2.2 : Conditional distribution and expectation :

Def : let X_1, X_2 be r.v's (both discrete or both continuous) :- we defined the conditional P.d.f of X_1 given $X_2 = x_2$ as follows

$$f_{1|2}(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}, \quad f_2(x_2) > 0$$

We defined the conditional P.d.f of X_2 given $X_1 = x_1$ as follows :

$$f_{2|1}(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}, \quad f_1(x_1) > 0$$

RMK : X_1, X_2 Continuous r.v's

1. $Pr(a < X_2 < b \mid X_1 = x_1) = \int_a^b f_{2|1}(x_2 \mid x_1)\, dx_2$.

2. $Pr(c < X_1 < d \mid X_2 = x_2) = \int_c^d f_{1|2}(x_1 \mid x_2)\, dx_1$.

3. $E(X_2 \mid X_1) = E(X_2 \mid X_1 = x_1) = \int_{-\infty}^{\infty} x_2\, f_{2|1}(x_2 \mid x_1)\, dx_2$.

4. $E(X_1 \mid X_2) = E(X_1 \mid X_2 = x_1) = \int_{-\infty}^{\infty} x_1\, f_{1|2}(x_1 \mid x_2)\, dx_1$.

5. $Var(X_2 \mid X_1) = Var(X_2 \mid X_1 = x_1) = E\left[(X_2 - E(X_2 \mid X_1))^2 \mid X_1 \right]$

$$= E(X_2^2 \mid X_1) - \left[E(X_2 \mid X_1) \right]^2 .$$

6. $Var(X_1|X_2) = Var(X_1|X_2 = x_2) = E\left[(X_1 - E(X_1|X_2))^2 \mid X_2\right]$

$$= E(X_1^2|X_2) - \left[E(X_1|X_2)\right]^2 .$$

7. $E(u(X_1)|X_2) = E(u(X_1)|X_2 = x_2) = \int_{-\infty}^{\infty} u(x_1)\, f_{1|2}(x_1|x_2)\, dx_1$.

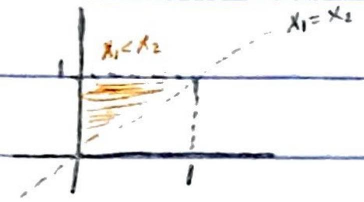
8. $E(w(X_2)|X_1) = E(w(X_2)|X_1 = x_1) = \int_{-\infty}^{\infty} w(x_2)\, f_{2|1}(x_2|x_1)\, dx_2$.

9. $E(u(X_1, X_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\, f(x_1, x_2)\, dx_1\, dx_2$.

$\Rightarrow$ similarly the above equation are used when X_1, X_2 are discrete r.v's By replacing the integrals with sums .

Example 1 : X_1, X_2 have joint p.d.f , $f(x_1,x_2) = \begin{cases} 2 & , 0<x_1<x_2<1 \\ 0 & , \text{elsewhere} \end{cases}$

Sketch :



1. $f_1(x_1) = \int_{-\infty}^{\infty} f(x_1,x_2)\, dx_2 = \int_{x_1}^{1} 2\, dx_2 = 2x_2 \Big|_{x_1}^{1} = 2 - 2x_1$, $0<x_1<1$

$\rightsquigarrow f_1(x_1) = \begin{cases} 2(1-x_1) & , 0<x_1<1 \\ 0 & , \text{elsewhere} \end{cases}$

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2. $f_2(x_2) = \int_{-\infty}^{\infty} f(x_1,x_2)\, dx_1 = \int_{0}^{x_2} 2\, dx_1 = 2x_1 \Big|_{0}^{x_2} = 2x_2$, $0<x_2<1$

$\rightsquigarrow f_2(x_2) = \begin{cases} 2x_2 & , 0<x_2<1 \\ 0 & , \text{elsewhere} \end{cases}$

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3. $f_{1|2}(x_1|x_2) = \dfrac{f(x_1,x_2)}{f_2(x_2)} = \begin{cases} \dfrac{2}{2x_2} = \dfrac{1}{x_2} & , 0<x_1<x_2 \\ \dfrac{0}{2x_2} = 0 & , \text{elsewhere} \end{cases}$

Where $0<x_2<1$

4. $E(X_1|X_2) = \int_{-\infty}^{\infty} x_1 \, f_{1|2}(x_1|x_2) \, dx_1 = \int_{0}^{x_2} x_1 \left(\frac{1}{x_2}\right) dx_1 \quad , \; 0 < x_2 < 1$

$= \frac{1}{x_2} \left(\frac{x_1^2}{2} \Big|_0^{x_2} \right) \quad , \; 0 < x_2 < 1$

