Distributed Forces, Centroids & Centers of Gravity

Chapter 5

Outlines

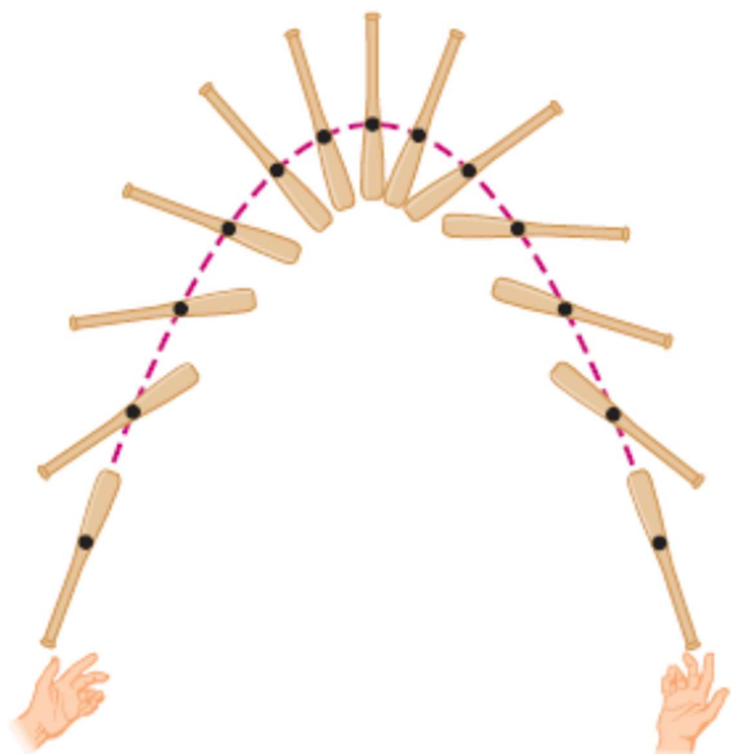
- Planar centers of gravity and centroids
- Determination of Centroids by Integration
- Composite Plates and Wires
- Theorems of Pappus-Guldinus
- Distributed Loads on Beams

Objectives

- Describe the centers of gravity of two dimensional bodies.
- Define the centroids of lines and areas.
- Consider the first moments of lines and areas, and examine their properties.
- Determine centroids of composite lines and areas by summation methods.
- Determine centroids of composite lines, and areas by integration.
- Apply the theorems of Pappus-Guldinus to analyze surfaces and bodies of revolution.
- Analyze distributed loads on beams.

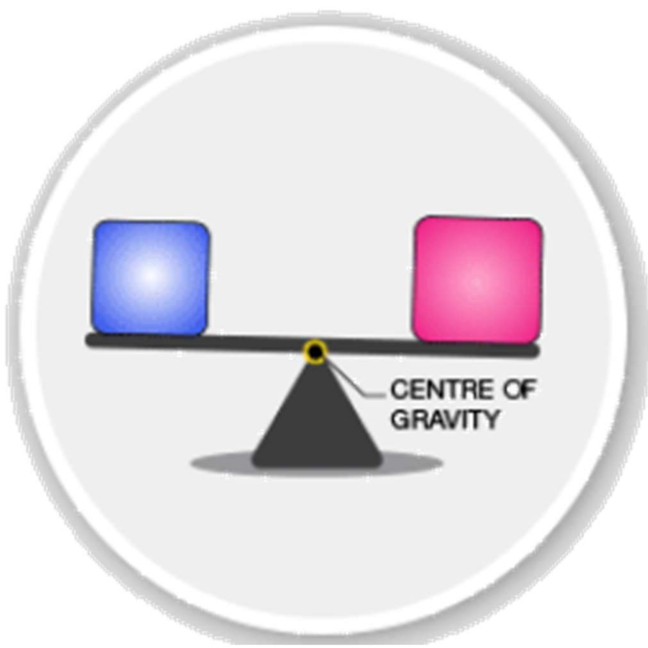
Centroid, center of mass and Center of Gravity

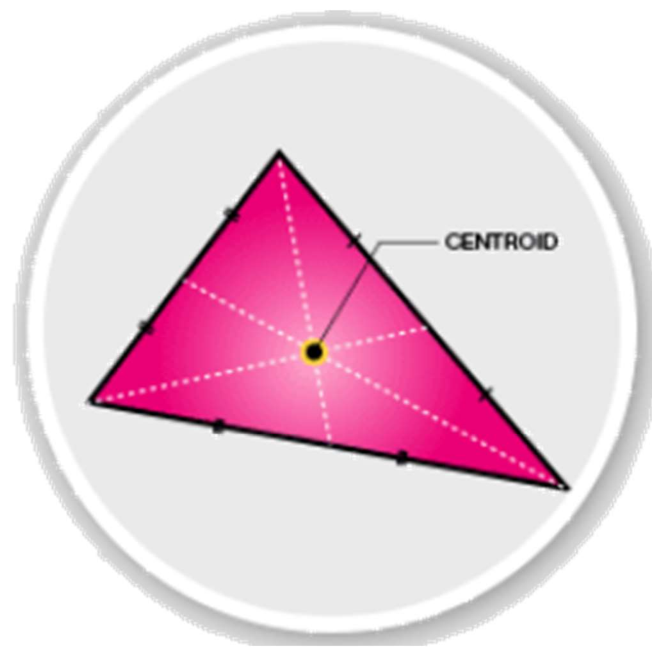
- In Physics, The center of mass of a system of particles is the point that moves as though on can assume that:
  - All of the system's mass were concentrated there and
  - All external forces were applied there.
- The center of gravity is the average position of a distribution of weight or, It is the point where the gravitational force (weight) acts on the body
- When the gravitational field is uniform across an object, center of mass is the same as the center of gravity.



