_0, q_0) = (2, 0.25)$, Exact $(1.90068, 0.311219)$

$$J = \left.\begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}\right|_{(2,0.25)} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} P_0 \\ q_0 \end{pmatrix} - J^{-1} \begin{pmatrix} f_1(P_0, q_0) \\ f_2(P_0, q_0) \end{pmatrix}$$

Note: $f_1(2, 0.25) = 0.25$ & $f_2(2, 0.25) = 0.25$

$$\Rightarrow \begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8} \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$$= \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}$$

Now
$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.008789 \\ 0.024414 \end{pmatrix}$$

$$= \begin{pmatrix} 1.900691 \\ 0.311213 \end{pmatrix}$$

Direct Method :

$$-\begin{pmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{pmatrix} = J\Big|_{(p_0, q_0)} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix}$$

$$-\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix}$$

Using Cramers :

$$\Delta p = \frac{\det \begin{pmatrix} -0.25 & -1 \\ -0.25 & 2 \end{pmatrix}}{\det \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}} = -\frac{0.75}{8} = -0.09375$$

$$\Rightarrow \quad p_1 - p_0 = \Delta p \Rightarrow p_1 = \Delta p + p_0 = -0.09375 + 2 = 1.90625$$

$$\& \quad \Delta q = \frac{\det \begin{pmatrix} 2 & -0.25 \\ 4 & -0.25 \end{pmatrix}}{\det \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}} = \frac{0.5}{8} = 0.0625$$

$$\Rightarrow \quad q_1 = q_0 + \Delta q = 0.3125 .$$