chapter (1):- Matrixes and system of equations:-

1.1 System of linear equation.

Df:- a linear equation in (n) unknows is of the form:-

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

$a_1, a_2, a_3, \dots, a_n, b$ are real number,

$x_1, x_2, x_3 \dots x_n$ are variables or (unknowns)

Ex:- $ax + by = c$ is linear equation of two variables.

- x and y are unknowns (variables)
- a, b and c are constant (real numbers)

Df:- A linear system of (m) eqs and (n) unknowns is of the form:

$$\begin{cases} a_{11}x_{11} + a_{12}x_{12} + \dots + a_{1n}x_{1n} = b_1 \\ a_{21}x_{21} + a_{22}x_{22} + \dots + a_{2n}x_{2n} = b_2 \\ \vdots \\ a_{m1}x_{m1} + a_{m2}x_{m2} + \dots + a_{mn}x_{mn} = b_m \end{cases} \quad *$$

$\Rightarrow a_{ij}$'s, b_i's are real number.

- we call the system as $m \times n$ linear system.

– m = number of eq's – n = number of variabele "unknowns"

Ex(1):- $\begin{cases} 2x_1 - x_2 = 5 \\ x_1 + 3x_2 = 6 \end{cases}$ is 2×2 linear system.

(2) $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 - x_2 + x_3 = 7 \end{cases}$ is 2×3 linear system

(3) $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \\ x_1 = 4 \end{cases}$ is 3×2 linear system.

=> non linear system: at lest one eqation is nonlinear.

ex:- $x + y = 1$ is non linear system.

$x^2 + y = 5$

- In this course we only study linear system.

=> by a soulution of $m \times n$ linear system (*)

we mean an ordered n-tuple.

$(x_1, x_2, x_3, \ldots, x_n)$

$(-,-,-)$ (x_1, x_2, x_3) in order

that satisfies all the equation,

Ex:- (3,1) is a soulution of :

$\begin{cases} 2x_1 - x_2 = 5 \\ x_1 + 3x_2 = 6 \end{cases}$ synse :- $(2 \cdot 3) - 1 = 5$

and $3 + 3 = 6$.

2×2 linear system:-

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad (**)$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

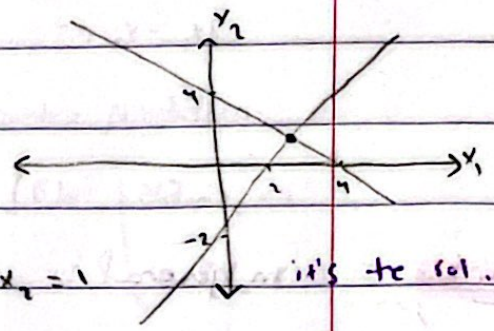
- each eq in $(*)$ is a line in the plane (x_1x_2 - plane) .
- (x_1, x_1) will be a <u>soluation</u> of $(**)$ iff it lies in both lines . نقطة تقاطع الخطان

Ex: solve the system:-

1) $\begin{cases} x_1 + \cancel{x_2} = 4 \\ x_1 - \cancel{x_2} = 2 \end{cases}$ +

$2x_1 = 6 \therefore x_1 = 3$ and $x_2 = 1$

$\therefore (3,1)$ is a soluation!



2) $\begin{cases} x_1 + 2x_2 = 4 \;/2 \\ -2x_1 - 4x_2 = 4 \end{cases} \Rightarrow + \begin{cases} 2x_1 + 4x_2 = 8 \\ -2x_1 - 4x_2 = 4 \end{cases}$

$0 = 12$ (contradiction)

$\therefore$ the system has no soluation . (خطان متوازيان)

3) $\begin{cases} 2x_1 - x_2 = 3 \;/2 \\ -4x_1 + 2x_2 = -6 \end{cases} \Rightarrow + \begin{cases} 4x_1 - 2x_2 = 6 \\ -4x_1 + 2x_2 = -6 \end{cases}$

$0 = 0$

$\therefore$ the system has a infinit soluation . (خطان منطابقان)

How to write it ??

$2x_1 - x_2 = 3 \quad \therefore x_2 = 2x_1 - 3$.

- let $x_1 = t$ free.

$x_2 = 2t - 3$

$\therefore S.S = \{ (x_1, x_2) = (t, 2t-3) : t \in \mathbb{R} \}$

or

$2x_1 - x_2 = 3 \quad \therefore x_1 = \frac{x_2 + 3}{2}$.

let $x_2 = t$ free

$\therefore S.S = \{ (x_1, x_2) = (\frac{t+3}{2}, t) : t \in \mathbb{R} \}$

Remark :- In general :-

there are 3 posibilites, for $\underset{2\times 2}{m\times n}$ linear system :-

(1) the lines intersect at a point (unique solution) (1)

(2) the line parallel (no solution) (2)

(3) both eqs represents the same line (Infinite solution). (3)

→ for $m\times n$ linear system.

Df:- A linear system is consistent if it has a solution and it's incosistent if it has no solution.

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Conclusion:-

$m \times n$ linear system.

- consistent
  - unique
  - Infinite solution.
- Inconsistent
  - No solution.

Equivalent systems:-

Same variables "unknowns" and same solution set

Ex:-

$$A) = \begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 6 \end{cases} \quad (4,-2)$$

$$B) = \begin{cases} x_1 + x_2 = 2 \\ 2x_2 = -4 \end{cases} \quad (4,-2)$$

A and B are equivalent since they have same variabels (x_1, x_2) and same S.S $= (4,-2)$.

Df:- An $m \times n$ linear system is <u>square</u> if $m = n$

Df:- square <u>$n \times n$</u> linear system is in <u>strict triangular form</u> if in the k^{th} equation the coefficient of the first $(k-1)$ variabels are all zero and the coef. of x_k is nonzero $k = 1, 2, 3, 4, \ldots, n$

eq(1) $\Rightarrow$ $x_1 + \ldots$ (0) ماقبلو, con $= x_1$ موجود

eq(2) $0x_1 + ax_2 + \ldots$ // // //, con $= x_2$ //

eq(3) $0x_1 + 0x_2 + bx_3 + \ldots$ // // //, con $= x_3$ //

Ex:- the system:-

$$\begin{cases} x_1 + x_2 + x_3 = 8 & x_1 = 4 \\ 2x_2 + x_3 = 5 & x_2 = 1 \\ 3x_3 = 9 & x_3 = 3 \end{cases}$$

the system is in STF ∴ the sol => $(x_1, x_2, x_3) = (4,1,3)$

→ the system is easy to solve and has a unique solution.
↓ in STF

and we solve it by back substitution as follows:-

Ex:- Solve the system:-

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 5 & k=1 \\ 3x_2 + x_3 - 2x_4 = 1 & k=2 \\ -x_3 + 2x_4 = -1 & k=3 \\ 4x_4 = 4 & k=4 \end{cases}$$

so it's STF.

$x_4 = 1$, $x_3 = 3$, $x_2 = 0$, $x_1 = -2$

in the sol for $(x_1, x_2, x_3, x_4) = \underline{(-2, 0, 3, 1)} = S.S$

Q:- How to transform a system in STF.

A:- we use the following elementary row operation:

I) Interchange two rows (eq's) type (1)

II) Multiply a row (eq's) by a nonzero constant type (2)

+ نجمع ← III) replace a row (eq's) by it's some with a multiple of other row (eq's) type (3).

Ex:- Solve :- the augmented matrix is $[A|b]$

$$\begin{cases} -x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases} \Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 6 \\ 0 & \boxed{-1} & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] = \begin{array}{c} \\ \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

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$$\begin{array}{c} \\ \\ 2R_2 + R_3 \\ -2R_2 + R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$-R_3 + 4$

is in STF

$$\text{STF} \Rightarrow \begin{cases} x_1 + x_2 + x_3 + x_4 = 6 \\ -x_2 - x_3 + x_4 = 0 \\ -3x_3 - 2x_4 = -13 \\ -x_4 = -2 \end{cases} \qquad \begin{array}{l} x_1 = 2 \\ x_2 = -1 \\ x_3 = +3 \\ x_4 = 2 \end{array}$$

$\therefore$ S.S $= (2, -1, 3, 2)$

1.2 row echelon Form (REF):

Df:- An $m \times n$ matrix is in REF iff:-

[1] the first nonzero entry in each non zero row is $\boxed{1}$ called leading one (pivot 1).

[2] the leading 1 in the k^{th} row is to the right of the leading 1 in the $(k-1)$ row.

[3] zero rows are below the non zero row.

Ex:- $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is in REF.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is REF too.

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not in REF.

$C = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is not in REF.

$D = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in REF.

$E = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & \underline{\underline{1}} \\ 0 & \underline{\underline{1}} & 3 \end{bmatrix}$ is not REF

• not on the right of the last one

$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in REF

• $G = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ REF

• $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ not REF

• $K = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ not REF

• $Y = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ REF

Rmk:- 1) Any matrix can be written in REF using the row operation

2) the process of using row operation I, II, III to transform a linear system into one whose augmented matrix is in REF is called Gaussian Elimination method.

$\Rightarrow [A|b] \sim REF$.

Ex:- use Gauss Elimination method to solve the following systems.

1) $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \\ -2x_1 + 2x_2 = -2 \end{cases}$

$$[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -2 & 2 & -2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ -R_1+R_2 \rightarrow \\ 2R_1+R_3 \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 0 \end{array}\right] -\frac{1}{2}R_2$$

$$\begin{array}{r} \\ \\ -4R_2+R_3 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] \frac{1}{4}R_3$$

∴ the equation system is:-

$\begin{cases} x_1 + x_2 = 1 \\ x_2 = -1 \\ 0 = 1 \Rightarrow \text{impossible} \end{cases}$

So the system is inconsistent (No sol)

2) $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 - x_2 + x_3 = 2 \\ 4x_1 + 3x_2 + 3x_3 = 4 \\ 3x_1 + x_2 + 2x_3 = 3 \end{cases}$ $[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right]$

$$= \begin{array}{c} \\ -2R_1 + R_2 \\ -4R_1 + R_3 \\ -3R_1 + R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & +\frac{1}{5} & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] \begin{array}{c} \\ -\frac{1}{5}R_2 \\ \\ \\ \end{array}$$

$$= \begin{array}{c} \\ \\ 5R_2 + R_3 \\ 5R_2 + R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

لأنه يوجد free .

∴ the equation system is :-

$x_1 + 2x_2 + 3x_3 = 1$

$x_2 + \frac{1}{5}x_3 = 0$

- infinite soluation
- x_1, x_2 leading.
- x_3 free.

$\Rightarrow$ let $x_3 = t$. ∴ $x_2 = -\frac{1}{5}t$, $x_1 = 1 - \frac{3}{5}t$

∴ the soluation set is :

$\{(1 - \frac{3}{5}t, -\frac{1}{5}t, t) : t \in \mathbb{R}\}$.

3) $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$ $[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $-2R_1 + R_2$

the system is inconsistent "has no solution".

$\Uparrow$ so

$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 0 = 1 \text{ (imposipple)} \end{cases}$

4) $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$ $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$ $-R_1 + R_2$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$ $-\frac{1}{2}R_2$

- infinite solution.
- x_1, x_2 leading.
- x_3 free

∴ the equivalent system is:

$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

$\Rightarrow$ let $x_3 = t$, $x_2 = -t$, $x_1 = 0$

∴ the solution set $= \{(0, -t, t) : t \in \mathbb{R}\}$

Reduced Row Echelon Form (RREF).

Df:- An $m \times n$ matrix is said to be in RREF if:-

1) the matrix is in REF.

2) the first nonzero entry in each row (1's) is the only nonzero entry in its column.

Ex:-

$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is RREF

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ not RREF

$B = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is RREF

$D = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ not RREF

$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is RREF

$F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ not RREF. So it's not the only non zero value in the coulmn

Gauss-Jordan Elimination method.

$[A|b] \sim$ RREF.

in the procese of using elementary row operation on the orgmented matrix $[A:b]$ of the system $AX=b$ to transform it into a system in RREF.

Ex:- Solving this equation by G.J.E.

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases} = \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -3R_1 + R_2 \\ -2R_1 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & +1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] = \begin{array}{c} -R_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \frac{1}{4}R_2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] = \begin{array}{c} R_1 + R_2 \\ \\ -R_2 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -\frac{1}{3}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} \\ R_2 + R_3 \\ \\ \end{array}$$

- x_1, x_2, x_3 leading
- x_4 free $= \alpha$

the system $\Rightarrow \begin{cases} x_1 - x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \end{cases}$

- infinite sol.

$x_2 = -\alpha$, $x_3 = +\alpha$, $x_1 = \alpha$

- $S.S = \{(\alpha, -\alpha, \alpha, \alpha) : \alpha \in \mathbb{R}\}$

Df:- An $m \times n$ linear system is called :-

underdetermind system if $m<n$
Square if $m=n$
overdetermined system if $m>n$

Rmks(1):- • An underdetermined system always has a free variable.
so it's either inconsistent "no solution" or has "infinite solution"
it's not possible to have a unique solution.

• An overdetermined system can not tell "All cases possible".

Rmk(2):- • Homogeneous system.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{mn}x_n + a_{mn}x_n + \dots = 0 \end{cases}$$

all value of b = zero.

• I) A homogeneous linear system is always consistant.
" $x = x_2 = \dots = x_n = 0$ is a solution
is called zero or trivial solution.

II) A homog. system is either has a unique sol or it has infinite sol if it has a free variable.

Rmks(3):- An under determined homog. linear system always has infinit solution.

Ex.-

II

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ -x_1 + x_2 + \beta x_3 = 0 \end{cases}$$

- for what value's of β dose the system have a:-

1) unique soluation.

2) infite soulation.

3) no soluation.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1 + R_2 \\ R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & (\beta+1) & 0 \end{array}\right] = \begin{array}{c} \\ \\ -3R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{(\beta-2)} & 0 \end{array}\right]$$

1) if $\beta \neq 2$

2) if $\beta = 2$

3) if there is no β "No soluation $\emptyset$".

2) $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + \alpha x_3 = \beta \end{cases}$

- For what value of α and β dose the system have :-
1) unique solution.
2) no ~~sa~~ solution.
3) infinite solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & \alpha & \beta \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1 + R_2 \\ -R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right] \frac{1}{3}R_2$$

3) has infinite sol if $\alpha = 1$ and $\beta = 2$.
2) has no sol if $\alpha = 1$ and $\beta \neq 2$.
1) has unique sol if $\alpha \neq 1$ and $\beta \in \mathbb{R}$.