Centroid, center of mass and Center of Gravity

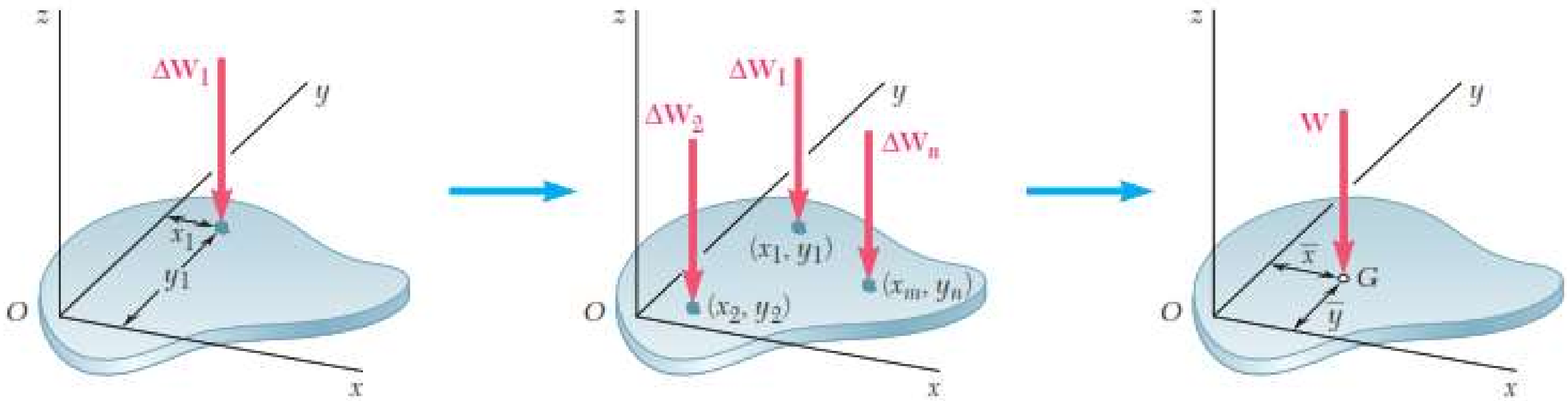
- Centroid is the average position of a distribution of shapes. It is referred to the geometrical center of a body.
- If the body is homogeneous (having constant density and thickness), then its center of gravity is equivalent with the centroid.





5.1A Center of Gravity of a 2D Body

- Applying the principle of equivalent system of forces, the moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body



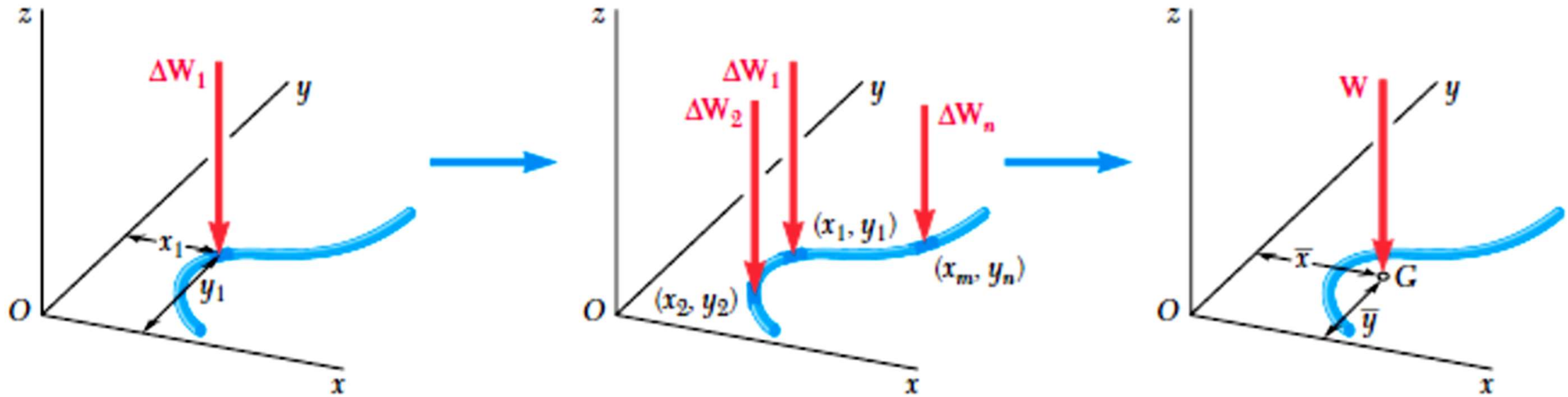
(a) Single element of the plate (b) Multiple elements of the plate (c) Center of gravity

$$W = \sum dw = \int dw$$

$$W\bar{x} = \sum x_i dw_i = \int x dw \quad \rightarrow \bar{x} = \frac{\int x dw}{W} \quad and \quad \bar{y} = \frac{\int y dw}{W}$$

Determining the Center of Gravity of Wire or Cable

- If the body is a wire or cable as shown below



(a) Single element of the wire (b) Multiple elements of the wire (c) Center of gravity

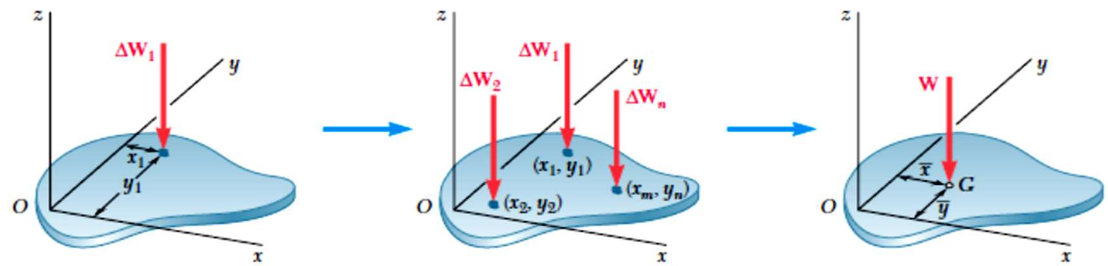
$$W = \sum \Delta w = \int dw$$

$$W\bar{x} = \sum x_i \Delta w_i = \int x dw \quad \rightarrow \bar{x} = \frac{\int x dw}{W} \qquad \bar{y} = \frac{\int y dw}{W}$$

- Note that the center of gravity may not actually be located on the wire.

5.1B Centroids of Areas

- In the case of a flat homogeneous plate of uniform thickness as shown, we can express the magnitude ΔW of the weight of an element of the plate as

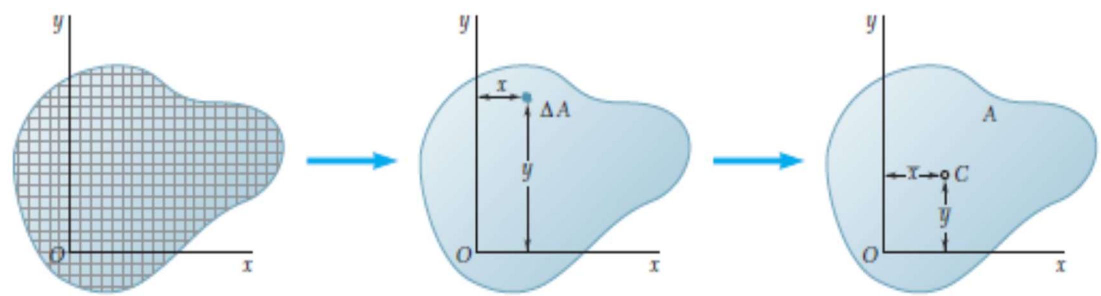


(a) Single element of the plate (b) Multiple elements of the plate (c) Center of gravity

$$W = \rho t A$$

$$\Delta W = \rho t \Delta A, and\ dw = \rho t dA$$

Where
ρ= specific weight (weight per unit volume) of the material
t = thickness of the plate, and
ΔA = area of the element



