MVT : $f$ cont. on $[a,b]$
$f$ diff on $(a,b)$

$\Rightarrow \exists$ at least one number $c \in (a,b)$

s.t $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Exp
$$f(x) = \begin{cases} 3 & , \ x = 0 \\ -x^2 + 3x + a & , \ 0 < x < 1 \\ mx + b & \ 1 \le x \le 2 \end{cases}$$

Assume $f$ satisfies MVT on $[0,2]$
Find constants $a, m, b$

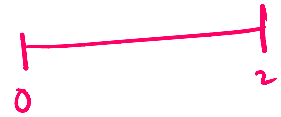
* $f$ cont. on $[0,2]$ $\Rightarrow$ $f$ cont. at $0$
$\Rightarrow$ $f$ cont. at $1$

مرحلة

$f$ cont. at $0$ $\Rightarrow$ $f(0) = \lim\limits_{x\to 0} f(x)$

$f(0) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$

$$3 = \lim_{x \to 0^+} (-x^2 + 3x + a)$$



$$\boxed{3 = a}$$

f cont. at $x = 1 \Rightarrow f(1) = \lim_{x \to 1} f(x)$

موجودة

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

$$\lim_{x \to 1^+} (mx + b) = \lim_{x \to 1^-} (-x^2 + 3x + a)$$

$$m + b = -(1)^2 + 3(1) + a \quad (a = 3)$$

$$m + b = -1 + 3 + 3$$

$$\boxed{m + b = 5} \Rightarrow m = 1 \Rightarrow 1 + b = 5$$

$$\boxed{b = 4}$$

f diff on $(0, 2)$

$$f'(x) = \begin{cases} 0 & x = 0 \\ -2x + 3 & 0 < x < 1 \\ m & 1 < x < 2 \end{cases}$$

$$f'(x) = \begin{cases} -2x+3 & \\ m & 1 < x < 2 \end{cases}$$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$$

$$f'_-(1) = f'_+(1)$$

$$-2+3 = m$$

$$\boxed{1 = m}$$

**Exp** show that $x^3 + 3x + 1 = 0$ has exactly one real root

$f(x) = 0 \Rightarrow \boxed{f(x) = x^3 + 3x + 1}$

cont. on IR since $f$ is poly.

$$[a,b] = [-1,0]$$

$$f(0) = 1 > 0$$

$$\cancel{f(1) = 1+3+1 = 5 > 0}$$

$$f(-1) = (-1)^3 + 3(-1) + 1 = -1 - 3 + 1 = -3 < 0$$

$f$ cont on $[-1, 0]$
$f(-1)\, f(0) < 0$ } Bolzano شروط تتحقق

النتيجة $\exists$ at least one root $c \in [a, b]$
such that $f(c) = 0$

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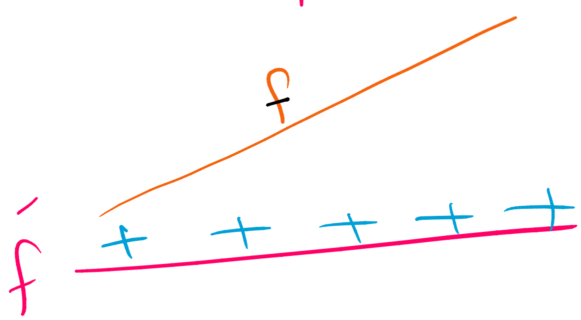
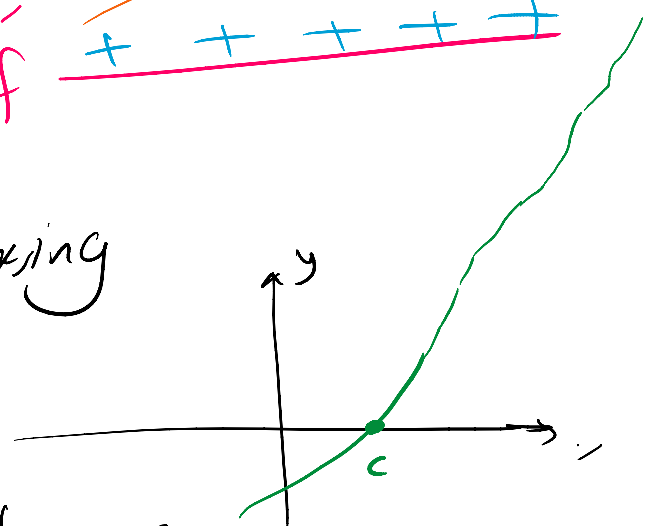
$f = x^3 + 3x + 1$

