Chapter 7.1, Problem 48E

Problem

Let *X* and *Y* be sets, let *A* and *B* be any subsets of *X*, and let *C* and *D* be any subsets of *Y*. Determine which of the properties are true for all functions *F* from *X* to *Y* and which are false for at least one function *F* from *X* to *Y*. Justify your answers.

Exercise

For all subsets *C* and *D* of *Y*,

F−1(*C* − *D*) = *F*−1(*C*) − *F*−1 (*D*).

Step-by-step solution

Step 1 of 3

Let *F* be a function from *X* to *Y*.

Assume that, $C \subseteq D$ and $D \subseteq y$ to prove,

$$F^{-1}(C-D) = F^{-1}(C) - F^{-1}(D)$$

The proof can be divided into two parts.

$$\text{Part-1:} F^{-1}(C-D) \subseteq F^{-1}(C) - F^{-1}(D)$$
$$\text{Part-2:} F^{-1}(C-D) \supseteq F^{-1}(C) - F^{-1}(D)$$

Step 2 of 3

Part -1:-

The proof of the part-1 is same as,

If $x \in F^{-1}(C-D)$, then $x \in F^{-1}(C) - F^{-1}(D)$.

By the definition of inverse image, $x \in X$ such that $f(x) \in C - D$

That means, $x \in X$ such that $f(x) \in C$ and $f(x) \notin D$.

By the definition of inverse image of the set,

$$x \in F^{-1}(C) \text{ and } x \notin F^{-1}(D)$$

By the definition of difference of two sets,

$$x \in F^{-1}(C) - F^{-1}(D)$$

Hence,

$$F^{-1}(C-D) \subseteq F^{-1}(C) - F^{-1}(D) \quad \ldots\ldots (1)$$

Step 3 of 3

Part II: -

The proof of the part-2 is same as,

If $x \in \left(F^{-1}(C) - F^{-1}(D)\right)$, then $x \in F^{-1}(C - D)$.

By the definition of difference of two sets and the inverse image,

$x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$

$F(x) \in C$ and $F(x) \notin D$

By the definition of difference of two sets and the inverse image,

$F(x) \in (C - D)$

$x \in F^{-1}(C - D)$

Hence,

$F^{-1}(C) - F^{-1}(D) \subseteq F^{-1}(C - D)$. …… (2)

Therefore, by equation (1) and (2), it can be conclude that,

$\boxed{F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)}$.