(a) Divide area into elements (b) Element ΔA at point x, y (c) Centroid located at

- Then

$$W\bar{x} = \rho t A\bar{x} = \sum x_i \Delta w_i = \int x dw = \rho t \int x dA \quad \rightarrow \bar{x} = \frac{\int x dA}{A} \quad and \quad \bar{y} = \frac{\int y dA}{A}$$

5.1B Centroids of Lines (wires/ cables)

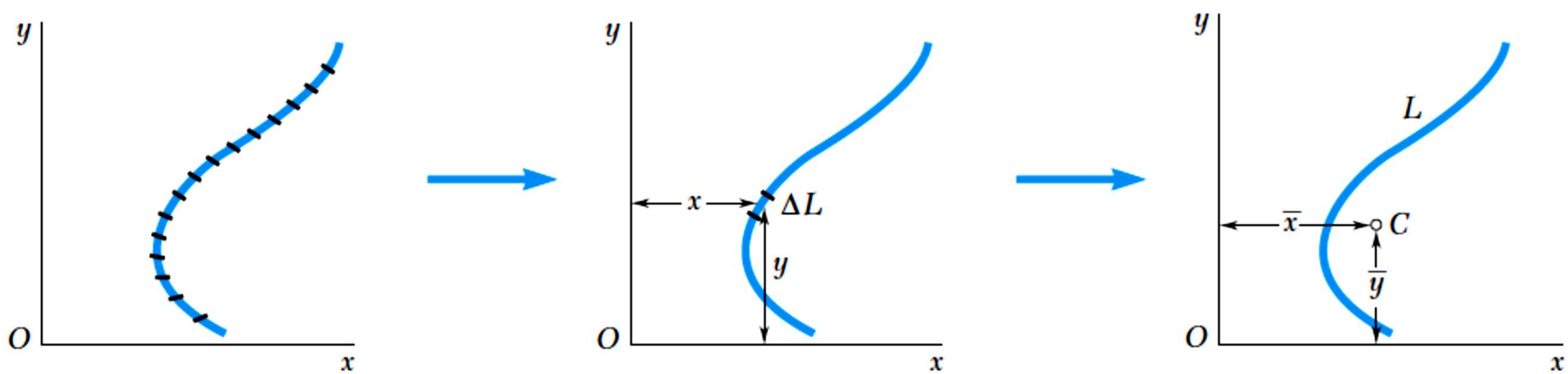
- It was shown that the coordinates of the center of gravity for a linear element are

$$\bar{x} = \frac{\int x dw}{W} \quad and \quad \bar{y} = \frac{\int y dw}{W}$$

- When the cross sectional area (a) and the density of the element (ρ) are constant the following equations can be written.

$$W = \rho a L, \Delta W = \rho a \Delta L, and\ dw = \rho a dL$$

$$W\bar{x} = \rho a L \bar{x} = \sum x_i \Delta w_i = \int x dw = \rho a \int x dL \quad \rightarrow \bar{x} = \frac{\int x dL}{L} \quad and \quad \bar{y} = \frac{\int y dL}{L}$$



(a) Divide line into elements (b) Element ΔL at point x, y (c) Centroid located at

5.1C First Moments of Areas and Lines

- The coordinates of the centroid of an area can be determined by

$$\bar{x} = \frac{\int x dA}{A}, \qquad \bar{y} = \frac{\int y dA}{A}$$

- First Moment Of Area. Here area similar to force is assumed to have moments about certain axis. The moment of the area is given by:

$$\boldsymbol{Q_y = \bar{x}A = \int xdA = First\ Moment\ of\ Area\ with\ Respect\ to\ y}$$

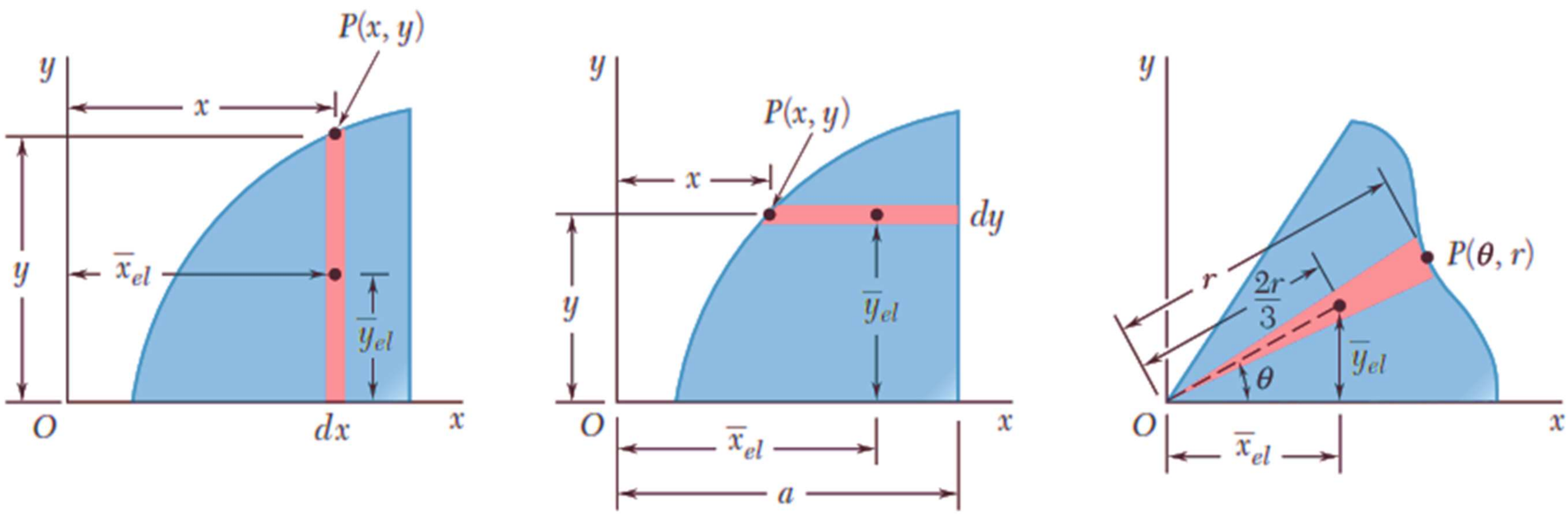
$$\boldsymbol{Q_x = \bar{y}A = \int ydA = First\ Moment\ of\ Area\ with\ Respect\ to\ x}$$

- Note that the First Moment Of Area about a centroidal axis shall be zero.