3) $\left[\begin{array}{ccc|c} x & x & x & x \\ x & x & x & x \\ 0 & 0 & \beta^2+1 & \alpha-1 \end{array}\right]$

Infinite there is no α and β ($\emptyset$.)
unique $\alpha, \beta \in \mathbb{R}$.
no sol $\emptyset$ there is no α and β.

4) $\left[\begin{array}{ccc|c} x & x & x & x \\ 0 & 0 & \alpha-1 & \beta \neq 1 \end{array}\right]$

- there is no unique sol cause it's underdetermined
- $\alpha = 1$, $\beta \in \mathbb{R}$ "no solution"
- $\alpha \neq 1$, $\beta \in \mathbb{R}$ "infinite solution"

1.3 Matrix Arithmetic:-

Def:- matrix $A, B, C, \ldots$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$ column

- $m \times n$ is the size "order" of A. rows
- a_{ij} the entries, $i = 1, 2, \ldots, m$
- مرطان $j = 1, 2, \ldots, n$
- For simplisity we write $A = (a_{ij})_{m \times n}$

$i = 1 \ldots m$

$j = 1 \ldots n$

Ex:-

$$\begin{bmatrix} 4 & -8 & 2 \\ 6 & 8 & 10 \end{bmatrix}_{2 \times 3}$$

- $a_{23} = 10$
- $a_{21} = 6$
- $a_{22} = 8$
- $a_{32} =$ undefined.

- column vector! is $m \times 1$ matrix.

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3 \times 1}$$

- row vector! is $1 \times n$ matrix.

$$\begin{bmatrix} 1 & 4 & 3 & 6 & 7 \end{bmatrix}_{1 \times 5}$$

- $\mathbb{R}^n$: All $n \times 1$ matrices with a real number "entries".

$x \in \mathbb{R}^3 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Leftrightarrow x_1, x_2, x_3 \in \mathbb{R}$.

• $\mathbb{R}^{1\times n}$: all $1\times n$ matrices with real entries.

$$x \in \mathbb{R}^{1\times 4} \Rightarrow x = [x_1\ \ x_2\ \ x_3\ \ x_4]_{1\times 4}$$

• $\mathbb{R}^{m\times n}$: all $m\times n$ matrices with real entries.

$$R^{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad a_{ij} \in \mathbb{R} \qquad \begin{matrix} i = 1, 2, 3 \\ j = 1, 2 \end{matrix}$$

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{matrix} \vec{a}_1 \\ \\ \vec{a}_m \end{matrix} \\ a_1 \qquad\qquad a_n \end{array}$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \end{bmatrix}_{2\times 3}$$

$$\vec{a}_1 = [1\ \ 2\ \ 3], \quad \vec{a}_2 = [0\ \ 4\ \ -1]$$

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\therefore A = [a_1\ \ a_2\ \cdots\ a_n] \Rightarrow A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} = (a_{ij})_{m\times n}$$

Df:- Equality of matrices :-

$A = B$ iff : 1) Size A = Size B

2) $a_{ij} = b_{ij}$, $\forall i,j$

Ex:-

1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2) If $\begin{bmatrix} 1 & 3 \\ 2x+1 & 3y^2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ Find x and y.

$x = 1$ and $y = \pm\sqrt{3}$

operation:- $(+, -, \times)$

Df:- • $A_{m \times n}$, $\alpha \in R$

$\alpha A = \alpha (a_{ij})_{m \times n}$

$A = \begin{pmatrix} -2 & 1 \\ 0 & 5 \end{pmatrix}$ $\Rightarrow 6A = \begin{bmatrix} -12 & 6 \\ 0 & 30 \end{bmatrix}$

• Addition and subtraction.

$A_{m \times n}$, $B_{m \times n}$ $\Rightarrow$ $A_{m \times n} \pm B_{mn} = (a_{ij} \pm b_{ij})_{mn}$ $\forall i,j$

- $A = \begin{bmatrix} 3 & 2 & 7 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$

$A + B$ = undefined.

- $2\begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} - 3\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$

$\begin{bmatrix} 4 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -9 & -6 & -3 \\ -12 & -15 & -15 \end{bmatrix} = \begin{bmatrix} -5 & -6 & 5 \\ -12 & -15 & -15 \end{bmatrix}$

- zero matrix 0: All entries are zero.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$$

Properties: $A_{m,n}$, $B_{m,n}$, $\alpha, \beta \in \mathbb{R}$.

1) $\alpha(A+B) = \alpha A + \alpha B$
2) $(\alpha\beta)A = (\alpha A)\beta = (\beta A)\alpha$.
3) $A+B = B+A$.
4) $A+(B+C) = (A+B)+C$
5) $A+0 = A = 0+A$.
6) $A + -A = 0$ $-A$: is additive inverse of A.

- matrix Multiplication.

$$A_{m\times n} B_{n\times k} = C_{m\times k}$$

Ex: 1) $A = \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \end{bmatrix}_{2\times3}$

BA = undefined. 2×3, 2×2

$AB = \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & -3 \end{bmatrix}$ 2×2, 2×3

$$= \begin{bmatrix} 1\times1 + 3\times0 & 1\times5 + 3\times1 & 1\times3 + 3\times-3 \\ 6\times1 + -1\times0 & 6\times5 + -1\times1 & 6\times3 + -1\times-3 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 1 & 8 & -7 \\ 6 & 29 & 15 \end{bmatrix}_{2\times3}$$

2) $A = [1 \quad 2]$ $\quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$AB = [1 \quad 2]_{1\times 2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$

$= 1+4 = [5]_{1\times 1}$

$BA = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} [1 \quad 2]_{1\times 2} = \cancel{1\times 1 \quad [5]} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2}$

$\therefore AB \neq BA$. in general.

$\Rightarrow$ Back to linear system :-

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$A_{m\times n} \in \mathbb{R}^{m\times n}$ $\quad$ $b \in \mathbb{R}^m$.

$x \in \mathbb{R}^n$

- $Ax = b$.

A: cofficants matrix.

$x \in \mathbb{R}^n$: unknown.

$b \in \mathbb{R}^m$: constant (known).

Ex:- write in a matrix notation:-

- $\begin{cases} 4x_1 + 2x_2 + x_3 = 1 \\ 5x_1 + 3x_2 + 7x_3 = 2 \end{cases}$ $\quad x_1\begin{bmatrix}4\\5\end{bmatrix} + x_2\begin{bmatrix}2\\3\end{bmatrix} + x_3\begin{bmatrix}1\\7\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$x_1 a_1 + x_2 a_2 + x_3 a_3 = b$.

$$\begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$A \in \mathbb{R}^{n\times m}$ $\quad x \in \mathbb{R}^3$ $\quad b \in \mathbb{R}^2$

$\Rightarrow$ In general we can write $Ax = b$

as $\boxed{x_1 a_1 + x_2 a_2 + \dots + a_n x_n = b}$

Rmks:- 1) a vector x_0 is a solution of $Ax = b$ iff $Ax_0 = b$.

2) x_1, x_2 are solution of $Ax = b$, then $\alpha x_1 + \beta x_2$ is a solution of $Ax = b$ iff $\alpha + \beta = 1$

proof:- Given $Ax_1 = b$, $Ax_2 = b$

$$A(\alpha x_1 + \beta x_2) = \alpha(Ax_1) + \beta(Ax_2)$$
$$= \alpha b + \beta b = b(\alpha + \beta)$$
$$= b \text{ iff } \alpha + \beta = 1$$

3) x_1, x_2 are solution of $Ax = 0$ then $\alpha x_1 + \beta x_2$ is a sol of $Ax = 0$ iff $\alpha, \beta \in \mathbb{R}$.

Df:- Linear combination:-

$a_1, a_2, \ldots \ldots a_n$ vectors in $\mathbb{R}^n$.

$c_1, c_2, \ldots \ldots, c_n$ scalars.

then:- $c_1a_1 + c_2a_2 + \ldots + c_na_n = V$

is called a linear combination of $a_1, a_2, \ldots a_n$.

Ex:- (1) is $b = \begin{pmatrix} 2 \\ 24 \end{pmatrix}$ a linear comb of $a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

let $c_1a_1 + c_2a_2 = b$.

$c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 24 \end{pmatrix}$, in $b = 2a_1 + 4a_2$

$$\begin{cases} c_1 = 2 \\ 2c_1 + 5c_2 = 24 \end{cases}$$

$\therefore b \Rightarrow$ is a linear comb

$\therefore c_2 = 4$.

(2) Is $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a linear comb.

of $a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$?

let $\alpha_1 a_1 + \alpha_2 a_2 = b$

$\alpha_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\alpha_1 + 2\alpha_2 = 1$

$2\alpha_1 + 4\alpha_2 = 1$

$0 = -1$ imposipel.

$\Rightarrow$ system is inconsistent

$\Rightarrow$ b is not a line comb of a_1 and a_2

Thm:- consistency of $Ax=b$ $\Leftrightarrow$

$Ax=b$ is consistent iff b is a linear combination of the columns of A.

that is, $(b = x_1a_1 + x_2a_2 + \cdots + x_na_n)$

Proof:-

$(\Rightarrow)$ $Ax=b$ is consistent.

$$\Rightarrow A\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\\ \alpha_n\end{pmatrix} = b$$

$\Rightarrow \alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3 + \cdots + \alpha_na_n = b$

b is a linear combination of $a_1, \ldots, a_n$

$(\Leftarrow)$ Suppose b is a lin. comb of the coulmns of $A\,(a_1 \ldots a_n)$

$b = \underbrace{c_1a_1 + c_2a_2 + c_3a_3 + \cdots + c_na_n}$.

$$b = A\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$ is a sol of $Ax=b$

$\Rightarrow Ax=b$ is consistencet

Application:-

Ex:- (I) $A_{3\times3}$, $Ax=b$, $b=4a_1-6a_2+3a_3$

is $Ax=b$ consistacal:-

since $b=4a_1-6a_2+3a_3$ then

one of the sol :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$$ is a sol of $Ax=b$

$\Rightarrow$ $Ax=b$ is consistacel

(2) $A_{3\times4}$ $b=a_1+\underline{a_2+a_3+a_4}$ $b=a_1$ then : 0 = ملاحظة البقية

1) is $Ax=b$ consistacel.

2) if yes what can conclude about the number of solution.

1) since $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a sol of $Ax=b$.

then $Ax=b$ is consistance.

2) since $Ax=b$ is under determined system and consistacel then it must have infinite sol.

③ $A_{3\times3}$, $a_3 = a_1 - a_2$. $a_1 - a_2 - a_3 = \boxed{0}$ b

How many sol of $Ax = b$

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is a sol of $Ax = 0$

Since $Ax = 0$ has a nontrivial solution $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ then $Ax = 0$ has infinite sols.

④ $A_{4\times3}$ with $a_1 = a_2$, $a_1 - a_2 = 0$ $(0 = a_3$ مثلا$)$

② Als $b = a_1 + a_2 + a_3$, $a_1 = a_2$

$b = 2a_1 + a_3 \Rightarrow$ $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is another sol.

If $\boxed{b = a_1 + a_2 + a_3}$.

① $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = b$ $\Rightarrow$ the sol is consistance.

Solvation is infinite one.

the transpose of a matrix (part of 1.4)

the transpose of $A_{m \times n}$ is $A^T = (a_{ij})^T_{m \times n} = (a_{ji})_{n \times m}$.

Ex:-

1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

4) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Df:-

- $A_{n \times n}$ is symetric iff $a^T = a$
- $A_{n \times n}$ is skew-symmetric iff $a^T = -a$.

properties of the transpose:-

1) $(A^T)^T = A$

2) $(A \pm B)^T = A^T \pm B^T$

3) $(\alpha A)^T = \alpha (A)^T$

4) $(AB)^T = B^T A^T$

5) if $A_{n\times n}$, $B_{n\times n}$ are sym. matrices then $A+B$ is also sym.

6) If $A_{n\times n}$ is sym, then αA also sym.

7) If A, B are sym, then $H = AB - BA$ is skew-sym. $H^T = -H$.

8) if A is $m\times n$ matrix then A^TA and AA^T are sym.

9) if A is sym and skew. sym, then A must be zero matrix

proof:-

$\boxed{5}$ Given $A^T = A$ and $B^T = B$, then $(A+B)^T$

$= A^T + B^T = A + B$ ($\because A, B$ sym)

$\therefore A + B$ sym

$\boxed{6}$ Given $A^T = A$

$(\alpha A)^T = \alpha (A)^T = \alpha (A) \therefore \alpha A$

$\therefore \alpha A$ is sym.

$\boxed{7}$ Given $A^T = A$ and $B^T = B$

$\boxed{H^T} = (AB - BA)^T = (AB)^T - (BA)^T$

$= B^TA^T - A^TB^T = BA - AB = \boxed{-H}$.

$\therefore$ it's skew-sym.

[8] • $(A^TA)^T = A^T(A^T)^T = \underline{\underline{A^TA}}$ ∴ it's sym.

• $(AA^T)^T = (A^T)^T A^T = \underline{\underline{AA^T}}$ ∴ it's sym.

[9] Given $A^T = A$ and $A^T = -A$

then $A^T + A^T = A - A$

$2A^T = 0$

$A^T = 0$

$(A^T)^T = (0)^T \Rightarrow A = 0$.

1.4 Matrix Algebra :-

theorem:- α, β scalars.

A, B and C matrices.

1) $A+B = B+A$.

2) $(A+B)+C = A+(B+C)$

3) $(AB)C = A(BC)$

4) $A(B+C) = AB+AC$

5) $(A+B)C = AC+BC$

6) $(\alpha\beta)A = \alpha(\beta A)$.

7) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

8) $(\alpha+\beta)A = \alpha A+\beta A$.

9) $\alpha(A+B) = \alpha A+\alpha B$.

10) $A^n = A \cdot A \cdot A \cdots A$ n-times.

Ex:- 46+47 in leacture note :-

in page 48:-

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{Find } A^{2023}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\underset{A^2 A}{A^3} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$\vdots$

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$$

$$\therefore A^{2023} = \begin{bmatrix} 2^{2022} & 2^{2022} \\ 2^{2022} & 2^{2022} \end{bmatrix}$$

The identity matrix:-

$n \times n$ - identity matrix is :-

$$I = \delta_{ij}, \text{ where } \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

ex:- $I_{3\times3} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_1 = [1]$

Rmk:- $B_{m\times n}$, matrix

$C_{n\times r}$ " , $I_{n\times n}$

then $BI = B$ and $IC = C$

Matrix Inversion:-

Df:- $A_{n\times n}$ is nonsingular or invertible if there exists a matrix B: $AB = BA = I$.

