# CH. 4
## Cyclic Group

Q.1:- The generators of $Z_6$ : 1, 5.

" " " $Z_8$ : 1, 3, 5, 7.

" " " $Z_{20}$ : 1, 3, 7, 9, 11, 13, 17, 19.

Q.2:- The generators of $\langle a \rangle$ : $a^1, a^5$.

" " " $\langle b \rangle$ : $b^1, b^3, b^5, b^7$.

" " " $\langle c \rangle$ : $c^1, c^3, c^7, c^9, c^{11}, c^{13}, c^{17}, c^{19}$

Q.4:- $\langle 3 \rangle = \{3, 6, 9, 12, 15, 0\}$ , $\langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}, a^{15}, a^0\}$

$\langle 15 \rangle = \{15, 12, 9, 3, 6, 0\}$ , $\langle a^{15} \rangle = \{a^{15}, a^{12}, a^9, a^3, a^6, a^0\}$

Q.6:- In any Group we show. $\langle a \rangle = \langle a^{-1} \rangle$.

proof:- $a^{-1} \in \langle a \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ — (1)

$a = (a^{-1})^{-1} \in \langle a^{-1} \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ — (2)

From (1) & (2) $\langle a \rangle = \langle a^{-1} \rangle$.

Q.8:- $|a| = 15$

a) $a^3, a^6, a^9, a^{12} \Rightarrow$ order = 5

b) $a^5, a^{10} \Rightarrow$ order = 3

c) $a^2, a^4, a^8, a^{14} \Rightarrow$ order = 15.

Q.10:- $Z_{24}$, subgroup of order 8 :- $24/8 = 3$

All subgroups :- 3, 9, 15, 21

$\rightarrow$ a :- $a^3, a^9, a^{15}, a^{21}$

$(a^3), (a^3)^3, (a^3)^5, (a^3)^7$

Q.11:- * let $a^{-1} \in \langle a \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ —① 

and $a = (a^{-1})^{-1} \in \langle a^{-1} \rangle \Rightarrow \langle a \rangle \subseteq \langle a^{-1} \rangle$ —② 

from ① & ② $\Rightarrow \langle a \rangle = \langle a^{-1} \rangle$.

Q.12:- All generators : $\langle 3 \rangle$ , $\langle -3 \rangle$

$\langle a^3 \rangle$ : $a^3, a^{-3}$.

Q.13:- $\langle 21 \rangle = \{0, 21, 18, 15, 12, 9, \underline{6}, 3\}$.

$\langle 10 \rangle = \{0, 10, 20, \underline{6}, 16, 2, 12, 22, 8, 18, 4, 14\}$.

$\langle 21 \rangle \cap \langle 10 \rangle = \langle 6 \rangle$ {smalest positive multiple}.

And:- $\langle a^{21} \rangle = \{a^0, a^{21}, a^{18}, a^{15}, a^{12}, a^9, a^6, a^3\}$.

$\langle a^{10} \rangle = \{a^0, a^{10}, a^{20}, a^6, a^{16}, a^2, a^{12}, a^{22}, a^8, a^{18}, a^4, a^{14}\}$.

$\langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^6 \rangle$.

Proof:- $\langle a^m \rangle \cap \langle a^n \rangle$, let $k = LCM(n,m)$

$k = nt_1$ , $k = mt_2$

$a^k = (a^n)^{t_1} \in \langle a^n \rangle$ and

$a^k = (a^m)^{t_2} \in \langle a^m \rangle$

$\Rightarrow \langle a^k \rangle \subseteq \langle a^n \rangle \cap \langle a^m \rangle$

let $x \in \langle a^n \rangle \cap \langle a^m \rangle$

then $x$ is multiple of $\underline{m}$ and $\underline{n}$

then $x$ is multiple of $k$

So $a^x \in \langle a^k \rangle$

$a^x \in \langle a^{LCM(m,n)} \rangle$.

Q.14:— $|G| = 49$.

$\Rightarrow$ let $|G| = n$, 7 and $n/7$ divisors of n.

If $\frac{n}{7} \neq 7$, then G has at least 4 divisors. X

So must $\frac{n}{7} = 7 \Rightarrow n = 7^2 = 49$ / In general $|G| = p^2$

Q.15:- G, abelian group, $H = \{g \in G, |g| \text{ divides } 12\}$.

~~*~~ One step:

$ab^{-1} \Rightarrow (ab^{-1})^{12} = a^{12} (b^{12})^{-1} = e \cdot e^{-1} = e \in H$

In general the same argument when 12 replace by any integer.

Q.16:- of $Z_{240}$ :- $\underline{240}$

$\langle 0 \rangle \subset \langle 120 \rangle \subset \langle 60 \rangle \subset \langle 30 \rangle \subset \langle 15 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$.

Q.18:- suppose $G = \langle g \rangle$ is cyclic of infinite order.

let $g^K$ ~~...~~ , $(g^K)^n = e$ { g has order dividing Kn }.

$\times$ (because g finite). unless $K = 0$, $g^0 = e$,

$\Rightarrow$ Is the only element of finite order in g.