5.2A Determination of Centroid by Integration

- When the area is bonded by defined curve, it is centroid can be determined using integration by defining several integration element as shown below.

$$\bar{x} = \frac{\int x dA}{A}; \quad \bar{y} = \frac{\int y dA}{A}$$



(a)

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = y/2 \qquad dA = y\,dx$$

(b)

$$\bar{x}_{el} = \frac{a+x}{2} \qquad \bar{y}_{el} = y \qquad dA = (a-x)\,dy$$

(c)

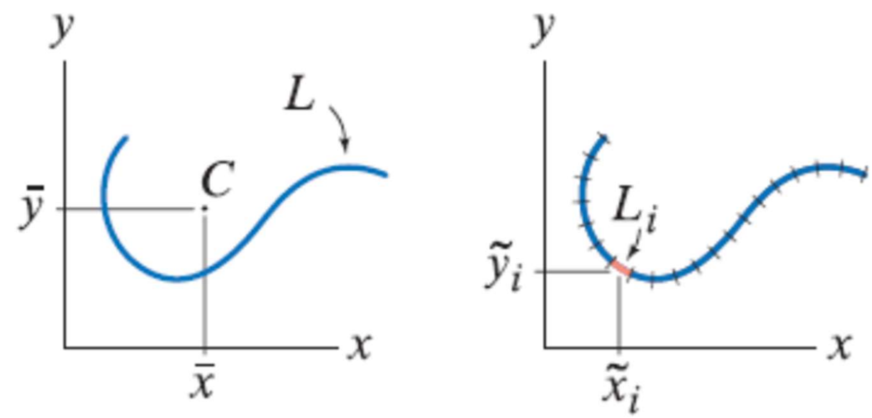
$$\bar{x}_{el} = \frac{2r}{3}\cos\theta \qquad \bar{y}_{el} = \frac{2r}{3}\sin\theta \qquad dA = \frac{1}{2}r^2\,d\theta$$

Centroids and areas of differential elements. (a) Vertical rectangular strip; (b) horizontal rectangular strip; (c) triangular sector.

5.2A Determination of Centroid by Integration

- When the line is defined by curve, it is centroid can be determined using integration by defining integration element similar to the previous procedure and as shown below.

$$\bar{x} = \frac{\int x\, dL}{L} \qquad \bar{y} = \frac{\int y\, dL}{L}$$

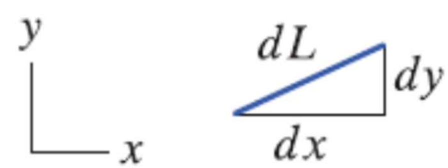


Where

$$dL = \sqrt{(dx)^2 + (dy)^2}.$$

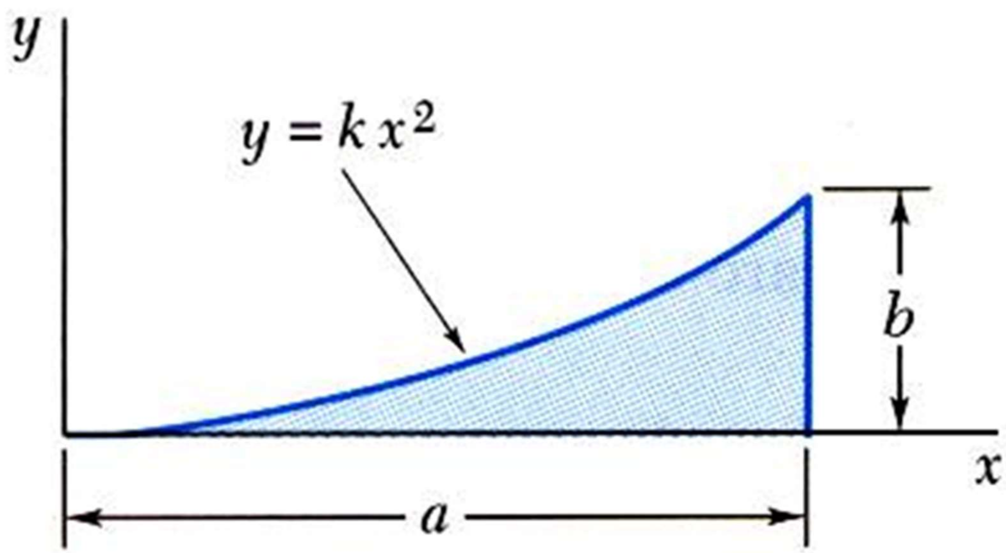
$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.$$



Sample Problem 5.4

Determine by direct integration the location of the centroid of a parabolic spandrel.

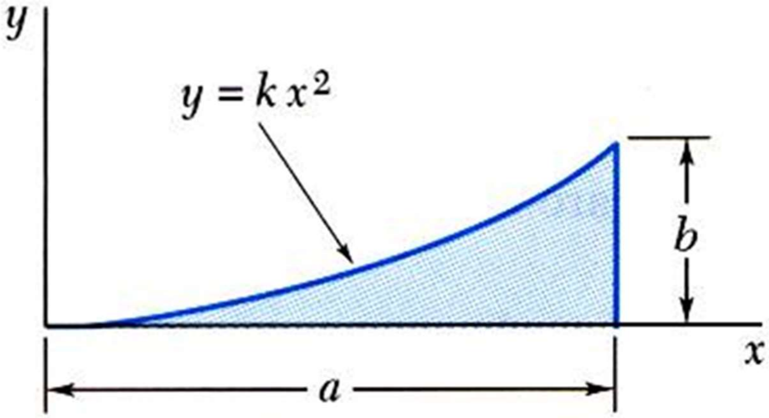


- First, estimate the location of the centroid by inspection. justify your answer.

