1.8: Expectation of a Random Variable.

r.v   p.d.f   c.d.f

X f F .

Def: The expectation of a Random variable is defined as follows:

- $E(X) = \sum_{x} x f(x)$, X discrete

- $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$, X continuous .

Assuming:
- $\sum_{x} |x| f(x)$ converges , X discrete .
- $\int_{-\infty}^{\infty} |x| f(x)\, dx$ converges , X continuous .

if it diverge $\Rightarrow$ $E(X)$ undefined .

Note: Expectation of X
expected value of X اسماء لـ $E(x)$
mathematical expectation of X

example 1: discrete since $A = \{1, 2, 3, 4\}$

X :	1	2	3	4
$f(x)$:	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

Find $E(X)$.

$$E(X) = \sum_{x} x f(x) = (1)\left(\frac{4}{10}\right) + (2)\left(\frac{1}{10}\right) + (3)\left(\frac{3}{10}\right) + (4)\left(\frac{2}{10}\right) = \frac{23}{10} = 2.3$$

Example 2 : continuous since $A = (0,1)$.

$$f(x) = \begin{cases} 4x^3 , & 0 < x < 1 \\ 0 , & \text{elsewhere} \end{cases}$$

Find $E(x)$?

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{1} x (4x^3)\, dx + \int_{1}^{\infty} x \cdot 0\, dx$$

$$= \int_{0}^{1} x (4x^3)\, dx = \frac{4x^5}{5} \Big|_0^1 = \frac{4}{5}.$$

Proposition : let X be a Random Variable

- $E(u(x)) = \sum_x u(x) f(x)$, X discrete.

- $E(u(x)) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$, X continuous.

Proposition :

1. if K is constant then $E(K) = K$.
2. if K constant , u function then $E(uK) = K E(u)$.
3. if K_1 , K_2 constants , u_1 , u_2 functions then $E(K_1 u_1 + K_2 u_2) = K_1 E(u_1) + K_2 E(u_2)$.

Proof of 3 (discrete) : X : discrete r.v , f : p.d.f , K_1, K_2 : constant , u_1, u_2 : functions

$$\Rightarrow E(K_1 u_1 + K_2 u_2) = E(K_1 u_1(x) + K_2 u_2(x))$$

$$= \sum_x (K_1 u_1(x) + K_2 u_2(x)) f(x)$$

$$= K_1 \sum u_1(x) f(x) + K_2 \sum u_2(x) f(x)$$

$$= K_1 E(u_1) + K_2 E(u_2)$$

example 3: X has a p.d.f :

$$f(x) = \begin{cases} 2(1-x) & , \quad 0 < x < 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

continuous

Find :

1. $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{1} x \cdot (2)(1-x)\, dx + \int_{1}^{\infty} x \cdot 0\, dx$

$$= \int_{0}^{1} 2x - 2x^2\, dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}.$$

2. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{0}^{1} x^2 (2)(1-x)\, dx = \int_{0}^{1} 2x^2 - 2x^3\, dx$

$$= \frac{2x^3}{3} - \frac{x^4}{2} \Big|_0^1 = \frac{1}{6}.$$

3. $E(6X + 3X^2) = \int_{-\infty}^{\infty} (6x + 3x^2)\, f(x)\, dx$ طويلة

OR

$$E(6X + 3X^2) = 6E(X) + 3E(X^2)$$

$$= 6\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right) \qquad \text{From 1, 2}$$

$$= 2 + \frac{1}{2}$$

$$= \frac{5}{2} \quad ✓$$

example 4 :

X is a Random Variable with p.d.f ,

$$f(x) = \begin{cases} \frac{x}{6} , & x = 1,2,3 \\ 0 , & \text{elsewhere} \end{cases}$$

discrete .

Find $E(X^3)$.

$\rightarrow E(X^3) = \sum_{x \in \mathbb{R}} x^3 f(x) = (1)^3 f(1) + (2)^3 f(2) + (3)^3 f(3)$

$= 1\left(\frac{1}{6}\right) + 8\left(\frac{2}{6}\right) + 27\left(\frac{3}{6}\right)$

$= \frac{98}{6} = 16.33$.

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example 5 : X has the p.d.f

$$f(x) = \begin{cases} \frac{1}{5} , & 0 < x < 5 \\ 0 , & \text{elsewhere} \end{cases}$$

continuous

Find :

1. $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{5} x \frac{1}{5}\, dx = \frac{x^2}{10}\Big|_0^5 = \frac{25-0}{10} = 2.5$.

2. $E(5-x) = E(5) + E(-x) = 5 - E(x) = 5 - 2.5 = 2.5$.

3. $E[x(5-x)] = \int_{-\infty}^{\infty} x(5-x)\, f(x)\, dx = \int_{0}^{5} x(5-x) \frac{1}{5}\, dx = \frac{25}{6} = 4.17$.

Note: $E(x(5-x)) \neq E(x)\, E(5-x)$ X

example 6 : X has the hypergeometric p.d.f .

$$f(x) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{2-x}}{\binom{5}{2}} & , \ x = 0, 1, 2 \ . \\ 0 & , \ \text{elsewhere} \ . \end{cases}$$

discrete .

Find $E(8-3x) = \sum_{x=0}^{2} (8-3x) f(x)$

$$= (8-3(0))\left(\frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}}\right) + (8-3(1))\left(\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}\right) + (8-3(2))\left(\frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}}\right)$$

$$= 8\left(\frac{1\times1}{10}\right) + 5\left(\frac{3\times2}{10}\right) + 2\left(\frac{3\times1}{10}\right)$$

$$= \frac{8}{10} + \frac{30}{10} + \frac{6}{10} = \frac{44}{10}$$