B is called the inverse of A, denoted by A^{-1}.

if A^{-1} dose not exists then A has no inverse or singular or not invertible.

Ex:- $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix}$

Show that $A^{-1} = B$.

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix} = \begin{bmatrix} \frac{-2}{10}+\frac{12}{10} & \frac{4}{5}+\frac{-4}{5} \\ \frac{-3}{10}+\frac{3}{10} & \frac{6}{5}-\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix} * \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q:- How to find $A^{-1}_{2\times 2}$?

A:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- if $\alpha = ad - bc = 0$. then A^{-1} dose not exists. (A singular)
- if $ad - bc \neq 0$, then A is nonsingular

$$A^{-1} = \frac{1}{\alpha}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof:- $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{\alpha} & \frac{-b}{\alpha} \\ \frac{-c}{\alpha} & \frac{a}{\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex:- 1) $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ Find A^{-1} if any.

$\alpha = 8 - 6 = 2$

$\therefore A^{-1} = \begin{bmatrix} \frac{2}{2} & \frac{-3}{2} \\ \frac{-2}{2} & \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-3}{2} \\ -1 & 2 \end{bmatrix}$

2) $A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$

$\alpha = 18 - 18 = 0$ so there is no A^{-1}.

(A is singular matrix).

Thm:- • if A and B are nonsingular then AB is also nonsingular. $\underline{(AB)^{-1} = B^{-1}A^{-1}}$.

proof:-

$\Rightarrow (AB)(AB)^{-1}$

$= (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= A I A^{-1}$ ($\because B^{-1}$ exists)

$= AA^{-1} = I$ (since $\because$ A nonsingular).

$\Rightarrow (AB)^{-1}(AB)$.

$(B^{-1}A^{-1})AB) = B^{-1}(A^{-1}A)B$

$= B^{-1} I B$

$= I$

Rules for Inverse:-

1) the A^{-1} if exists is unique.

2) $(A^{-1})^{-1} = A$.

3) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$, $\alpha \neq 0$

Proof:- $(\alpha A)(\alpha A)^{-1}$

$(\alpha A)\frac{1}{\alpha} A^{-1}$.

$(\alpha \frac{1}{\alpha})(AA^{-1}) = I$.

$\therefore (\alpha A)(\alpha A)^{-1} = I$.

4) if A is invertible, then A^T is invertible

and $(A^T)^{-1} = (A^{-1})^T$.

Proof:- • $(A^T)(A^T)^{-1} = (A^T)(A^{-1})^T$.

$= (A^{-1}A)^T = I^T$

$= I$

• $(A^T)^{-1}(A^T) = (A^{-1})^T (A)^T = (AA^{-1})^T = I^T = I$.

5) $[(AB)^T]^{-1} = (A^{-1})^T (B^{-1})^T$

6) $A_1, \dots, A_k$ are non singular $\Rightarrow A_1 \cdot A_2 \cdot \dots \cdot A_k$ is non singular

$(A_1 \dots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \dots A_2^{-1} A_1^{-1}$.

Exercises:- prove or disprove.

1) if A and B are nonsingular, then $A+B$ is also nonsing.
False.

2) the sum of singular is singular. False.

3) $A^2 - B^2 = (A-B)(A+B)$ A, B matrices False.

3*) if A, B are commute. $AB = BA$ then $A^2 - B^2 = (A-B)(A+B)$

4) $(A+B)^2 = A^2 + 2AB + B^2$ False.

5) if $AB = 0$, then A or $B = 0$ False.

6) if $A^2 = 0$, then $A = 0$ False.

7) if $AB = AC$, then $B = C$ False.

7*) if $AB = AC$ and A^{-1} exists then $B = C$.

8) if $A^2 = A$ then $A = I$ or $A = 0$ False.

9) $A_{n\times n}$, $A^2 = A$, then $(I+A)^{-1} = I - \frac{1}{2}A$ True.

proof:- • $(I+A)(I-\frac{1}{2}A)$

$= I^2 + AI - I\frac{1}{2}A - \frac{1}{2}AA$

$= I + A - \frac{1}{2}A + A - \frac{1}{2}A = I + 0 = I$

• $(I - \frac{1}{2}A)(I+A) = I$

$I^2 + AI - \frac{1}{2}AI - \frac{1}{2}A^2$

$I + 0 = I$.

10) $A_{n\times n}$. $A^2 = 0$, then $(I-A)^{-1} = I + A$ true.

proof:- • $(I-A)(I+A) = I$

$I^2 - AI + AI - A^2$

$= I - A + A - 0 = I$

• $(I+A)(I-A)$

$I^2 - IA + AI - A^2$.

$I^2 - 0 = I$.

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ non singulare.

1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ singulare.

11) $A_{n\times n}$, $B_{n\times n}$. if $AB = A$ and $B \neq I$, then A must be singular. True.

proof:- if A were non singular

$\Rightarrow AB = A \quad / A^{-1}$

$AB A^{-1} = A A^{-1}$

$IB = I$

$\therefore I = B \longleftarrow$ contradiction تناقض.

$\therefore A$ must be singular.

1.5 Elementary matrices.

Df:- A matrix E is an elementary matrix if it is obtained from $\boxed{I_n}$ by performing exactly one row operations.

there are three type.

- type I :- Interchanging any two row. $E^{(1)}$
- type II :- multiply by a non zero constant $E^{(2)}$
- type III :- adding a multiple of one row of I_n to other $E^{(3)}$

Ex:

$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(1)}$ تبديل 1 , 2

$E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(3)}$ $\quad -3R_3 + R_1$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2023 \end{bmatrix}$ is $E^{(2)}$ $\quad R_3 = \frac{R_3}{2023}$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ it's not elementary matrix.

note:- $E \cdot E^{T} \neq E$.

Ex:- • $A_{3\times3}$, $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(1)}$.

تبديل صفوف $EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$ R_2 R_1

بالعمليات of type $E^{(1)}$

تبديل اعمدة $AE = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}$

بالاعمدة of type $E^{(1)}$

Thm:- If E is an elementary matrix, then E is nonsingular (invertible) and E^{-1} is an elementary of the same type.

Ex:- $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is $E^{(2)}$

توضيح.

$E^{-1} =$ $\quad \alpha = 2 - 0 = 2$.

it's also $E^{(2)}$ $\Leftarrow$ $\therefore E^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

Df:- A matrix B is a row equivalent to a matrix A, if there exists element matrices, $E_1, E_2, \dots, E_n$:-

$$B = (E_k E_{k-1} \dots E_1) A.$$

or B is row equivalent to A if B can be obtained from A by a finite of row equivalent.

Ex:- if $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix} R_{A3} + R_{A1}$

$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$ it's $(R_2 - R_3)$ at B

a) find an element matrix $E : EA = B$ "that is B is row equivalent to A"

$\therefore E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ it's come from $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\Downarrow$
it's $R_1 + R_3$

b) find an elementary matrix $E \mid E : EB = C$ " C is a row equiv to B"

$\therefore E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ check if $EB = C$.

c) Is C row equiv to A?

yes because $B = E_1 A$ and $C = E_2 B$

$\therefore C = E_2 E_1 A$.

$\therefore$ C is row equiv to A.

Remark: I) If A is row equiv to B, then B is row equiv to A.

II) If $A \overset{row}{\sim} B$ and $B \overset{row}{\sim} C$

$\therefore A \overset{row}{\sim} C$. "proof page 57, lecture note"

very important.

Theorem: Let $A_{n\times n}$ matrix then the following are equivalent

(a) A is nonsingular.

(b) $Ax=0$ has only trivial (zero) solution.

(c) A is row equivalent to I.

$$\begin{array}{ccc} & a & \\ \Uparrow & & \Downarrow \\ c & \Leftarrow & b \end{array}$$ **Proof**

- $a \Rightarrow b$: suppose A is nonsingular. let y be a solution of $Ax=0$ $\therefore Ay=0$

$\Rightarrow A^{-1}(Ay) = (A^{-1})(0) = Iy = 0$

$\Rightarrow y=0$

$\Rightarrow Ax=0$ has only zero sol

- $b \Rightarrow c$: suppose $Ax=0$ has only zero sol we need to prove $A \overset{row}{\sim} I$

Suppose $A \overset{row}{\not\sim} I$ so, the RREF of A has a free variable.

$Ax=0$ has infinite sol which is a contradiction

$\therefore A \overset{row}{\sim} I$

• c ⟹ a : suppose $A \overset{row}{\sim} I$. Show A is nonsingular. Indeed,

$A \overset{row}{\sim} I \Rightarrow A = \underbrace{(E_k \dots E_1)}_{\text{clearly}} I$.

$A = (\overset{\text{nonsingular}}{E_k} \dots \overset{\text{nonsingular}}{E_1})$

∴ A is nonsingular, is a product of nonsingular matrices :)

انتهى البرهان (Q.E.D).

خُلِع ايرلاينيزيه ☺.

نتيجة Corollary: $Ax = b$, $A_{n\times n}$ has a unique sol iff A is nonsingular.

$A^{-1}Ax = A^{-1}b$

$x = A^{-1}b$

Ex:- True or False. if $A_{4\times4}$ and $a_1 + a_2 = a_3 + 2a_4$, then A must be singular. T "so we need proof".

$a_1 + a_2 - a_3 - 2a_4 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \\ -2 \end{pmatrix}$ is a non zero sol of $Ax = 0$

so the $\underbrace{\text{homog. system}}_{Ax=0}$ has infinite sol.

So, A is singular.

Remember How to find $A^{-1}_{n\times n}$?

$$[A_{n\times n} \mid I_{n\times n}] \xrightarrow[\text{RREF}]{\text{row operation}} [I_{n\times n} \mid A^{-1}]$$

if we have a zero row at I or A

then it's don't have or A^{-1}.

Ex:- $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ Find A^{-1} "if any".

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] -2R_1+R_3$$

$$\begin{array}{l} -R_2+R_1 \\ R_2 \\ 2R_2+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$-2R_3+R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & -2 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] \checkmark$$

$\underbrace{\hphantom{000}}_{I}$ $\underbrace{\hphantom{000}}_{A^{-1}}$ $(A^{-1}A) = (AA^{-1}) = I$.

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

Ex: solve using inverses :-

$$\begin{cases} x_1 + x_2 + 2x_3 = -2 \\ x_2 + 2x_3 = 3 \\ 2x_1 + x_3 = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}}_{b}$$

- $x = A^{-1} \cdot b$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ -17 \\ 10 \end{bmatrix}$$

it's unique sol (i.e. $x_1 = -5$, $x_2 = -17$, $x_3 = 10$)

to find A^{-1} $\Longrightarrow$ like the last exampel. so it's $\begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

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Diagonal and triangular matrices:-

Df:- A $n \times n$ matrix:-

1) if $a_{ij} = 0$, $\forall i > j$

$\Rightarrow$ A is upper tringular.

2) if $a_{ij} = 0$, $\forall i < j$

$\Rightarrow$ A is lower tringular.

3) A is triangular if it is lower or upper

4) $a_{ij} = 0$, $\forall j \neq i$ $\Rightarrow$ Diagonal.

Ex:-

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$ upper triangular.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 0 & 0 \\ 0 & 9 & 3 \end{bmatrix}$$ lower tringular.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ Diagonal and tringular.

every Diagonal is tringular.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Both diagonal and tringular.

- **LU - factorization:-** • Triangular factorization.

Df:- if A is reduced into an $\boxed{\text{upper triang using Type III only}}$.

then A has a triang factorization.

$A = LU$ where U is upper triangular.

L is a unit lower triangular.

$$\bullet\ L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}_{3\times 3}$$

Remark:- Not every matrix has an LU factorization.

Ex:- compute LU factorization of:-

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \Rightarrow \begin{array}{c} \\ 2R_1+R_2 \\ -3R_1+R_3 \end{array} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{c} \\ \\ 2R_2+R_3 \end{array} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \text{ upper tringular.}$$

$$\bullet\ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bullet\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\bullet\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

so $E_3 E_2 E_1 A = U$.

$A = (E_3 E_2 E_1)^{-1} U$

$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$

$E_1^{-1} \Rightarrow$

$-2R_1+R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$E_2^{-1} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ نضرب الرقم المختلف بـ (-1).

$E_3^{-1} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\Rightarrow L : E_1^{-1} E_2^{-1} E_3^{-1}$

$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} = L$

$U = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$2R_1 + R_2$

$-3R_1 + R_3$

$2R_2 + R_3$.

L نضرب (-1) نتكون

Ex:- $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$ Find LU - fac.

$$\begin{array}{l} \\ -\frac{1}{2}R_1+R_2 \\ -2R_1+R_3 \end{array}\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

$\Downarrow$ $+3R_2+R_3$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 & 3 \\ 2 & 4 & -1 \\ -2 & 2 & -4 \end{bmatrix}$ has no LU-fact.

$a_{11} + a_{21}$ لانه لا نستطيع تبديل

U-tri وبالتالي لا نستطيع تحويلها إلى

→ T or F:-

- if A has LU-fact, then :-

1) A is non-sing iff L is nonsingular. F

2) A 〃 〃 〃 U 〃 〃 T "L always non singular".

3) A is row equivalent to U . T "$A = (E_k \cdots E_1)^{-1} U$".

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

chapter(2):- Determinants.

2.1 the determinant of a matrix:-

Df:- $A_{n\times n}$, the determinant of A is denoted by $\det(A)$ or $|A|$.

- case 1×1, $(a_{11}) = A$

$\det(A) = a_{11}$

- ex:- $A = [5]_{11} \rightarrow \det(A) = 5$.

$B = [-5]_{11} \rightarrow \det(A) = -5$.

- case 2×2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = (a_{11} \times a_{22}) - (a_{12}\, a_{21}).$$

- ex:- $A = \begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}$

$$|A| = -10 - 4 = -14$$

Thm:-

$A_{n\times n}$ is nonsingular iff $\det(A) \neq 0$

$A_{n\times n}$ is singular iff $\det(A) = 0$

- ex:- $A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ is singular

since $|A| = 18 - 18 = 0$.

Ex:-

- is $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ singular ?

$\det(A) = -14$ so it's non singular , invertabel.

- if $A = \begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix}$ is singular , find λ .

$\det(A) = 0 = (2-\lambda)(3-\lambda) - 12$.

$12 = 6 - 2\lambda - 3\lambda + \lambda^2$

$\lambda^2 - 5\lambda - 6 = 0$.

$(\lambda - 6)(\lambda + 1) = 0$

$\therefore \lambda = 6 , -1$.