SOLUTION:

- Determine the constant k.
- Select either vertical or horizontal strips.
- Evaluate the total area.
- Using selected strip, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem 5.4



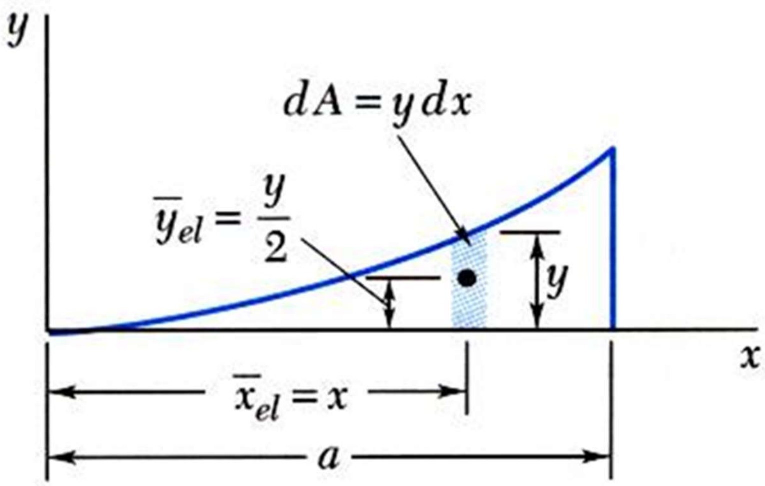
1. Determine k

$$y = k\,x^2$$

$$b = k\,a^2 \Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

2. Select a strip



3. Evaluate the total area.

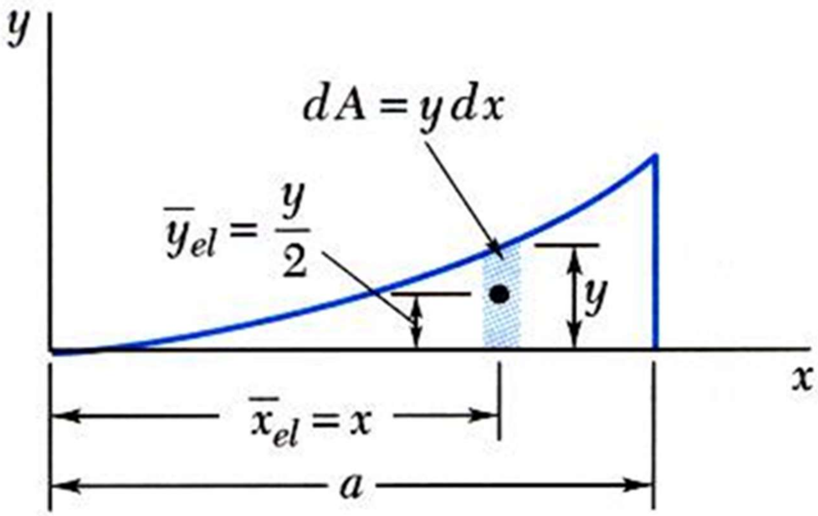
$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem 5.4

4. Using vertical strips, perform a single integration to find the first moments.

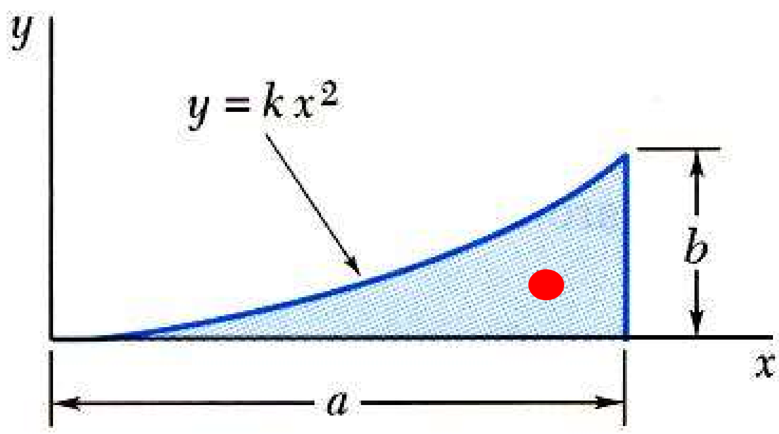


$$Q_y = \int \bar{x}_{el} dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$



5. Evaluate the centroid coordinates.

$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4}$$

$$\bar{x} = \frac{3}{4}a$$

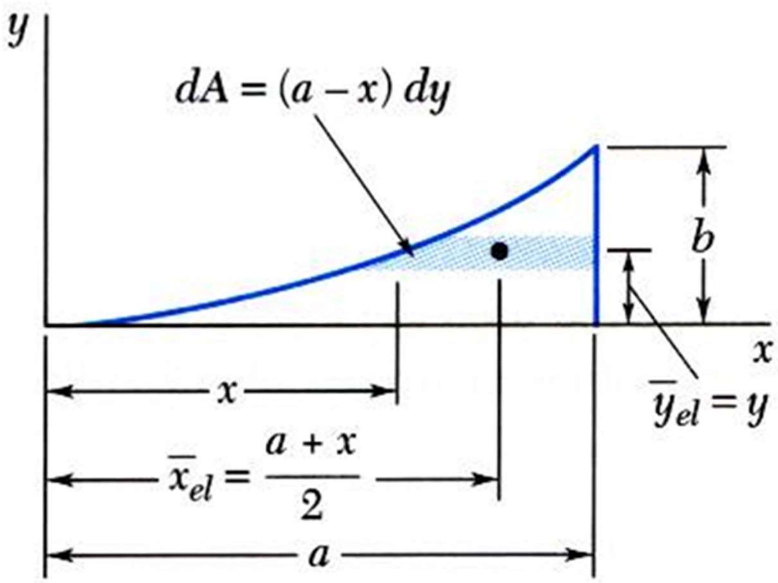
$$\bar{y}A = Q_x$$

$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\bar{y} = \frac{3}{10}b$$

Sample Problem 5.4

- Or, using horizontal strips, perform a single integration to find the first moments.



$$Q_y = \int \overline{x}_{el} dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2-x^2}{2} dy$$

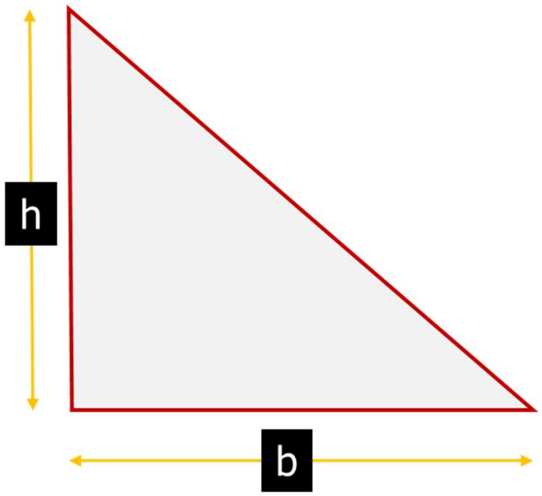
$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y\right) dy = \frac{a^2 b}{4}$$