To show $c$ is unique




$f'(x) = 3x^2 + 3$

$f$ is always increasing

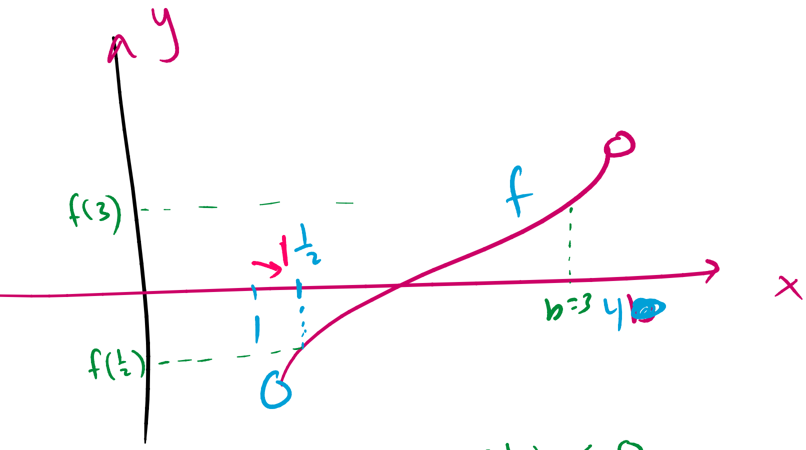



$f$ cross~~es~~ x-axis
only one time at $\underline{x = c}$

Bolzano

$f$ cont. on $[a,b]$




$a = \frac{1}{2}$

$b = 3$

$[a,b] = [1\frac{1}{2}, 3]$

$\left\{ \begin{array}{l} f \text{ cont. on } [1\frac{1}{2}, 3] \\ f(1\frac{1}{2}) f(3) < 0 \end{array} \right.$ $\Rightarrow$ $f(1\frac{1}{2}) < 0$, $f(3) > 0$

$\exists$ root.

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**Exp** $f(x) = x\sqrt{1-x^2}$

① Domain $\Rightarrow D(f)$

$1 - x^2 \geq 0$

$1 \geq x^2$

$\sqrt{1} \geq \sqrt{x^2}$

$1 \geq |x|$

$|x| \leq 1$

$\boxed{-1 \leq x \leq 1}$

$D(f) = [-1, 1]$

② CP's $f(x) = x\sqrt{1-x^2} = x(1-x^2)^{\frac{1}{2}}$

$f'$ ... $)^{-\frac{1}{2}}(-2x) + \sqrt{1-x^2}$

$$f'(x) = \cancel{x} \cdot \frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}}(-\cancel{2}x) + \sqrt{1-x^2}$$

$$= \frac{-x^2}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}\,\sqrt{1-x^2}}{\sqrt{1-x^2}} \quad 1-x^2$$

$$= \frac{-x^2+1-x^2}{\sqrt{1-x^2}}$$

$$f' = \frac{1-2x^2}{\sqrt{1-x^2}} = 0$$

$$1-2x^2=0 \Rightarrow 2x^2=1$$

$$x^2 = \frac{1}{2}$$

in domain $\leftarrow$ $x = \pm\frac{1}{\sqrt{2}}$

$x = \pm\frac{1}{\sqrt{2}} \in [-1,1]$

$x = \pm\frac{1}{\sqrt{2}}$ critical point

$$\left(\frac{1}{\sqrt{2}}, f\left(\frac{1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$$\left(-\frac{1}{\sqrt{2}}, f\left(-\frac{1}{\sqrt{2}}\right)\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$$