- cofactor method :-

$A_{n\times n}$, $M_{ij} = (n-1)\times(n-1)$ matrix

obtained from A by deleting the row and column

containing a_{ij} . then M_{ij} = the minor of a_{ij}

$\det(M_{ij})$.

A_{ij} = the cofactor of a_{ij}

$= (-1)^{i+j} \det(M_{ij})$.

Ex:. $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ find M_{13}, A_{32} and A_{21}

- $M_{13} = \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 7$. the minor of $a_{13} = 4$.

- $A_{32} = (-1)^{3+2} M_{32}$

  $= -\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = +8$.

- $A_{21} = -\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} = -(30-16)$

  $= -14$

- A_{34} : undefined

Df:

$A_{n \times n}$. then:-

$$\det(A) = \begin{cases} a_{11} & n=1 \\ a_{11}A_{11} + a_{12}A_{12} - \cdots + a_{1n}A_{1n} & n>1 \end{cases}$$

the expansion of $|A|$ along the first row of A.

Ex:-

- $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$

$\det(A) = 3\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$

$= 3(-4-9) - 2(2-6) + 4(3+4)$

$= (3)(-13) + 8 + (4)(7)$

$= -39 + 8 + 28$

$= -3$ so it's non singular.

- Find $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 7 & 8 & 2 & 0 \\ -2 & 1 & 5 & 6 \end{vmatrix}$

$\therefore \det = 1\begin{vmatrix} 6 & 0 & 0 \\ 8 & 2 & 0 \\ 1 & 5 & 6 \end{vmatrix} = 6\begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix}$

$= 6(12)$

$= 72 \neq 0$

$\therefore$ non singular.

is the product of the main diagonal entries

this the case in general for triangular matricies.

Thm:- If $A_{n\times n}$, $n \geq 2$, then det(A) can be expressed as a new factor expansion using any row or column of A.

Ex:- Find det of $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$.

$$-2 \begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (-2)(3) \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= (-2)(3)(10-12)$$

$$= 4 \times 3 = 12.$$

Thm:-
- $A_{n\times n}$ then $|A^T| = |A|$.
- $A_{n\times n}$ tringular matrix then det(A) = the product of the diagonal entries.
- $A_{n\times n}$:-
  1) if A has a zero row or a zero column, then $|A| = 0$.
  2) if A has two identical rows or columns then det(A) = 0.

صفين او عامودين متشابهين

Ex:-

- $\begin{vmatrix} \boxed{1 & 2 & 4 & 6} \\ 5 & -1 & 0 & 4 \\ 6 & 7 & -9 & 10 \\ \boxed{1 & 2 & 4 & 6} \end{vmatrix} = 0$ (thm).

- $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$ (thm)

2.2 Row operations.

Thm:- Let A be a square matrix and B is obtained from A by only one row operation.

- Type I : change row $\Rightarrow |B| = -|A|$.

ex:- $A = \begin{bmatrix} 2 & 4 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}$

$A = -14$, $B = 14$

- Type II : multiplication $\Rightarrow |B| = \alpha|A|$.

"multiple one row from A by α".

ex:- $A = \begin{bmatrix} 2 & 4 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$

$A = -14$, $|B| = -7$

- Type III : multiplication and addition. $\Rightarrow |B| = |A|$.

ex:- $A = \begin{bmatrix} 1 & 4 \\ 5 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 0 & -25 \end{bmatrix}$ $-5R_1 + R_2$

$|A| = -25$ $|B| = -25$

Thm:- Let E an elementary matrix then $\det(A) = \begin{cases} -1 & \text{Type I} \\ \alpha & \text{Type II} \\ 1 & \text{Type III} \end{cases}$

if $\alpha \neq 0$

Rmrk:- $\det(A) \neq 0 \Rightarrow E$ is nonsingular.

Thm: • $E_{n\times n}$ elem, $A_{n\times n}$ matrix.

$|EA| = |E|\,|A|$.

• $E_1, \dots, E_k$ elem

$|E_1 \cdots E_k| = |E_1| \cdots |E_k|$.

• $A_{n\times n}$ is nonsingular iff $\det(A) \neq 0$.

$A_{n\times n}$ is singular iff $\det(A) = 0$.

• A, B $n\times n$ matrices then, $\det(AB) = \det(A)\det(B)$.

Ex:- T or F:-

- $\det(AB) = \det(BA)$ $A_{n\times n}$, $B_{n\times n}$.

T, $|AB| = |A||B| = \underline{\underline{|B||A|}}$

$|BA| = \underline{\underline{|B||A|}}$ integers.

Ex:- if $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, Find

$$2 \xleftarrow{①} \begin{vmatrix} \boxed{2a \quad 2b \quad 2c} \\ d \quad e \quad f \\ g+a \quad h+b \quad i+c \end{vmatrix} \xleftarrow{③} -R_1 + R_3 \Rightarrow 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 10$$

Ex:- prove or disprove:-

1) $|A+B| = |A| + |B|$, $A_{n\times n}$, $B_{n\times n}$

F, $\begin{array}{c} A \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} + \begin{array}{c} B \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$ $\quad |A| = |B| = 1$

$|A+B| = 0$

but $|A| + |B| = 2$

2) $|A^n| = |A|^n$, $n = 0, 1, 2, 3, \ldots$

T. $|A^n| = |AA\,AA \ldots|$ n times.

$|A||A||A| \ldots$ n times

$= |A|^n$.

3) $A_{n\times n}$, $|kA| = k^n|A|$. T

4) A $\underbrace{\text{nonsingular}}$ ($|A| \neq 0$) $\Rightarrow |A^{-1}| = \frac{1}{|A|}$

T, $AA^{-1} = I$.

$|A||A^{-1}| = |I| = 1 \quad \therefore |A^{-1}| = \frac{1}{|A|}$.

5) if $A^2 = A$, then $|A| = 0$ or 1

T, $|A^2| = |A|$

$|A||A| = |A|$.

$|A|(|A|-1) = 0$

$\therefore |A| = 0$ / 1.

6) $A^T A = I$, then $|A| = \pm 1$.

T, $|A^T A| = |I|$

$|A^T||A| = 1$.

$|A|\,|A| = 1 \quad \therefore |A| = \pm 1$

7) $A_{n\times n}$ skew-sym, n is odd, A must be singular.

T. $A^T = -A$

$|A^T| = |-A|$

$|A| = (-1)^n |A|$.

$|A| = -|A|$

$2|A| = 0 \therefore |A| = 0$, $\Rightarrow$ A is singular.

- if it's sym $\Rightarrow$ F
  if n is even $\Rightarrow$ F

لا يتم لأي استنتاج

9) $A_{n\times n}$, $B_{n\times n}$, then AB is nonsingular iff A and B are both nonsingular.

T, AB nonsingular iff $|AB| \neq 0$

$\Leftrightarrow |A||B| \neq 0$

$\Leftrightarrow |A| \neq 0$ and $B \neq 0$

$\Leftrightarrow$ A and B are nonsingular.

10) A, B, C (3x3) matrices $|A|=9$, $|B|=2$, $|C|=3$.

then $|4C^TBA^{-1}| = \frac{128}{3}$

T, $(4)^3 |C^T||B||A^{-1}| = 64 \frac{|C||B|}{|A|} = \frac{64(3)(2)}{\cancel{9}\,3}$

$= \frac{128}{3}$

2.3 Additional Topics and Applications.

The adjoint of a matrix.

$A_{n\times n}$, then $\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$

where $A_{ij} = (-1)^{i+j} M_{ij}$ cofactor

Ex:- Find adj(A) where $A = \begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix}$

$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$

- $A_{11} = \det([6]) = 6$
- $A_{12} = -\det([4]) = -4$
- $A_{21} = -\det([3]) = -3$
- $A_{22} = \det([-1]) = -1$

$\operatorname{adj}(A) = \begin{bmatrix} 6 & -4 \\ -3 & -1 \end{bmatrix}^T$

$= \begin{bmatrix} 6 & -3 \\ -4 & -1 \end{bmatrix}$

Thm:- $A_{n\times n}$ matrix then $\underline{A \, adj(A) = |A| I_{n\times n}}$ either if A sing or not.

if A is non singular, $\underbrace{A^{-1} = \frac{1}{|A|} adj(A)}_{\text{adjoint formula.}}$

Ex:- $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ Find $|A|$, $adj(A)$, A^{-1}.

- $|A| = 5 \neq 0$ ∴ non singular.
- $adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T$

$+\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}$

$= \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$ ∴ $A^{-1} = \frac{1}{|A|} adj(A)$

$= \frac{1}{5} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$

Cramer's Rule

$A_{n\times n}$ nonsingular, $b \in \mathbb{R}^n$

A_i = the matrix A obtained by replacing i column of A by b.

if X is the unique sol of $Ax = b$, $x_i = \frac{|A_i|}{|A|}$, $i = 1, \dots, n$.

Ex: use cramer's rule to solve:-

$$\begin{cases} x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 6 \\ x_1 + 2x_2 + 3x_3 = 9 \end{cases}$$

$$A \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$|A| = -4$, $|A_1| = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} = -4$

$|A_2| = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} = -4$

$|A_3| = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} = -8$

$x_1 = \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1$

$x_2 = \frac{|A_2|}{|A|} = \frac{-4}{-4} = 1$

$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-4} = 2$.

- Let $A_{n \times n}$ be nonsingular with $n>1$. Show that

$$|\operatorname{adj}(A)| = |A|^{n-1}$$

$$|\operatorname{adj}(A)| = \left| |A| A^{-1} \right|$$
$$= |A|^n |A^{-1}| = \frac{|A|^n}{|A|} = |A|^{n-1}.$$

- Show that if A is nonsingular, then $(\operatorname{adj}(A))^{-1} = \operatorname{adj}(A^{-1}) = |A^{-1}| A$

and $\operatorname{adj}(A)$ is nonsingular. $\operatorname{adj}(A) = |A| A^{-1}$.

Proof: $\Rightarrow$ Since A is nonsingular $\Rightarrow |A| \neq 0$

$\operatorname{adj} A = \underbrace{|A|}_{\neq 0} A^{-1}$ $\therefore$ adj of A is nonsingular.

Next:- $(\operatorname{adj} A)^{-1} = (|A| A^{-1})^{-1} = \frac{1}{|A|} (A^{-1})^{-1}$

$$= \frac{1}{|A|} A = |A^{-1}| A$$

$\therefore (\operatorname{adj} A)^{-1} = |A^{-1}| A$ ... (1)

and $\operatorname{adj}(A^{-1}) = |A^{-1}| (A^{-1})^{-1}$

$$= |A^{-1}| A \quad \ldots (2)$$

(1) and (2) give it.

- Let $A_{n\times n}$ "nonsingular", $n>1$ show $adj(adj\ A) = |A|^{n-2} A$

$adj(adj(A))$

$|adj(A)|\,(adj(A))^{-1}$

$\left|\,|A|A^{-1}\right|\,(|A|A^{-1})^{-1}$

$\frac{|A|^n}{|A|}\ \frac{1}{|A|} A = |A|^{n-2} A$.

- show that if $|A|=1$ then $adj(adj\ A) = A$

from the last example :-

$adj(adj\ A) = |A|^{n-2} A$

$= |1|^{n-2} A$

$= A$

- Homework:-

think of $adj(adj(adj\ A))$, $A_{n\times n}$

- if $A_{1\times1}$ matrix then adj (adj A) = A.

T or F:-

If $|adj\, A| = |A|$, then A is 2X2 matrix and it's nonsingular.

$|A|^{n-1} = |A|^{2-1} = |A|^1 \quad \therefore n-1=1 \quad \therefore n=2$

it's F.

Ex:- $A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $|adj\, A| = |A| = 1$ but $A = 3\times3$

$\cdot |A|^{n-1} = |A|^2 = \boxed{1^2}$ not same.

chapter(3):- Vector Spaces.

3.1 Definition and examples.

Df:- A vector space V is a set of elements with the operation of addition and scalar multiplication such that the following satisfy.

1) If $x \in V$ and α is scalar (real or complix) then $\alpha x \in V$ "closed under scalar multiplication".

2) If $x, y \in V$, then $x + y \in V$ "closed under addition".

3) $x + y = y + x$, $\forall x, y \in V$.

4) $(x+y) + z = x + (y+z)$, $\forall x, y, z \in V$.

5) $\exists\ 0 \in V : x + 0 = 0 + x = x \quad \forall x \in V$.

6) $\forall x \in V,\ \exists -x \in V : x + -x = 0$.

7) $\alpha(x+y) = \alpha x + \alpha y$, α scaler, $x, y \in V$.

8) $(\alpha+\beta)x = \alpha x + \beta x$, α, β scaler, $x \in V$.

9) $(\alpha\beta)x = \alpha(\beta x)$, $\forall x \in V$, α, β scaler.

10) $1x = x$, $\forall x \in V$.

Notation $(V, \oplus, \odot)$ addition, multiplication.

- Zero is not a natural number.
- just the real number and matrix is a vectorspace.

Ex:- 1) $(\mathbb{R}, +, \cdot)$ is a vector space.

2) $(\mathbb{R}^2, +, \cdot)$ is a vector space $(a,b)+(c,d)$

$= (a+c, b+d)$

$\alpha(a,b) = (\alpha a, \alpha b)$

Proof:- $\Longrightarrow$ $\boxed{1}$ α scalar, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \in \mathbb{R}^2$

$\boxed{2}$ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \in \mathbb{R}^2$

$\boxed{3}$ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \cancel{\begin{pmatrix} c \\ a \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$

$\boxed{4}$ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} (a+c)+e \\ (b+d)+f \end{pmatrix}$

$= \begin{pmatrix} a+(c+e) \\ b+(d+f) \end{pmatrix}$

$= \begin{pmatrix} a \\ b \end{pmatrix} + \left[\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \right]$

$\boxed{5}$ $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\boxed{6}$ $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Rightarrow \begin{pmatrix} -x \\ -y \end{pmatrix} \in \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} x-x \\ y-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{7}$ $\alpha\left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right] = \alpha\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} z \\ w \end{pmatrix}$

8 + 9 + 10 from the note.

- In general $(\mathbb{R}^n, +, \cdot)$ is a vector space.

Ex:-

- $M_{m\times n} = \mathbb{R}^{m\times n}$ is the set of all $m\times n$ matrices with real enteries under addition and scalar multiplication is a vector space.

- the set of all real value function under + and $\cdot$

  $(f+g)(x) = f(x) + g(x)$

  $(\alpha f)(x) = \alpha f(x)$

  is a vector space.

- $C[a,b] = \{ f : [a,b] \to \mathbb{R}$ ·
  f is continuose on $[a,b]$ under $+, \cdot$ is a vector space.