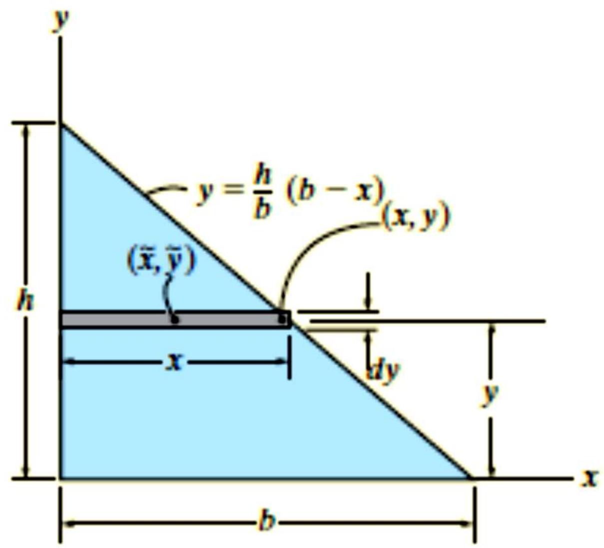
$$Q_x = \int \overline{y}_{el} dA = \int y(a-x)dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2}\right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2}\right) dy = \frac{ab^2}{10}$$

Sample Problem

- Determine the distance y measured perpendicular to leg b to the centroid of the area of the triangle shown



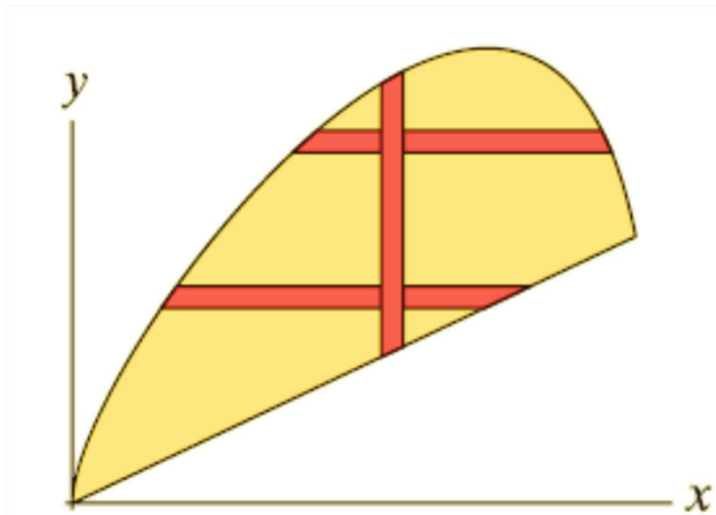


- The area of the element is $dA = x\,dy = \frac{b}{h}(h - y)dy$, and its centroid distance $\tilde{y} = y$ from the x axis.

$$\bar{y} = \frac{\int_0^h y dA}{A} = \frac{\int_0^h y\left[\frac{b}{h}(h - y)dy\right]}{\frac{1}{2}bh} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} = \frac{h}{3}$$

NOTE: Selection of integration element

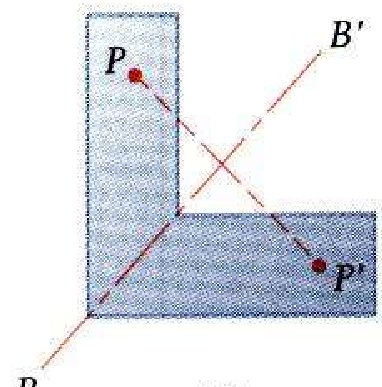
- Usually, the choice between using a vertical or horizontal strip is equally good, but in some cases, one choice is much better than the other. For example, for the area shown below, is a vertical or horizontal strip a better choice, and why?



- Vertical strip is better choice because of continuity. If horizontal step to be selected two integration functions are needed to cover the area.

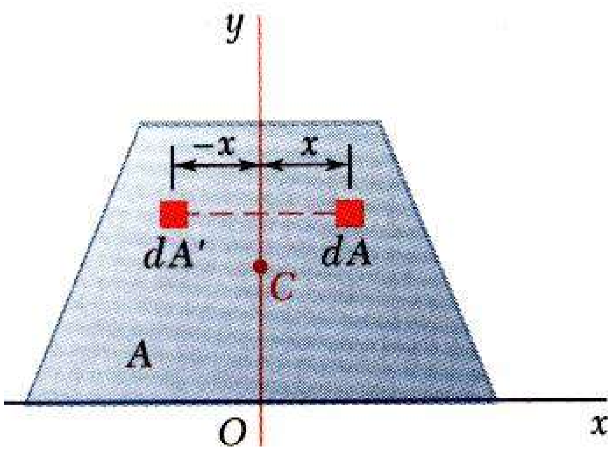
Sections Symmetry

- An area is symmetric with respect to an axis *BB′* if for every point *P* there exists a point *P′* such that *PP′* is perpendicular to *BB′* and is divided into two equal parts by *BB′.*



(a)

- The first moment of an area with respect to a line of symmetry is zero.

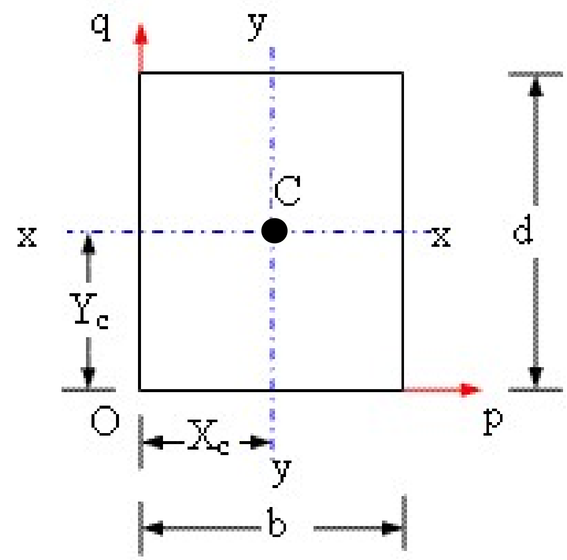


About y

$$Q_y = \bar{x}A = \int x dA = 0$$

About x

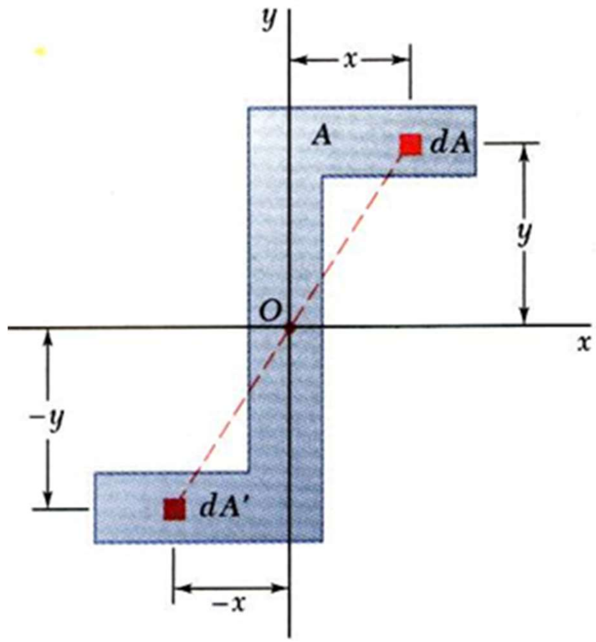
$$Q_x = \bar{y}A = \int y dA = 0$$



- If an area possesses a line of symmetry, its centroid lies on that axis and If an area possesses two lines of symmetry, its centroid lies at their intersection.