$= \frac{x_2}{2} \quad , \; 0 < x_2 < 1$

5. $var(X_1|X_2) = var(X_1|X_2 = x_2)$

$= E\left[(X_1 - E(X_1 \backslash X_2 = x_2))^2 \,|\, X_2 = x_2 \right]$

$= E(X_1^2 \backslash X_2 = x_2) - (E(X_1 \backslash X_2 = x_2))^2$

$= \int_{-\infty}^{\infty} (x_1 - \frac{x_2}{2})^2 \, f_{1|2}(x_1/x_2) \, dx_1 \quad , \quad 0 < x_2 < 1$

$= \int_{0}^{x_2} (x_1 - \frac{x_2}{2})^2 \left(\frac{1}{x_2}\right) dx_1 \quad , \quad 0 < x_2 < 1$

$= \int_{0}^{x_2} \left(x_1^2 - x_1 x_2 + \frac{1}{4} x_2^2\right)\left(\frac{1}{x_2}\right) dx_1 = \int_{0}^{x_2} \left(\frac{1}{x_2} x_1^2 - x_1 + \frac{1}{4} x_2\right) dx_1$

$= \frac{1}{x_2} \cdot \frac{x_1^3}{3} - \frac{x_1^2}{2} + \frac{1}{4} x_1 x_2 \Big|_0^{x_2}$

$= \frac{1}{x_2} \cdot \frac{x_2^3}{3} - \frac{x_2^2}{2} + \frac{1}{4} x_2^2$

$= \frac{4x_2^2 - 6x_2^2 + 3x_2^2}{12}$

$= \frac{1}{12} x_2^2 .$

$\Rightarrow var(x_1|x_2) = \frac{1}{12} x_2^2 \quad , \; 0 < x_2 < 1$

6. $Pr(0 < X_1 < \frac{1}{2} \mid X_2 = \frac{3}{4}) = \int_0^{\frac{1}{2}} f_{1|2}(x_1 \mid \frac{3}{4})\, dx_1$

$\rightsquigarrow f(x_1 \mid \frac{3}{4}) = \begin{cases} \frac{4}{3}, & 0 < x_1 < \frac{3}{4} \\ 0, & \text{elsewhere} \end{cases}$ $\quad = \int_0^{\frac{1}{2}} \frac{4}{3}\, dx_1 = \frac{4}{3} x_1 \Big|_0^{\frac{1}{2}}$

$= \frac{4}{3} \cdot \frac{1}{2}$

$= \frac{2}{3}$

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7. $Pr(0 < X_1 < \frac{1}{2}) = \int_0^{\frac{1}{2}} f_1(x_1)\, dx_1$

$= \int_0^{\frac{1}{2}} 2(1 - x_1)\, dx_1$

$= 2x_1 - \frac{2x_1^2}{2} \Big|_0^{\frac{1}{2}} = 1 - \frac{1}{4} = \frac{3}{4}$

Remark: refer to exp.1

$E(X_1 \mid X_2) = \frac{X_2}{2}, \quad 0 < X_2 < 1.$

Define $y = E(X_1, X_2)$ then

1. $E(y) = E(E(X_1 \mid X_2)) = E(X_1).$

2. $Var(y) = Var(E(X_1 \mid X_2)) \leq Var(X_1).$

3. $Var(X_1) = var(E(X_1 \mid X_2)) + E(var(X_1 \mid X_2)).$

check exp2, exp 3.

exp 2 : let X_1 and X_2 have the joint p.d.f $f(x_1, x_2) = \begin{cases} 6x_2 & , \ 0 < x_2 < x_1 < 1 \\ 0 & , \ \text{elsewhere} \end{cases}$

Find :

1. the marginal p.d.f of X_1 is ?

$$f_1(x_1) = \int_0^{x_1} 6x_2 \, dx_2 = 3x_2^2 \Big|_0^{x_1} = 3x_1^2 \quad , \ 0 < x_1 < 1$$

$$\rightarrow f_1(x_1) = \begin{cases} 3x_1^2 & , \ 0 < x_1 < 1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

2. the conditional p.d.f of X_2 given $X_1 = x_1$:

$$f_{2|1}(x_2 \backslash x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} = \begin{cases} \frac{6x_2}{3x_1^2} = \frac{2x_2}{x_1^2} & , \ 0 < x_2 < x_1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

3. the conditional mean of X_2 given $X_1 = x_1$:

$$E(x_2 | x_1) = \int_0^{x_1} x_2 \left(\frac{2x_2}{x_1^2} \right) dx_2 = \frac{2}{3} x_1 \quad , \ 0 < x_1 < 1$$

$$= \frac{2x_2^3}{3} \cdot \frac{1}{x_1^2} \Big|_0^{x_1} = \frac{2}{3} x_1$$

[illegible]

check exp 3 .