- $C^n[a,b] = \{ f : [a,b] \to \mathbb{R} :$ $\quad c^0 = C \Rightarrow$ the function.

$f^{(n)}$ is continuous on $[a,b] \}$ $\quad c' \Rightarrow$ first diff.

under $+, \cdot$ is a vector space. $\quad C^{(n)} \Rightarrow$ [illegible]

- P_n = all polynomial of degree less than n, that is

$f(x) = a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$

$a_0, a_1, \dots, a_{n-1} \in \mathbb{R} \}$,

is a vector space under $+, \cdot$

$\boxed{Ex:-}$ $P_3 = \{ f(x) = ax^2 + bx + c, \; a,b,c \in \mathbb{R}.$

$P_2 = \{ f(x) = ax + b, \; a,b \in \mathbb{R}.$

Ex:-

- $Q = \{ \frac{a}{b} : b \neq 0. \; a,b$ integer.

it's not a vector space

, $\alpha = \sqrt{2}$, $x = \frac{1}{3}$

$\alpha x = \frac{\sqrt{2}}{3} \notin Q$

- Q^c irrational like $\sqrt{2}, \sqrt{3}, \pi, e, \dots$

it's not a vector space.

$-\sqrt{2}, \sqrt{2} \in Q^c$

but $-\sqrt{2} + \sqrt{2} = 0 \notin Q^c$

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

is not a vector space.

$\alpha = \frac{1}{2}$, $x = 5 \in \mathbb{Z}$

$\alpha x = \frac{5}{2} \notin \mathbb{Z}$

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

is not a vector space.

- $V = \{f : \text{degree}(f) = 3\}$

is not a vector space.

$1 - x^3$, $1 + x + x^3 \in V$

but $1 - x^3 + 1 + x + x^3 = 2 + x$ it's not $\in V$.

- $S = \{\begin{pmatrix} 1 \\ y \end{pmatrix} : y \in \mathbb{R}\}$

is not a vector space

since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \neq 1$ is $\notin S$.

Thum: Let V be a vector space then :-

I) $0\vec{v} = \vec{0}$, $\forall \vec{v} \in V$.

II) if $\vec{x} + \vec{y} = \vec{0}$ then $y = -\vec{x}$.

III) $-1\vec{v} = -\vec{v}$, $\forall \vec{v} \in V$.

Proof:- I) $0 = 0 + 0$.

$0\vec{v} = (0+0)\vec{v}$

$0\vec{v} = 0\vec{v} + 0\vec{v}$

$-0\vec{v} + 0\vec{v} = -0\vec{v} + 0\vec{v} + 0\vec{v}$.

$\vec{0} = \vec{0} + 0\vec{v}$

$\vec{0} = 0\vec{v}$.

II) $\vec{x} + \vec{y} = 0$.

$\underbrace{-\vec{x} + \vec{x}} + \vec{y} = -\vec{x} + 0$

$\vec{0} + \vec{y} = -\vec{x} + 0$.

$\vec{y} = -\vec{x}$

III) Exercise.

$0 = 1 + -1$, so $(1 + -1)v = 0v = \vec{0}$.

thus, $1v + -1v = 0$ so $v + -1v = \vec{0}$

$\Rightarrow$ $-v + v + -1v = -v + \vec{0} = -v$

$\Rightarrow$ $\vec{0} + -1v = -v$ thus, $-1v = -v$.

3.2 Subspace and spanning sets.

Def: A non empty ($S \neq 0$) subset S of a vector space V is a subspace of V iff :-

nonempty • اثبات انه

1) $x+y \in S$, $\forall x,y \in S$. • اخذ عنصرين مرتين، اثبات $[x+y] \in S$

2) $\alpha x \in S$, $\forall \alpha \in \mathbb{R}$, $x \in S$

Thm: Let S be a Subspace of V, then $\vec{0} \in S$.

if S contains a zero matrix then it's non empty subset

Remark: $0 \notin S \rightarrow S$ is not a sub space .

Ex:-

1) $S = \left\{ \begin{array}{l} (a,b)^T : a+b = 1 \\ a,b \in \mathbb{R} \end{array} \right.$

is S a subspace of $\mathbb{R}^2$?

$(0,0) \notin S$ since $0+0 \neq 1$

$\therefore S$ is not a subspace of $\mathbb{R}^2$.

2) $S = \left\{ \begin{pmatrix} 1 \\ b \end{pmatrix} : b \in \mathbb{R} \right\}$ $V = \mathbb{R}^2$.

S is not a subspace of $\mathbb{R}^2$ since $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$.

3) $S = \{ A_{n\times n} : |A| \neq 0 \}$, $V = \mathbb{R}^{n\times n}$.

S is not a subspace of $\mathbb{R}^{n\times n}$ since

$A = 0_{n\times n} \notin S$ " $|0| = 0$ " .

4) $S = \{ A_{n\times n} : |A| = 0 \}$.

$0_{n\times n} \in S \Rightarrow S \neq \phi$.

but $A = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \in S$ " $|A| = 0$ " .

$B = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \in S$ " $|A| = 0$ "

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$ " $|A+B| \neq 0$ " .

5) $S = \{ (0, y, z) : z, y \in \mathbb{R} \}$ اذا معطى "0"، الثاني والثالث أي شي

$V = \mathbb{R}^3$, show that S is a subspace of $\mathbb{R}^3$.

1) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S \Rightarrow S \neq \phi$

2) Let $\begin{pmatrix} 0 \\ x \\ y \end{pmatrix} , \begin{pmatrix} 0 \\ z \\ w \end{pmatrix} \in S$, then

$\begin{pmatrix} 0 \\ x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ x+z \\ y+w \end{pmatrix} \in S$.

3) let $\alpha \in \mathbb{R}$, $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \in S$ then $\alpha \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$

$= \begin{pmatrix} 0 \\ \alpha y \\ \alpha z \end{pmatrix} \in S$

$\Rightarrow S$ is a subspace of V .

6) $S = \{A_{n\times n} : \underline{\underline{A^T = A}}\}$, $V = \mathbb{R}^{n\times n}$

So it's symmetric.

Show that S is a subspace of V.

1) $0^T_{n\times n} = 0_{n\times n} \Rightarrow 0_{n\times n} \in S$ then S is nonempty $\neq \phi$

2) let $A, B \in S$. then $A^T = A$, $B^T = B$

$(A+B)^T = A^T + B^T$

$= A + B$

$\therefore A+B \in S$.

3) $\alpha \in \mathbb{R}$, $A \in S$ such that $A^T = A$.

$(\alpha A)^T = \alpha(A^T) = \alpha A$.

$\therefore \alpha A \in S$.

$\Rightarrow$ S is a subspace of $(V)(\mathbb{R}^{n\times n})$.

7) $S = \{A_{2\times n} : a_{12} = -a_{21}\}$, $V = \mathbb{R}^{2\times n}$

Show that S is a subspace of $\mathbb{R}^{2\times n}$.

1) $\begin{bmatrix} 0 & +0 \\ -0 & 0 \end{bmatrix} \in S$ $\therefore S$ is nonempty $\neq \phi$.

2) let $A, B \in S$, $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ -y & z \end{bmatrix}$

$A + B = \begin{array}{c c} & \\ -a_{12} & \end{array}\begin{bmatrix} a+x & b+y \\ -b-y & c+z \end{bmatrix} a_{12}$ So it's $\in S$.

3) $\alpha \in \mathbb{R}$, $A = \begin{bmatrix} x & y \\ -y & z \end{bmatrix} \in S$, $\alpha A = \begin{bmatrix} \alpha x & \alpha y \\ -\alpha y & \alpha z \end{bmatrix} \in S$

$\Rightarrow$ it's a subspace.

8) $S = \{ p(x) \in P_4 : p(0) = 0 \}$

Polynomial . من الدرجة الرابعة

Show that S is a subspace of P_4.

1) $O(x) = 0$, $\forall x \Rightarrow 0 \in S$

$\therefore S \neq \emptyset$

2) let $p, q \in S$, $p(0) = 0$, $q(0) = 0$.

$(p+q)(0) = 0 + 0 = 0$

$\therefore p+q \in S$

3) $\alpha \in \mathbb{R}$, $p \in S$ $(p(0) = 0)$

$(\alpha p)(0) = \alpha p(0) = \alpha(0) = 0$.

$\therefore \alpha p \in S$.

9)

10) $S = \{ A_{n\times n} : A \text{ is triangular} \}$ $V = \mathbb{R}^{n\times n}$.

S is not a subspace of $\mathbb{R}^{n\times n}$ since:-

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} \in S.$$

lower upper

but $A+B = \begin{bmatrix} 5 & 5 \\ 2 & 9 \end{bmatrix} \notin S$.

Thm:- let S and T be subspace of a vector space V, that-

I) $S \cap T$ is a subspace.

II) $S \cup T$ is not always a subspace of V

III) $S+T = \{ x+y, x \in S, y \in T \}$.

is a subspace of V.

proof (i):- ① since $0 \in S$ and $0 \in T$

∴ subspace

② let $x, y \in S \cap T$.

$\Rightarrow x, y \in S$ and $x, y \in T$

$\Rightarrow x, y \in S$ and $x+y \in T$

∴ S, T subspace

③ $\alpha \in \mathbb{R}$, $x \in S \cap T$.

$\alpha \in \mathbb{R}$, $x \in S$, $x \in T$

$\Rightarrow \alpha x \in S$ and $\alpha x \in T$ " S, T subspace ".

$(\alpha x \in S \cap T)$

∴ $S \cap T$ is a subspace of V.

(ii) :- $S = \{(x,0) : x \in \mathbb{R}\}$ subspace.

$T = \{(0,y) : y \in \mathbb{R}\}$ subspace.

notice that S and T are subspace of $\mathbb{R}^2$

but $S \cup T = \{(x,y) : x$ or y is zero$\}$.

is not supspace for exampel :-

$(0,1), (1,0) \in S \cup T$, but $(0,1) + (1,0) = (1,1)$

$\notin S \cup T$

The Null space

Df: let A be $m\times n$ matrix the null space of A is

$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

Ex:- if $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$.

Find $N(A)$.

$N(A) = \{x \in \mathbb{R}^4 : Ax = 0\}$.

$-2R_1 + R_2$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$ $-R_2$

$R_1 = R_1 - R_2$ $= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & +1 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{array}\right]$

x_1, x_2 leading, $x_3 = \alpha$, $x_4 = \beta$.

$x_1 - x_3 + x_4 \Rightarrow x_1 = \alpha - \beta$.

$x_2 + 2x_3 - x_4 = 0 \Rightarrow x_2 = \beta - 2\alpha$.

$\therefore N(A) = \{(\alpha-\beta, \beta-2\alpha, \alpha, \beta)^T : \alpha, \beta \in \mathbb{R}\}$.

Thm:- let $A_{m\times n}$. then $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof:- I) $A0 = 0 \Rightarrow 0 \in N(A)$ $A[\,] = 0$

$\Rightarrow N(A) \neq \emptyset$. . the null space.

II) let $x, y \in N(A)$ then $Ax = 0$ and $Ay = 0$

$A(x+y) = Ax + Ay = 0 + 0 = 0$

III) let $\alpha \in \mathbb{R}$, $x \in N(A)$ $(Ax = 0)$

$A(\alpha x) = \alpha A(x) = \alpha 0 = 0$

$\therefore \alpha x \in N(A)$ $\square$ ✓.

linear combinations:-

Def:- Let V be a vector space and $v_1, v_2, \ldots, v_k \in V$.

$c_1, c_2, \ldots, c_k$ skalers.

then $\boxed{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k}$

is called a linear combination of $v_1, \ldots, v_k$.

the set of all linear combinations of $v_1, \ldots, v_k$

is called the span of $v_1, \ldots, v_k$

- denoted by $\text{span}(v_1, \ldots, v_k)$.

Ex:- 1) is $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in$ span $\left[v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$?

let $v = \alpha v_1 + \beta v_2$.

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \beta \end{pmatrix} \Rightarrow \alpha + \beta = 2$$

$\beta = 3$ $\therefore \alpha = -1$.

$\boxed{v = -v_1 + 3v_2}$ $\therefore v \in$ span (v_1, v_2).

2) is $f(x) = x \in$ span $(1, 3x)$?.

let $x = \alpha(1) + \beta(3x)$.

x: $1 = \beta 3$ $\Rightarrow \beta = \frac{1}{3}$.

x^0: $0 = \alpha(1)$ $\Rightarrow \alpha = 0$

$\therefore x = 0.(1) + \frac{1}{3}(3x)$.

$\therefore x \in$ span $(1, 3x)$.

3) Find span $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \therefore x = \alpha$$

$\beta = y$

$0 = z$

$\therefore \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}x\\y\\0\end{pmatrix} : x, y \in \mathbb{R}\right\} = xy\text{-plane}$

4) In $\mathbb{R}$, find $\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

let $\begin{pmatrix}x\\y\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \quad \therefore x = \alpha$

$\beta = y$

$\therefore \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \left\{\begin{pmatrix}x\\y\end{pmatrix} : x, y \in \mathbb{R}\right\}$

$= \mathbb{R}^2$.

so it's $\boxed{\text{~~sperik~~}}$ set. Spanning set

thm:- let V be a vector space and $v_1, v_2, \dots v_k \in V$.

then span $(v_1, v_2, \dots v_k)$

is a subspace of V.

proof:- at page 112 lecture note.

Spanning set.

Def:- Let V be a vector space. A set $v_1, v_2, \dots, v_k \in V$ is called a spanning set iff $\text{span}(v_1, \dots, v_k) = V$.

Ex:- 1) $\left\{ e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$.

is a spanning set for $\mathbb{R}^3$.

in general, A spanning set for $\mathbb{R}^n$ is :-

$e_1 = \begin{pmatrix}1\\0\\0\\0\\ \vdots\\0\end{pmatrix}, \dots, e_n = \begin{pmatrix}0\\0\\0\\ \vdots\\ 1\end{pmatrix}$.

2) $P_n = \{ f(x) = a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \}$ $a_0, \dots, a_{n-1} \in \mathbb{R}$.

$= \text{span}\{1, x, x^2, x^3, \dots, x^{n-1}\}$.

$\therefore \{1, x, \dots, x^{n-1}\}$ is spanning set for P_n.

ex:- $\{1, x, x^2\}$ is a spaning set for P_3.

ex:- $\{1, x\}$ " " " " " P_2.

3) $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $v_3 \begin{pmatrix}0\\0\\1\end{pmatrix}$.

is $\{v_1, v_2, v_3\}$ a spaning set for $\mathbb{R}^3$?