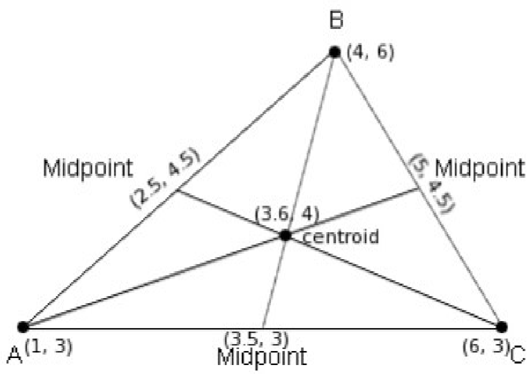
Sections Symmetry

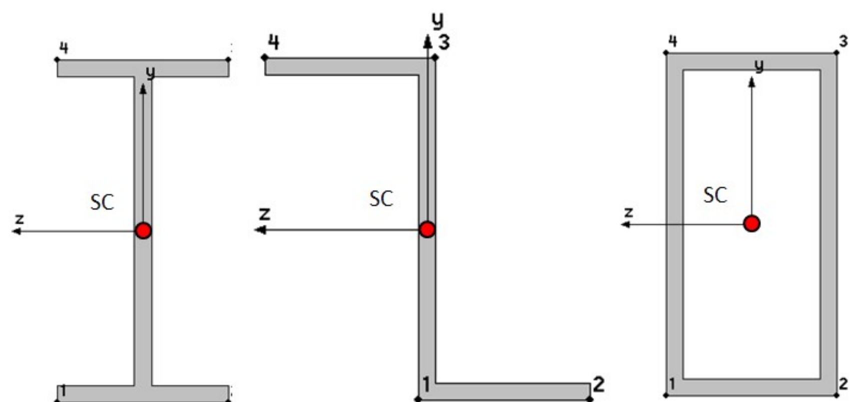
- **Symmetry about point.** An area is symmetric with respect to a center O if for every element dA at (x,y) there exists an area dA' of equal area at (-x,-y).



- The centroid of the area that area symmetrical about a point coincides with the center of symmetry.

- The centroid of symmetrical areas can be directly determined from the conditions of symmetry (see examples below).



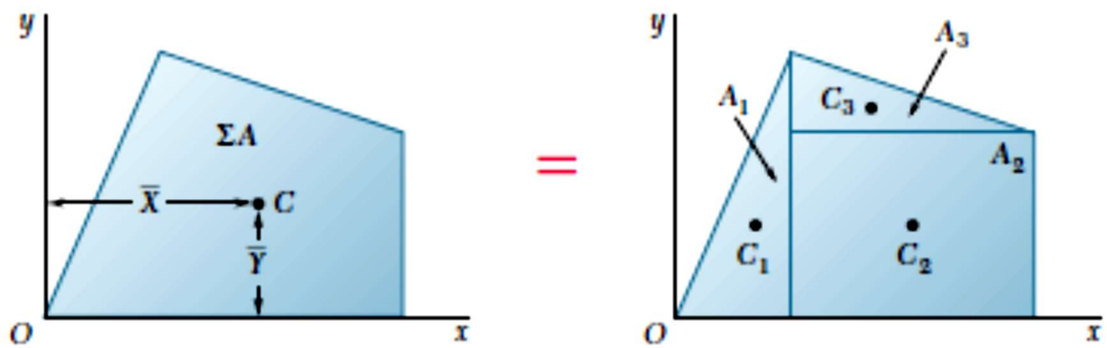


5.1D Composite Plates and Wires

- Rather than integration, If the area consists of several basic shapes of known centroids combined together, the first moment of area of the composite shape can be calculated as follows.

$$Q_y = \bar{X}\sum A_i = A_i\bar{x_i}$$

$$Q_x = \bar{Y}\sum A_i = A_i\bar{y_i}$$



- Therefore $\bar{X} and\ \bar{Y}$ of these areas can be calculated as:

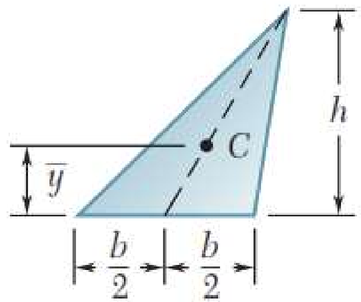
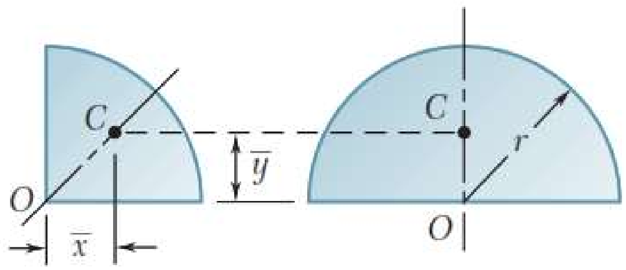
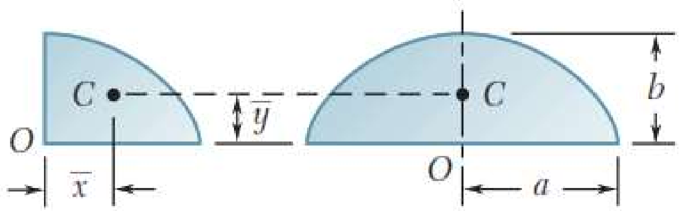
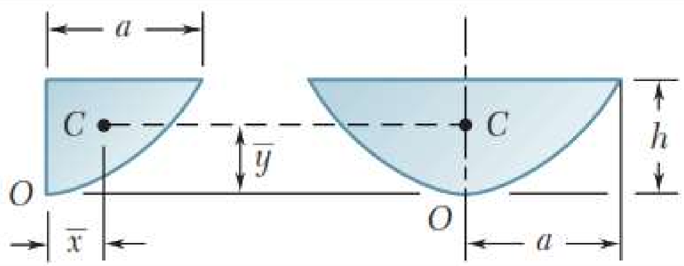
$$\bar{X} = \frac{\sum A_i x_i}{\sum A_i}; \quad \bar{Y} = \frac{\sum A_i y_i}{\sum A_i}$$