$f(x) = x\sqrt{1-x^2}$

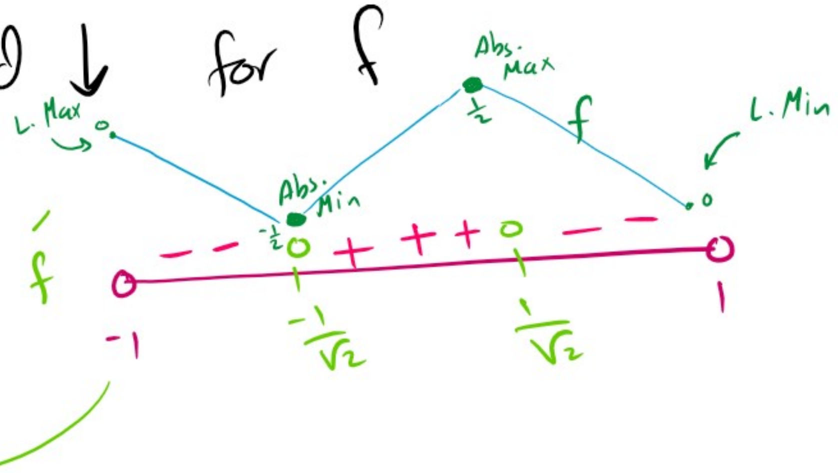
$f(x) = x\sqrt{1-x^2}$

$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}} = \frac{1}{2}$

$f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}} = -\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}} = \frac{-1}{2}$

(3) intervals of ↑ and ↓ for f

$f' = \frac{1 \ominus 2x^2}{\sqrt{1-x^2}}$




$f$ ↑ on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

$f$ ↓ on $\left[-1, -\frac{1}{\sqrt{2}}\right)$ and $\left[\frac{1}{\sqrt{2}}, 1\right]$

(4) EV's

→ check CP's : $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$

→ check Endpoints $(-1, 0)$, $(1, 0)$

$f(x) = x\sqrt{1-x^2}$ on $[-1, 1]$

f has L. Max of $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$f(x) = x\sqrt{1-x^2}$ on $[-1,1]$

$f(-1) = -1\sqrt{1-(-1)^2} = (-1)(0) = 0$

$f(1) = (1)\sqrt{1-(1)^2} = (1)(0) = 0$

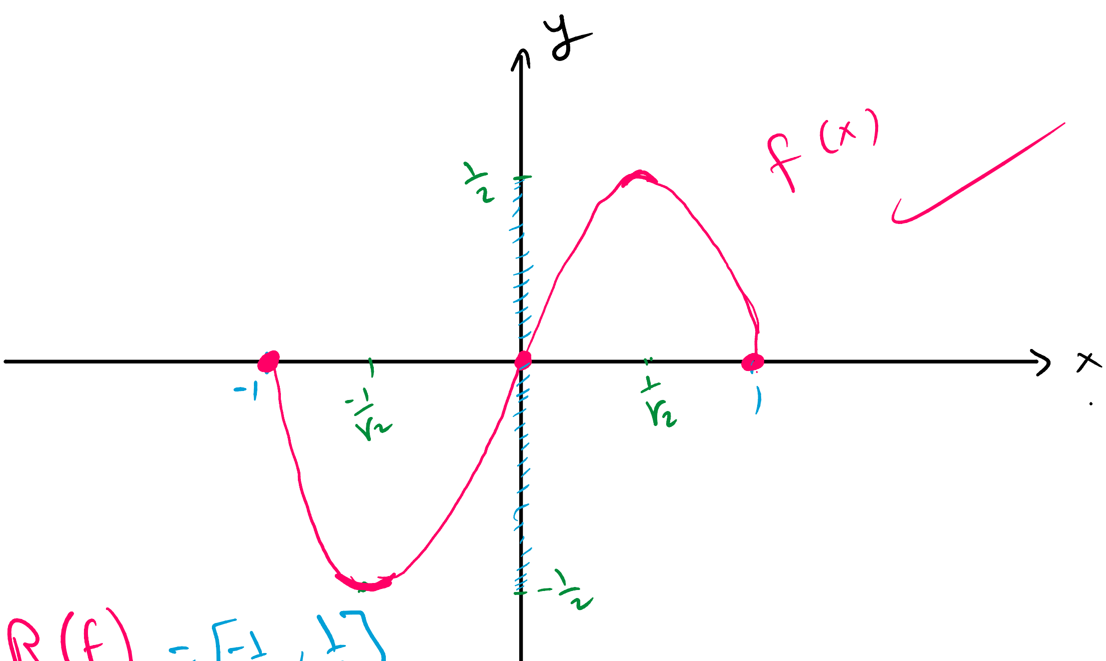
$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$f$ has L. Min of $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

5 sketch $f$

$f(x) = x\sqrt{1-x^2}$

keypoint $(0,0)$




6 $R(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$