Let $\begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha \begin{pmatrix}1\\2\\3\end{pmatrix} + \beta \begin{pmatrix}1\\2\\2\end{pmatrix} + \gamma \begin{pmatrix}0\\0\\1\end{pmatrix}$.

$\alpha + \beta = x$

$2\alpha + 2\beta = y$

$3\alpha + 2\beta + \gamma = z$

$$\begin{array}{l} -2R_1 + R_2 \\ -3R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 2 & 2 & 0 & y \\ 3 & 2 & 1 & z \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ \boxed{0 & 0 & 0} & -2x+y \\ 0 & -1 & 1 & -3x+z \end{array}\right]$$

is not always consistent.

$\therefore \{v_1, v_2, v_3\}$ is not a spaning set for $\mathbb{R}^3$.

Q) Is $\left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$ a spaning set for $\mathbb{R}^2$?

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = \alpha + \beta + \gamma$

$y = 2\alpha + \gamma$

$$-2R_1 + R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 2 & 0 & 1 & y \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -1 & -2x+y \end{array}\right] -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & \frac{1}{2} & \frac{2x-y}{2} \end{array}\right]$$ is always consistent

$\therefore \{v_1, v_2, v_3\}$ is a spaning set for $\mathbb{R}^2$.

5) is $\{x, 1, 2x-1\}$ a spaning set for P_3 ?

let $ax^2+bx+c = \alpha_1 x + \alpha_2 1 + \alpha_3(2x-1)$.

$ax^2+bx+c = 0(x^2) + (\alpha_1+2\alpha_3)x + (\alpha_2-\alpha_3)$

$x^2: \quad 0 = a$

$x: \quad \alpha_1+2\alpha_3 = b \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a \\ 1 & 0 & 2 & b \\ 0 & 1 & -1 & c \end{array}\right]$

$x^0: \quad \alpha_2-\alpha_3 = c$

is not always consitant.

$\Rightarrow$ not spaning set.

if it's P_2 it's become always consistent.

Linear system Revisited.

Thm: $A_{m\times n}$ and $Ax=b$ is consistent with X_0 a solution then y is a solution of $Ax=b$ iff $y = X_0 + Z$, $y - x_0 = Z \in N(A)$.

$Z \in N(A)$. $AZ = 0$.

Proof:- • Given $AX_0 = b$ and $Ay = b$.

then: $Ay - AX_0 = b - b = 0$

$\therefore y - x_0 = 0$

$y - x_0 = Z \in N(A)$

$y = X_0 + Z$, $Z \in N(A)$

• Given $y = X_0 + Z$, $Z \in N(A)$, $AX_0 = b$.

$Ay = A(X_0 + Z)$

$Ay = AX_0 + AZ$

$= b + 0$

$= b$.

$\therefore Ay = b \implies y$ is a solution of $Ax = b$.

3.3 Linear Independence.

Def: Let V be a vector space. A set $v_1, v_2, \dots, v_k \in V$ is called <u>linearly independent</u> if $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$

$$\Rightarrow c_1 = c_2 = \dots = c_k = 0.$$

If there exist scalers $c_1, c_2, \dots, c_k$ not all zero such that $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$, the set is linearly dependent.

Ex: 1) is $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ lin. indep?

let $c_1 \begin{pmatrix}1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$c_1 + c_2 = 0$

$-$

$c_1 + 2c_2 = 0$ $\Rightarrow$ $-c_2 = 0$ $\therefore c_2 = 0$

$\therefore c_1 = 0$.

$\therefore \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ lin. indep.

2) $v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 \begin{pmatrix}0\\1\\0\end{pmatrix}$ is $\{v_1, v_2\}$ lin. indep ?

مستحيل أن يصبح ١

$c_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$c_1 = 0$, $c_2 = 0$, $c_1 = 0$ $\therefore c_1 = c_2 = 0$.

$\therefore$ set is line. indep.

3) $v_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\end{pmatrix}$

is $\{v_1, v_2, v_3\}$ lin. indep ?

$c_1 + c_2 + c_3 = 0$

$2c_1 + 0 + c_3 = 0$ under + homog.

this system has infinite sol since it's an under determined homog. system.

So this set is linearly dep.

4) is $\{\overset{f}{x}, \overset{g}{1}, \overset{h}{2x-1}\}$ lin. indep? P_2

$c_1 x + c_2 + c_3(2x-1) = 0$.

- x: terms :-

$c_1 + 2c_3 = 0$. this system has infinite. so under + homog

- x^0: terms :- $\Rightarrow \{x, 1, 2x-1\}$ lin. dep.

$c_2 - c_3 = 0$

notice $h = 2f - g$.

thmc(1):- A set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ are lin. indep iff

$A = [v_1, v_2, \ldots, v_n]_{n \times n}$ is nonsingular ($|A| \neq 0$).

ex:- 1) $\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right]$ is lin dep since.

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix} \begin{matrix} - \\ + \\ - \end{matrix}$$

$$-1 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0.$$

2) $\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$ $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$

so it's lin. indep in $\mathbb{R}^2$.

3) $[2x^2 + x + 8,\ 1x^2 + 8x + 7,\ 1x^2 - 2x + 3]$ lin dep or indep?

$c_1(2x^2 + x + 8) + c_2(x^2 + 8x + 7) + c_3(x^2 - 2x + 3) = 0$

x^2 : $2c_1 + c_2 + c_3 = 0$

x^1 : $c_1 + 8c_2 - 2c_3 = 0$

x^0 : $8c_1 + 7c_2 + 3c_3 = 0$.

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 8 & -2 \\ 8 & 7 & 3 \end{vmatrix} \begin{matrix} x^2 \\ x \\ x^0 \end{matrix} = 0 \quad \therefore \text{Given}$$

$\therefore$ it's has infinite sol

$\Rightarrow$ Lin dep.

theorem:- A set of vectors $v_1, v_2, \dots, v_k$ in V is lin. dep iff one of them is a linear combination of the remaining vector.

ex:-

1) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ with labels v_1, v_2, v_3, is lin. dep since $v_3 = v_1 + v_2$.

2) $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\}$ with labels v_1, v_2, v_3, is lin. dep since $v_3 = v_1 + v_2$.

$v_3 = 3v_1$.

$v_2 = 2v_1$.

3) $\{\underset{f}{1}, \underset{g}{x}, \underset{h}{2-5x}\}$ is lin. dep $h = 2f - 5g$.

4) $\{\underset{f}{e^x}, \underset{g}{e^{-x}}, \underset{h}{\cosh x}\}$ is lin dep $h = \frac{1}{2}f + \frac{1}{2}g$.

5) $\{\underset{f}{1}, \underset{g}{\cos^2 x}, \underset{h}{\sin^2 x}\}$ " " " $f = g + h$.

Thm(3):- $v_1, v_2, \ldots, v_k \in V$ are lin. indep iff every $v \in span(v_1, \ldots, v_k)$ is uniquely written as a linear combination of $v_1, \ldots, v_k$.

ex:- 1) $\left\{ \underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}2\\0\end{pmatrix}} \right\}$, $v = \begin{pmatrix}5\\0\end{pmatrix}$

(dep)

$v \in span(v_1, v_2)$ since

$v = \begin{pmatrix}5\\0\end{pmatrix} = 5v_1 + 0v_2$

$= 2v_2 + v_1$

$= 3v_1 + v_2$

is not uniquely written.

2) $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right\}$ $v = \begin{pmatrix}5\\0\end{pmatrix}$

linear indep "الحالة $\neq 0$"

$v = \begin{pmatrix}5\\0\end{pmatrix} = \overset{c_1}{5}v_1 + \overset{c_2}{0}v_2$.

the vector space $C^{n-1}([a,b])$.

<u>Df:-</u> Let $f_1, f_2, \dots, f_n \in C^{n-1}([a,b])$, then:

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ \vdots & & \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

$W(f_1, \dots, f_n)$ is called the wronskian of $f_1, f_2, \dots, f_n$.

Ex:- 1) $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

2) $W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$

3) $W(1,x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$

4) $W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix}$, $x>0$.

$= x + x\ln x - x\ln x = x$.

5) $W(1,x,x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$ indep.

6) $W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = \underline{\underline{0}}$

$\cdot\ x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$

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$= \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases} = 2|x|$

Theorem(4):- Let $f_1, \dots, f_n \in C^{n-1}[a,b]$.

if $\exists$ $x_0 \in [a,b]$ such that $W(f_1, \dots, f_n) \neq 0$ then $f_1, \dots, f_n$ are Lin indep.

Rmks:- 1) If $f_1, \dots, f_n$ are Lin. dep then $W(f_1, \dots, f_n)(x) = 0$. $\forall x \in [a,b]$.

2) if $W(f_1, \dots, f_n)(x) = 0$ $\forall x \in [a,b]$, we cannot tell anything about the indep or dep.

In this case we use the def:-

ex:- $\cdot W(x^2, x|x|)$ $[-1,1]$

$W(x^2, x|x|) = 0$ $\forall x \in [-1,1]$. thm(4) fails.

we use the def:- $c_1x^2 + c_2 x|x| = 0$, $[-1,1]$

$x = +1 \Rightarrow c_1 + c_2 = 0$

$x = -1 \Rightarrow c_1 - c_2 = 0$

$\therefore c_1 = c_2 = 0$

by def $\Rightarrow$ Lin indep.

• $W(x^2, \underset{-x^2}{\underline{x|x|}}) = 0$. $x[-1, 0]$ . ...

∴ they are lin dep since $-x^2 = (-1)x^2$

• $\{1, x, x^2\}$ lin. indep on $(-\alpha, \alpha)$.

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = \underline{\underline{2 \neq 0}}, \forall x.$$

• $\{e^x, e^{-x}\}$ lin indep on $(-\alpha, \alpha)$

$$\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^{-x} e^x = -2e^{(-x+x)} = -2 \neq 0 . \underline{\underline{\forall x}}$$

$\therefore W(e^x, e^{-x})(2033) = -2 \neq 0$

$\Rightarrow$ lin. Indep.

• $\{x, x\ln x\}$ $(0, \alpha)$

$W(x, x\ln x) = x$, $x \in (0, \alpha)$.

$W(x, \ln x)(2) = 2 \neq 0$

∴ $[x, x\ln x]$ lin. indep in $(0, \alpha)$

3.4 **Basis and Dimension.**

Def:- A set $v_1, v_2, \ldots, v_n$ form a basis for V iff:-

(1) $v_1, \ldots, v_n$ span V.

i.e. $\{v_1, v_2, \ldots, v_n\}$ spanning set for V.

(2) $v_1, v_2, \ldots, v_n$ are linearly independent.

Ex:- $\{e_1, e_2, \ldots, e_n\}$ for a basis for $\mathbb{R}^n$ "standard basis".

- $\left\{e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$ since:

(i) let: $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$: $\begin{pmatrix}x\\y\\z\end{pmatrix} = xe_1 + ye_2 + ze_3$

spanning set.

(ii) $c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$c_1 = c_2 = c_3 = 0 \Rightarrow$ lin indep.

- $\{1, x^2, \ldots x^{n-1}\}$ is a standard basis for P_n

$\Rightarrow \{1, x, x^2\}$ standard basis for P_3.

$\Rightarrow \{1, x\}$ " " " P_2.

- $E_{ij} = (e_{ij})$ where $e_{ij} = 1$ and 0 otherwise is standard basis for $\mathbb{R}^{m \times n}$. For example:-

$\Rightarrow$ the standard basis for $\mathbb{R}^{2\times 3}$

$$\underline{\underline{E_{11}}} \qquad \underline{\underline{E_{12}}} \qquad \underline{\underline{E_{13}}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so on. will be 6 of them

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21} \qquad E_{22} \qquad E_{23}.$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = aE_{11} + bE_{12} + cE_{13} + dE_{21} + eE_{22} + fE_{23}.$$

$$\Rightarrow c_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} +$$

$$c_5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_6\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}.$$

$\therefore c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0 \quad \therefore$ lin. indep.

for $M_{2\times2}$ $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

Ex:- • $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ show that $\{v_1, v_2, v_3\}$ form a basis for $\mathbb{R}^3$.

• Spanning set :-

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{cases} c_1 + c_2 + c_3 = x \\ 2c_1 + c_2 = y \\ 3c_1 = z \end{cases}$$

$$\begin{array}{l} -2R_1 + R_2 \\ -3R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 2 & 1 & 0 & y \\ 3 & 0 & 0 & z \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -1 & -2 & y-2x \\ 0 & -3 & -3 & -3x+z \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & 2x-y \\ 0 & -3 & -3 & z-3x \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & 2x-y \\ 0 & 0 & 3 & 6x-3y+z-3x \end{array}\right]$$

3 بيفوتس

is always consistent.

• Linear indep :-

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} = +3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$$

lin. indep.

$\therefore \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

- $\{1+x, x\}$ is a basis for P_2.
- Spanning set:

$c_1(1+x) + c_2(x) = ax + b.$

$c_1 + c_1 x + c_2 x = ax + b.$

$c_1 + c_2 = a$

$c_1 = b$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & a-b \end{array}\right]$$

ازا كان ال سيستم له حل دائما يكون هنا شغلة بار دوس.

so it's always consistent $\Rightarrow$ spanning set.

- linearly indep:

$c_1 + c_1 x + c_2 x = 0x + 0.$

$c_1 + c_2 = 0$

$c_1 = 0$ $\quad\quad \therefore c_1 = c_2 = 0$

$\therefore$ it's linearly indep.

$\therefore \{1+x, x\}$ is a basis for P_2.

df: Let V be non zero vector space $\{v_1, \dots, v_n\}$ is a basis for V, then V called <u>finite dimention</u>,

dimention $= n$ $(\dim V = n)$

if $V = \{0\}$ zero vector space.

$\dim(\vec{0}) = 0$ with basis $\emptyset$

otherwise, V is infinite dimentional $(\dim V = +\infty)$

اي set تحتوي صفر zero vector تكون dep.

ex:-
- $\dim \mathbb{R}^n = n$ (finite)
- $\dim P^n = n$ (finite)
- $\dim \mathbb{R}^{m \times n} = m \times n$ (finite) ضرب
- $\dim \{\vec{0}\} = 0$ (finite)
- $\dim \mathbb{R}^1 = 1$ (finite) يمكن تمثيلها $x, 1$
- $\dim C^n[a,b] = +\infty$ (infinite).

Examples:- ① Find a basis and "bases / جمع". dimension for $N(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$$-2R_1 + R_2 \leftarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

x_1, x_2 leading.
x_3 free. "$x_3 = \alpha$"
- $x_2 + 2x_3 \Rightarrow x_2 = -2\alpha$.
- $x_1 + \alpha - 2\alpha = 0$

$x_1 = \alpha$.