~~Q.~~

Q.27:- let complex number $Z \Rightarrow Z^n = 1$

let H subgroup $= \{ Z \in C^* : Z = (1)^{\frac{1}{n}} \} = \{ Z \in C^* : Z = (\cos 0 + \sin 0 i)^{\frac{1}{n}} \}$

$= \{ Z \in C^* : Z = (\cos(0 + 2K\pi) + i\sin(0 + 2\pi K))^{\frac{1}{n}}, K = 0, 1, \ldots n-1 \}$.

$= \{ Z \in C^* : Z = (e^{i2K\pi})^{\frac{1}{n}}, K = 0, 1, \ldots n-1 \}$.

$= \{ Z \in C^* : Z = (e^{i(\frac{2\pi}{n})})^K, K = 0, 1, \ldots, n-1 \}$.

$= \{ Z \in C^* : Z = w^K, K = 0, 1, \ldots, n-1$, where $w = e^{i(\frac{2\pi}{n})} \}$.

$= \{ w^0, w^1, w^2, \ldots w^{n-1} \} = \langle w \rangle \rightarrow$ is the cyclic group.

and order $= n$.

Q.28:- $\langle a^i\rangle = \langle a^j\rangle \Longleftrightarrow i = \pm j$

If $\langle a^i\rangle = \langle a^j\rangle$ then $a^i = (a^j)^k = a^{jk}$, since $a$ has inf order

$i = jk$ $\rightarrow$ $j$ divides $i$ $\rightarrow$ like wise $i$ divides $j$ so $i = \pm j$

$j = im$

$i = jk = imk \Rightarrow mk = 1 \Rightarrow m = k = 1 \rightarrow i = j$

or $m = k = -1 \rightarrow i = -j$.

Q.30:- $a, b \in G$, $a$ odd order, $aba^{-1} = b^{-1}$, show $b^2 = e$ :-

$(aba^{-1}) \Rightarrow a^2 b a^{-2} = a(aba^{-1})a^{-1} = ab^{-1}a^{-1}$

$(aba^{-1})^{-1} = (b^{-1})^{-1} = b$

So $a^2 \in C(b)$, $\langle a^2\rangle = \langle a\rangle$, $a$ odd order.

$\Rightarrow aba^{-1} = b$, therefore $b = b^{-1}$ and $b^2 = b \cdot b^{-1} = e$.

Q.34:- lattic of $\mathbb{Z}_8$:-

1x2x4

1, 2, 4, 8

$\langle 1\rangle$
|
$\langle 2\rangle$
|
$\langle 4\rangle$
|
$\langle 0\rangle$

Q.40:- $\langle m\rangle \cap \langle n\rangle \Rightarrow$ The generators are $LCM(m,n)$.

let $\langle m\rangle \cap \langle n\rangle = \langle LCM(m,n)\rangle = K$

if $a \in \langle m\rangle \cap \langle n\rangle$, the $K$ divides $a$.

let $a = qK + r$, $q, r \in \mathbb{Z}$ and $0 \leq r < K$

$\Rightarrow r = a - qK$, but $K = LCM(m,n)$

So $r = 0$ and $qK = a$, $K$ divides $a$.

$\Rightarrow \langle m\rangle \cap \langle n\rangle = \langle K\rangle$.

Q.41 :- $a, b$ are group, orders $m, n$, If $\langle a\rangle \cap \langle b\rangle = \{e\}$.

$\Rightarrow$ let $t = LCM(m, n)$ and $|ab| = s$, then.

$(ab)^t = a^t b^t = e$ and $s$ divides $t$.

Also $e = (ab)^s = a^s b^s$ so $\rightarrow$ $\underbrace{a^s = b^{-s}}$ and $m$ divides $s$.

belong to $\langle a\rangle \cap \langle b\rangle = \{e\}$.

and $n$ divides $s \rightarrow t$ divides $s$.

* $R_{120}$ : rotation of $120°$, and $F$ Reflection in $D_3$ :-

$\langle R_{120}\rangle \cap \langle F\rangle = \{R_0\}$, $|R_{120}| = 3$, $|F| = 2$

but $D_3$ its no element of order 6.

Q.53:- $\langle a\rangle \cap \langle b\rangle$ is a subgroup.

$\langle a\rangle \cap \langle b\rangle \subset \langle a\rangle$ and $\langle a\rangle \cap \langle b\rangle \subseteq \langle b\rangle$

$|\langle a\rangle \cap \langle b\rangle|$ is a common divisor of 10, 21 then

$|\langle a\rangle \cap \langle b\rangle| = 1 \longrightarrow |\langle a\rangle \cap \langle b\rangle| = \{e\}$.

Q.54:- let $x \in \langle a\rangle \cap \langle b\rangle$

$x \in \langle a\rangle \rightarrow |x| / |a|$ and $x \in \langle b\rangle \longrightarrow |x| / |b|$

$|x|$ is a common divisor of 2 relatively prime integers

$|a|$ and $|b|$ so it must be $|x| = 1$.

The only element with order 1 is the identity so $x = e$.

Q.55:- $|\langle a\rangle \cap \langle b\rangle|$ must divide both 24 and 10

So $|\langle a\rangle \cap \langle b\rangle| = 1$ or $2$