$N(A) = \{(\alpha, -2\alpha, \alpha)^T : \alpha \in \mathbb{R}\}$

$\{\alpha(1,-2,1)^T\} = \operatorname{Span}\{(1, -2, 1)^T\}$ and dim $= 1$

$\boxed{2}$ $S = \left\{ \begin{pmatrix} a - b + c \\ 2b - 3c \\ 4a + 2c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

$x \in S$, $x = a\underbrace{\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}}_{v_1} + b\underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{v_2} + c\underbrace{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}_{v_3}$.

$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\}$

$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 4 & 0 & 2 \end{vmatrix} = 8 \neq 0$.

so it's indep.

$\Rightarrow \{v_1, v_2, v_3\}$ is a basis for S and $\dim S = 3$.

3) $S = \left\{ \begin{pmatrix} a + 3b + c \\ 2a + 6b \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

$x \in S$

$x = a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}3\\6\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\1\end{pmatrix}$

$= \text{Span}\left\{ \underbrace{\begin{pmatrix}1\\2\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}3\\6\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{v_3} \right\}$

$v_2 = 3v_1$ so it's lin. dep.

$\rightarrow \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ it's lin. indep.

$\therefore \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ is a basis for (S) and the dim = 2.

[4] $S = \{ p(x) \in P_3 : p(0) = 0 \text{ and } p'(1) = 0 \}$

$p(x) = ax^2 + bx + c$

$p(0) = 0$, $\therefore c = 0$

$p'(x) = 2ax + b$

$p'(1) = 2a + b = 0$

$\therefore b = -2a$

$\therefore S = \{ p(x) = ax^2 - 2ax + 0 \}$

$= \{ p(x) = a(x^2 - 2x) \}$

$\therefore$ Span $\{ x^2 - 2x \}$ $\therefore$ dim $= 1$

Linearly indep

$\therefore \{x^2 - 2x\}$ is a basis for S dim $S = 1$.

[5] Find a basis and dimension of $S = \{ p(x) \in P_3 : p''(x) = 0 \}$

$p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b$

$p''(x) = 2a = 0$ $\therefore a = 0$

$S = \{ p(x) = bx + c(1) \}$

Span $= \{ x, 1 \}$

- $\{x, 1\}$ is lin indep. set since

iff $\alpha x + \beta . 1 = 0 \Rightarrow \alpha = 0, \beta = 0$.

$\{x, 1\}$ is a basis for S and dim $S = 2$.

= de second one is $\{1, x\}$.

thm(1):- let $\{v_1, v_2, \ldots, v_n\}$ be a spanning set for V. If $w_1, w_2, \ldots, w_k \in V$, $k>n$, then $w_1, w_2, \ldots, w_k$ are linearly dependent.

ex:-

$\{\binom{2}{1}, \binom{4}{3}\}$ is a spanning set for $\mathbb{R}^2$.

then $\{\binom{2}{1}, \binom{4}{7}, \binom{-7}{3}\}$ are linearly dep by thm(1).

thm(2):- let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_k\}$ be two bases for a vector space V, then $k=n$.

thm(3):- let V be a vector space with dim $V = n > 0$ then the following are equivalent.

i) $\{v_1, \ldots, v_n\}$ is a basis.

ii) $\{v_1, \ldots, v_n\}$ span V.

iii) $\{v_1, \ldots, v_n\}$ linearly indep.

ex:-

$S = \{\binom{1}{2}, \binom{4}{3}\}$ is a basis for $\mathbb{R}^2$ since $\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 3-5 \neq 0$
"lin. indep".

thm (3) $\Rightarrow$ S is a basis for $\mathbb{R}^2$.

Rmk: (summary) let V be a vector space with $\dim V = n > 0$ then

1) A set $v_1, \dots, v_k$, $k > n$ lin. dep.

2) A set $v_1, \dots, v_k$, $k < n$ can not span V.

3) if $k = n$ and $v_1, \dots, v_k$ are lin. indep or span V, then $\{v_1, v_2, \dots, v_k\}$ is a basis for V.

4) A spanning set of $v_1, v_2, \dots, v_k$, $k > 0$ can be reduced (pared down) to a basis for V.

5) A lin. inde set $v_1, \dots, v_k$ $k < n$ can be extended to a basis for V.

Ex:- 1) $x_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $x_2 = \begin{pmatrix}2\\5\\4\end{pmatrix}$, $x_3 = \begin{pmatrix}1\\3\\2\end{pmatrix}$, $x_4 = \begin{pmatrix}2\\7\\4\end{pmatrix}$, $x_5 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

Span for $\mathbb{R}^3$.

$$\begin{array}{ccc} x_1 & x_2 & x_5 \end{array}$$
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -2 \neq 0$$

$\Rightarrow \{x_1, x_2, x_5\}$ is a basis for $\mathbb{R}^3$

$\Rightarrow \{x_2, x_3, x_4\}$ is not a basis for $\mathbb{R}^3$ $(|\ | = 0)$

2) $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix} \right\}$ Lin. indep in $\mathbb{R}^3$.

we can extend it to a basis for $\mathbb{R}^3$.

$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\end{pmatrix} \right\}$

Since $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$. "Lin. indep".

3.5 change of basis.

Def:- V: vector space, $E = \{v_1, \ldots, v_n\}$ basis for V then $v \in V$:-

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots \alpha_n v_n$, $\alpha_1, \alpha_2 \ldots \alpha_n \Rightarrow$ scalers

the vector $(\alpha_1, \alpha_2, \ldots \alpha_n)^T \in \mathbb{R}^n$ is called the coordinate of v with respect to a basis E denoted by $[v]_E$ or v_E

$$= \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Ex:- 1) $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$, $E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ basis, Find $[v]_E$.

$v = \alpha_1 v_1 + \alpha_2 v_2$

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\alpha_1 = 2, \quad \alpha_2 = 5$

$\therefore \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix} \right]_E = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

in general, $[v]_E = v$ if E is standard basis, $v \in \mathbb{R}^n$

2) $f(x) = x^2 + 2 \in P_3$, $E = \{1, x, x^2\}$ basis for P_3 Find $[f(x)]_E$.

$x^2 + 2 = \alpha_1 + \alpha_2 x + \alpha_3 x^2$

$\alpha_1 = 2, \quad \alpha_2 = 0, \quad 1\alpha_3 =$

$\therefore [x^2 + 2] = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = (2, 0, 1)^T$. (X ا, ا ن ا ل ت ر ت ي ب)

3) $E = \left[\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right]$ basis for $\mathbb{R}^2$, $v = \begin{pmatrix}7\\4\end{pmatrix}$ Find $[v]_E$.

$$\begin{pmatrix}7\\4\end{pmatrix} = \alpha \begin{pmatrix}3\\2\end{pmatrix} + \beta \begin{pmatrix}1\\1\end{pmatrix}$$

$3\alpha + \beta = 7$ $\qquad \alpha = 3,\ \beta = -2$.

$-$

$2\alpha + \beta = 4$

$$\left[\begin{pmatrix}7\\4\end{pmatrix}\right]_E = \begin{pmatrix}\alpha_1\\\alpha_1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$$

def: Transition Matrix S.

$E = \{u_1, u_2\}$ basis $\quad U_2$, $[e_1, e_2]$ standard basis , $U_1 = [u_1, u_2]$

$S = U_2^{-1} U_1$ $\qquad U_2^{-1}$

$F = [v_1, v_2]$ basis.

def: V, vector space, $\dim V = n > 0$

$E = [v_1, \dots, v_n]$

$F = [w_1, \dots, w_n]$ two basis. then the transition matrix from the basis E into the basis F is the $n \times n$ nonsingular matrix

$$S_{E \to F} = ([v_1]_F, [v_2]_F, \dots, [v_n]_F)$$

Ex:- $E = \left[\binom{5}{2}, \binom{7}{4}\right]$

$F = \left[\binom{3}{2}, \binom{1}{1}\right]$

two basis for $\mathbb{R}^2$.

1) Find te transition matrix from E to $[e_1, e_2]$.

2) From $[e_1, e_2]$ to F.

3) From E to F inverse $F \times E$.

4) Find $\left[\binom{1}{-1}\right]_F$ "two ways".

1. $S_E \to [e_1, e_2]$ • basis ⇒ stand الحركة

$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix} = U_1$ • stand ⇒ basis inv.

2. $S_{[e_1, e_2] \to F}$

$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3-2}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = U_2$.

3. $S_{E \to F} = \underset{\to (F)^{-1}}{U_2^{-1}} \; \underset{\to E}{U_1}$

$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & 3 \\ -4 & -2 \end{bmatrix} = S$

(4) $\left[\begin{pmatrix}1\\-1\end{pmatrix}\right]_F = S_{E\to F}\left[\begin{pmatrix}1\\-1\end{pmatrix}\right]_E \Rightarrow \begin{pmatrix}1\\-1\end{pmatrix} = \alpha_1\begin{pmatrix}5\\2\end{pmatrix} + \alpha_2\begin{pmatrix}7\\4\end{pmatrix}$

$= \begin{bmatrix}3 & 3\\-4 & -2\end{bmatrix}\begin{bmatrix}\frac{11}{6}\\ \frac{-7}{6}\end{bmatrix}$ $\quad \alpha_1 = \frac{11}{6}, \ \alpha_2 = -\frac{7}{6}$.

$= \begin{bmatrix}2\\-5\end{bmatrix}$

Ex:- $E = \{3x+6, 9\}$, $F = [2x+1, x-4]$ basis for P_2.

a) $S_{E\to[1,x]}$, $U_1 = \begin{bmatrix}6 & 9\\3 & 0\end{bmatrix}$

b) $S_{[1,x]\to F}$, $U_2 = \begin{bmatrix}1 & -4\\2 & 1\end{bmatrix}$

c) $S_{E\to F}$

d) $[3x+15]_F$

$U_2^{-1}U_1 = \begin{bmatrix}1 & -4\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}6 & 9\\3 & 0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 4\\-2 & 1\end{bmatrix}\begin{pmatrix}6 & 9\\3 & 0\end{pmatrix}$

$= \begin{bmatrix}2 & 1\\-1 & -2\end{bmatrix}$

$S_{E\to F}[3x+15]_E$

$\begin{bmatrix}2 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix}$

3.6 Row space and column space.

Df:- $A_{m\times n}$ then:-

1) the row space of A is $R(A) = \text{Span}(\vec{a}_1, \dots \vec{a}_m)$

2) the column space of A is $C(A) = \text{Span}(a_1, \dots a_n)$

3) the NULL space of A is $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.

4) nullity of A is nullity of A = dim N(A).

5) the rank of A is rank(A) = dim R(A) = dim C(A).

them(1):- A, B are equivalent then R(A) = R(B).

them(2):- $A_{m\times n}$, dim R(A) = dim C(A).

them(3):- rank nullity theorem.

$A_{m\times n}$ rank(A) + nullity(A) = n. - A أعمدة المصفوفة

Remark:- to find rank(A) we do:-

I) U is the REF or RREF of A.

II) the non zero row of U.

form a basis for R(A).

III) the colums of A that correspond to the leading 1's in U is a basis for C(A).

EX:- Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}_{3\times 4}$.

Find:-

a) a basis for $R(A)$, $C(A)$ and $N(A)$.

b) rank(A) and nullity of A.

c) the dependancy relation.

$$\begin{array}{c} \\ -2R_1+R_2 \\ -R_1+R_3 \end{array} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \begin{array}{c} R_1+R_2 \\ =-R_2 \\ +2R_2+R_3 \end{array} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & +2 & +4 \end{bmatrix}$$

$$= \begin{array}{cccc} u_1 & u_1 & u_2 & u_4 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

a) $\{(1,2,0,3),(0,0,1,2)\}$, a basis for row

a_1 and a_3 , a basis for column.

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \right\}$.

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ x_1, x_3 leading. $\underline{\underline{x_2}} = \alpha$ $\underline{\underline{x_4}} = \beta$. free

$x_3 = -2\beta$, $x_1 = -2\alpha - 3\beta$.

$N(A) = \left\{ \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$

$$= \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \cdot \alpha, \beta \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}}_{\text{indep.}} \right\}.$$

$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ is a basis for $N(A)$

b) $\text{rank}(A) = \dim R(A) = \dim C(A) = 2.$

$\text{nullity}(A) = \dim N(A) = 2.$

c) dependency relation.

$$U = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In U, $u_2 = 2u_1$. ∘ A ياخذ نفس الشي

$u_4 = 3u_1 + 2u_3.$

Sol :- In A. $a_2 = 2a_1$

$a_4 = 3a_1 + 2a_3.$

Recall:- $Ax = b$ is consistent iff b is a linear combination of the columns of A.

$\Rightarrow$ that is, $Ax = b$ is consistent iff $b \in C(A)$.

Thm:- $A_{m \times n}$, $b \in \mathbb{R}^m$ then:-

1) $Ax = b$ is consistence $\forall b \in \mathbb{R}^m$ iff $C(A)$ span $\mathbb{R}^m$.

2) $Ax = b$ has at most one solution $\forall b \in \mathbb{R}^n$ iff $C(A)$ are lin. indep.

Cor:- $A_{n \times n}$ is nonsingular iff $C(A)$ from a basis for $\mathbb{R}^n$.

note • $Ax = b$ $A_{n \times n}$ $\Leftrightarrow$ A nonsinguar. a has uniq. sol.

$\Leftrightarrow$ $|A| \neq 0$.

$\Leftrightarrow$ $C(A)$ basis for A.

Chapter (4): Linear Transformation.

4.1 Definition and Examples:

Def: A mapping L from a vector space V into a vector space W is said to be linear transformation iff

(1) $L(v_1+v_2) = L(v_1)+L(v_2)$, $\forall\, v_1, v_2 \in V$.

$L(\alpha v) = \alpha(Lv)$, $\forall\, v \in V$, $\alpha \in \mathbb{R}$.

Notation: $L: V \to W$

If $V=W$, then $L: V \to V$ is said to be linear operator.

Ex: $L: \mathbb{R}^2 \to \mathbb{R}^2$, $L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x\\3y\end{pmatrix}$

Show that L is a linear operator. see as transformation.

(1) let $\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix} \in \mathbb{R}^2$, then

$$L\left(\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) = L\left(\begin{pmatrix}a+c\\b+d\end{pmatrix}\right) = \begin{pmatrix}3(a+c)\\3(b+d)\end{pmatrix}$$

$$= L\begin{pmatrix}3a\\3b\end{pmatrix} + L\begin{pmatrix}3c\\3d\end{pmatrix}$$

$$= L\begin{pmatrix}a\\b\end{pmatrix} + L\begin{pmatrix}c\\d\end{pmatrix}$$

(2) let $L\begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.

$$L\left(\alpha\begin{pmatrix}a\\b\end{pmatrix}\right) = L\begin{pmatrix}\alpha A\\\alpha b\end{pmatrix} = \begin{pmatrix}3\alpha A\\3\alpha b\end{pmatrix} = \alpha\begin{pmatrix}3A\\3b\end{pmatrix}$$

$$= \alpha L\begin{pmatrix}A\\b\end{pmatrix}.$$

i.e. lin. transformation.

Ex:- $L: C[a,b] \to \mathbb{R}$.

$L(f(x)) = \int_a^b f(x)\,dx$ Show that L is a lin. tran.

(1) $L(f(x)+g(x)) = \int_a^b (f(x) + g(x))\,dx$

$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$= L(f(x)) + L(g(x))$

(2) $L(\alpha f(x)) = \int_a^b \alpha f(x)\,dx$

$= \alpha \int_a^b f(x)\,dx$

$= \alpha L(f(x))$

$\therefore$ lin. transformation.

Ex:- $L: C'[a,b] \to C[a,b]$.

$L(f(x)) = f'(x)$.

Show that L is a lin. trans.

(1) $L(f(x)+g(x)) = (f(x)+g(x))'$

$= f'(x) + g'(x)$

$= L(f(x)) + L(g(x))$

(2) $L(\alpha f(x)) = (\alpha f(x))'$

$= \alpha f'(x) = \alpha L(f(x))$

$\therefore L$ is a linear transformation.

Ex:- $L: \mathbb{R}^2 \to \mathbb{R}$

$L\begin{pmatrix} x \\ y \end{pmatrix} = x+y$ show that L is a lin. transformation.

(1) $L\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = L\begin{pmatrix} a+c \\ b+d \end{pmatrix} = (a+c)+(b+d).$

$= (a+b)+(c+d)$

$= L\begin{pmatrix} a \\ b \end{pmatrix} + L\begin{pmatrix} c \\ d \end{pmatrix}.$

(2) $L\left(\alpha\begin{pmatrix} a \\ b \end{pmatrix}\right) = L\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \alpha a + \alpha b$

$= \alpha(a+b)$

$= \alpha L\begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore L$ is a linear transformation.

Ex:- $L: \mathbb{R}^2 \to \mathbb{R}^3$

$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is L a lin. trans?

(1) $L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\neq$

not the same so it's not a lin. trans.

$\to L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \neq L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

* $L\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow L(0_{\mathbb{R}^2}) \neq 0_{\mathbb{R}^3}$

Ex:- $L = P_2 \to P_3$

$L(p(x)) = p(x) + x^2$. Show that L is not a lin. tran.

$p(x) = x+1$, $q(x) = 1-x$.

$L(p(x)+q(x)) = L(x+1+1-x) = L(2) = 2+x^2$

$L(p(x)) + L(q(x)) = 1+x+x^2 + 1-x+x^2 = 2+2x^2$

$\neq$

$\therefore L(p(x)+q(x)) \neq L(p(x)) + L(q(x))$

$\Rightarrow$ L is not a lin. trans

Thm:- $L: V \to W$ lin trans.

a) $L(0_V) = 0_W$

b) $L(v_1 - v_2) = L(v_1) - L(v_2)$.

c) $L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$

$= \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n)$

$\forall\, v_1, \ldots, v_n \in V$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Rmk:- if $L(0_V) \neq 0_W$, then L is not lin. trans.

Kernal and Images

Def:- $L: V \to W$ lin. transformation.

a) the kernal of L is:-

$\ker(L) = \{v \in V : L(v) = 0_W\}$. sub space of V

b) the image (range) of L is:- // // of W.

$I_m(L)$ or $L(V)$ or R_L is:

$L(V) = \{w \in W : w = L(v)$ for some $v \in V\}$.

c) If $L(V) = W$, then L is said to be onto.

d) If $\ker(L) = \{0_V\}$, then L is said to be one to one.

Ex: $L: P_3 \to \mathbb{R}^2$ lin. trans.

$$L(p(x)) = \begin{pmatrix} p''(x) - p'(1) \\ p(0) \end{pmatrix}$$

a) Find $\ker(L)$ and its dimension.

b) Find R_L " " "

c) Is L onto or one to one?

$p(x) = ax^2 + bx + c$, $p(0) = c$

$p'(x) = 2ax + b$, $p'(1) = 2a + b$

$p''(x) = 2a$

$\therefore L(ax^2 + bx + c) = \begin{pmatrix} 2a - 2a - b \\ c \end{pmatrix} = \boxed{\begin{pmatrix} -b \\ c \end{pmatrix}}$. الة صيغة!

$L(x^2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$L(x+1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

a) $\ker(L) = ??$

$L(ax^2+bx+c) = \begin{pmatrix}-b\\c\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\boxed{\therefore b = c = 0}$

$\therefore \ker(L) = \{ax^2 : a \in \mathbb{R}\}$

$= \text{Span}(x^2)$, $\{x^2\}$ lin. indep.

$\therefore \{x^2\}$ is a basis for $\ker(L)$ and $\dim(\ker(L)) = 1$.

Since $\ker(L) \neq \{0\}$, then L is not 1-1.

b) $R_L = \text{Im}(L)$

$L(ax^2+bx+c) = \begin{pmatrix}-b\\c\end{pmatrix}$

$= b\begin{pmatrix}-1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix}$

$= \text{Span}\left(\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$ lin. indep.

$\therefore \left\{\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis for $\text{Im}(L)$.

$\dim(\text{Im}(L)) = 2 = \dim \mathbb{R}^2$

$\therefore L$ is ONTO.

Rmk:- $\dim \ker(L) + \dim R_L$

$1 + 2 = 3$ $\dim P_3$

- in general $L: V \to W$, $\dim V < \infty$,

$\dim \ker(L) + \dim R_L = \dim V$.

Ex:

$L: \mathbb{R}^4 \to \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix}$$ lin. transformation.

Find: a) Ker (L) and Image of L

b) is L 1-1 or onto.

$\boxed{Ker(L) \; = ?}$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

$x_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_4 = 0$ $\Leftarrow$ x_1, x_4 leading.

$x_1 = -\alpha - \beta$ x_2, x_3 free.

$\to x_2 = \alpha, \quad x_3 = \beta$

$$Ker(L) = \left\{ \begin{pmatrix} -\alpha-\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$ lin. indep.

$\Rightarrow \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\ker(L)$ and $\dim = 2 \neq 0$

$\therefore$ not 1-1.

$\boxed{L_2 = ??}$

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ lin. indep.}$$

$\therefore \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\text{Im}(L)$ and dim $=$ dim $\mathbb{R}^2 = 2$

$\therefore L$ is onto.

Q(4):- $L: \mathbb{R}^2 \to \mathbb{R}^2$ lin. operator.

If $L\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$, $L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$

find $L\begin{pmatrix}7\\5\end{pmatrix}$.

$\begin{pmatrix}7\\5\end{pmatrix} = \alpha\begin{pmatrix}1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\-1\end{pmatrix}$.

$7 = \alpha + \beta$

$5 = 2\alpha - \beta$

$\therefore \alpha = 4, \ \beta = 3$.

$\Rightarrow \begin{pmatrix}7\\5\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}1\\-1\end{pmatrix}$.

$L\begin{pmatrix}7\\5\end{pmatrix} = 4L\begin{pmatrix}1\\2\end{pmatrix} + 3L\begin{pmatrix}1\\-1\end{pmatrix}$

$= 4\begin{pmatrix}-2\\3\end{pmatrix} + 3\begin{pmatrix}5\\2\end{pmatrix}$.

$= \begin{pmatrix}-8\\12\end{pmatrix} + \begin{pmatrix}15\\6\end{pmatrix}$

$= \begin{pmatrix}7\\18\end{pmatrix}$.

chapter(6):- Eigenvalues.

6.1 Eigenvalues and eignevectors.

Df:- let A be an $n \times n$ matrix. A scelar λ is said to be an eigenvalue or a characterstic value of A is there exists a nonzero vector v such that $\boxed{Av = \lambda v}$. $A\binom{1}{2} = 5\binom{1}{2}$.

the vector v is said to be an eigenvector or charcteristic vector belonging λ. $v \neq 0$.

ex:- $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, $v = \binom{2}{1}$, $\lambda = 3$. then.

$$Av = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda v.$$

$\therefore \lambda = 3$ is an eigenvalue of A

and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector belonging to $\lambda = 3$.

Q:- How to find the eigenvalue and corresponding eignvalue of a squer matrix A?

Ans:- $Av = \lambda v$, $v \neq 0$

$Av - \lambda v = 0$ $\Rightarrow$ $A - \lambda I_n$ is singular.

$(A - \lambda I)v = 0$ $\Rightarrow$ $\det(A - \lambda I_n) = 0$.

$\underbrace{A - \lambda I = 0}_{\text{homog. system.}}$, $v \neq 0$. characteristing equ.

$\Rightarrow$ $P(\lambda) = \det(A - \lambda I_n)$ characteristing Polynomial of A.

to find an eignvector belonging to λ, we find:

$N(A-\lambda I)$ fuct ir $\{A-\lambda I \mid 0\}$ ......

Examples:- Find the eignevalues and the corresponding eignvector of the given matrix.

[1] $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

the characteristic eq is $|A - \lambda I| = 0$ $\quad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0.$

$(3-\lambda)(-2-\lambda) - 6 = 0$

$-6 - 3\lambda + 2\lambda + \lambda^2 - 6 = 0.$

$\lambda^2 - \lambda - 12 = 0$

$(\lambda-4)(\lambda+3) = 0$

$\therefore \lambda = 4 \,/\, \lambda = -3.$ are the eignvalues of A.

- for $\boxed{\lambda_1 = 4}$, to find an eignvector belonging to $\lambda_1 = 4$

we find $N(A-\lambda I) = N(A-4I)$

$[A-4I : 0] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -6 & 0 \end{array}\right]$

$\Rightarrow N(A-4I) = \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} : t \in \mathbb{R} \right]$ $\quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$= \text{span}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]$ $\quad x_1 = 2x_2$

$\therefore v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eignvector corr to $\lambda = 4$ $\quad x_2 = t \Rightarrow x_1 = 2t$

- For $\boxed{\lambda_2=-3}$, we find $N(A+3I)$

$$[A+3I \mid 0] = \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = r,\ x_1 = -\frac{1}{3}r.$

$N(A+3I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}r \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$ eignspace.

$\text{span}\left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \right\} \therefore \text{span}\left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$.

$\therefore v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is an eignvector belonging to $\lambda_2=-3$.

$\boxed{2}$ $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

the characterstic equation is $|A-\lambda I| = 0$.

$$\begin{vmatrix} \overset{+}{2-\lambda} & \overset{-}{-3} & \overset{+}{1} \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix}$

$(2-\lambda)\left[(-2-\lambda)(2-\lambda)+3\right] + 3\left[2-\lambda-1\right] + -3+2+\lambda$

$-\lambda(\lambda-1)^2 = 0.$

$-\lambda = 0$ or $(\lambda-1)^2 = 0$ $\therefore \lambda_1 = 0$ or $\lambda_2 = \lambda_3 = 1$

are the eignvalues of A.

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$\boxed{\lambda_1 = 0}$, we find $N(A - 0I) = N(A)$.

$-R_2 + R_1$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 + R_2 \\ -R_1 + R_3. \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 leadings.

$x_3 = \alpha$ free

$$N(A) = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eiginvector corr. to $\lambda_1 = 0$.

For $\lambda_2 = \lambda_3 = 1$, we find $N(A-I)$

$= N(A-I)$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 leading.

x_2, x_3 free.

$x_2 = \alpha$, $x_3 = \beta$.

$x_1 = 3\alpha - \beta$.

$\therefore N(A-I) = \left\{ \begin{pmatrix} 3\alpha-\beta \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

$$\left\{ \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

lin. indep.

$v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are an eignvecotrs belonging to $\lambda_2 = \lambda_3 = 1$.

$\dim(N - I) = 2$.

[3] $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$p(\lambda) = |A - \lambda I|$

$= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(2-\lambda) = 0$

$\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 4$. are the eignvalues of A.

For $\lambda_1 = \lambda_2 = 2$, we find $N(A - 2I)$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$ $x_2 = 0$, $x_1 = 0$, $x_3 = \alpha$ free

$\therefore N(A-2I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} . \alpha \in \mathbb{R} \right\}$

, Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

is an eignvectors corr to $\lambda_1, \lambda_2 = 2$.

$\dim N(A - 2I) = 1$

For $\lambda_3 = 4$, we find $N(A-4I)$ ⟩⟩ the Answer.

$$N(A-4I) = \left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \text{span} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eignvectors corr to $\lambda_3 = 4$

the product and the sum of the Eignvalues

Def:- let A be $n \times n$ matrix, then the trace of A denoted by $tr(A)$ is the sum of all entries on the main diagonal.

ex:- $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 0 & 2 & 8 \end{pmatrix} \Rightarrow tr(A) = 1 - 5 + 8 = 4.$

thm:- let A be an $n \times n$ matrix with eignvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then:-

1) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

2) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

ex:- $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ $\lambda_1 = 4$, $\lambda_2 = -3$.

- $\det(A) = -6 - 6 = -12 = \lambda_1 \times \lambda_2$
- $tr(A) = 3 - 2 = 1 = \lambda_1 + \lambda_2$.

thm:- $A_{n\times n}$, then A is singular iff 0 is an eignvalue of A.

thm:- $A_{n\times n}$, then A and A^T have the same eignvalues.

thm:- $A_{n\times n}$. λ is an eignvalue of A, then
λ^n " " " " A^n, $n \in \mathbb{Z}^+$, with the same eignvector

ex:- $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ $\lambda_1 = 4$, $\lambda_2 = -3$.

the eignvalues of A^3, $\lambda_1 = 4^3$, $\lambda_2 = -3^3$.

$\lambda_1 = 64$, $\lambda_2 = -27$.

thm:- $A_{n\times n}$ nonsingular, if λ is an eignvalue of A then $\frac{1}{\lambda}$ is an eignvalue of A^{-1} with the same eignvector

ex:- $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$, $\lambda_1 = 4$, $\lambda_2 = -3$

A is nonsingular then the eignvalues of A^{-1} are $\frac{1}{4}$, $\frac{-1}{3}$.