- Similarly for lines/ wires

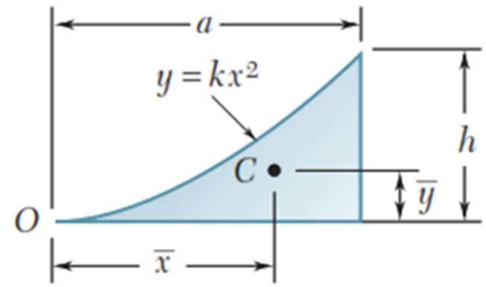
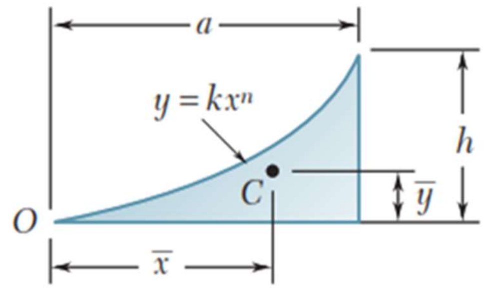
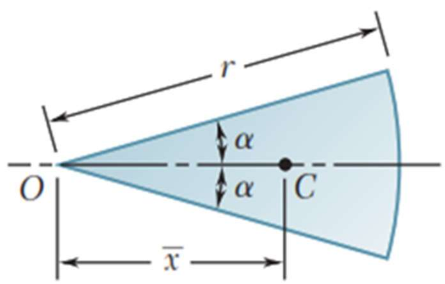
$$\bar{X} = \frac{\sum L_i x_i}{\sum L_i}; \quad \bar{Y} = \frac{\sum L_i y_i}{\sum L_i}$$

- There are ready made tables for the most frequent or basic sections - see next pages – that can be used for such purpose.

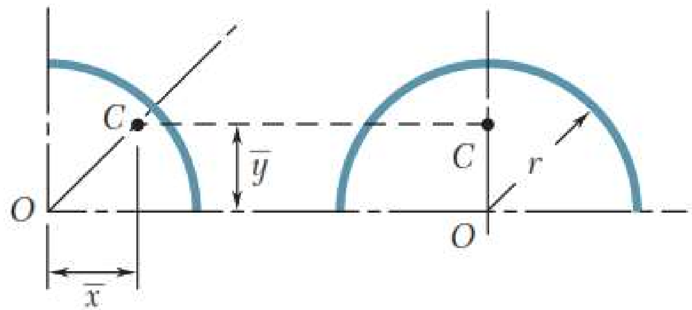
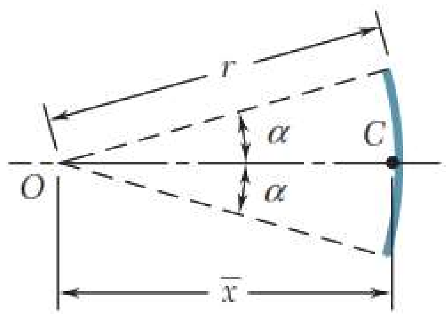
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$

Centroids of Common Shapes of Areas

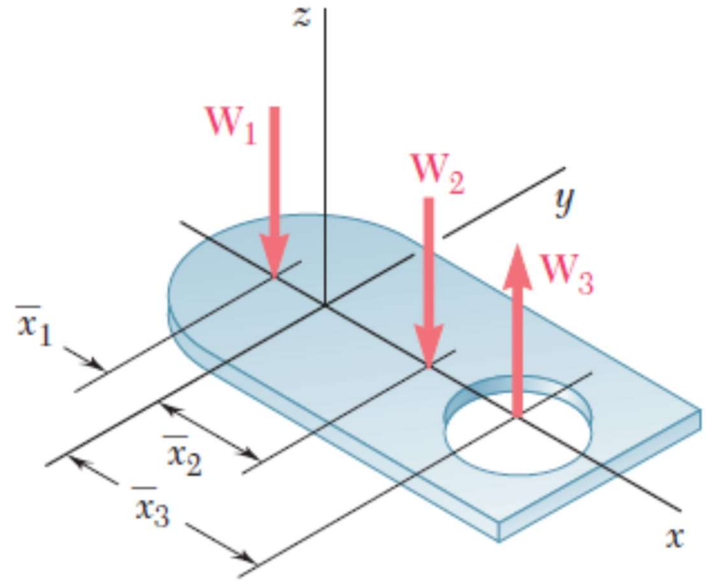
Shape		$\bar{x}$	$\bar{y}$	Area
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

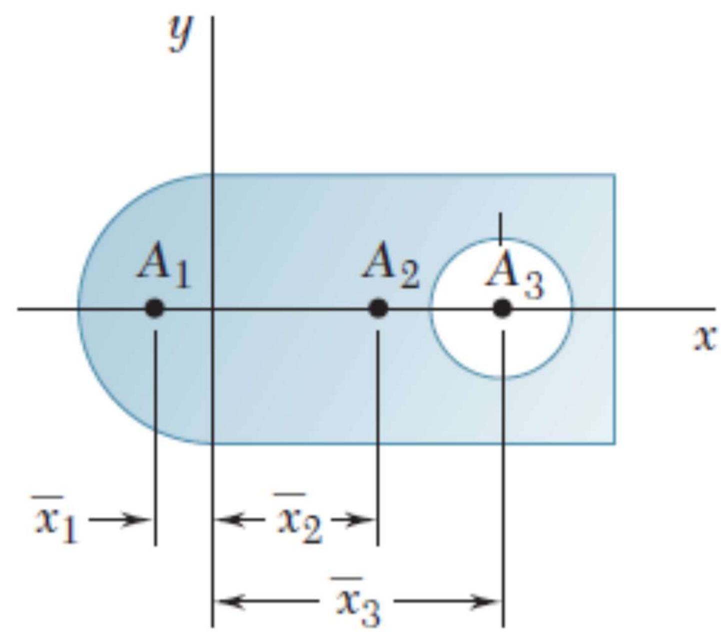
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Notes on composite area approach

- When calculating the centroid of a composite area, note that if the centroid of a component area has a negative coordinate distance relative to the origin, or if the area represents a hole, then the first moment is negative.

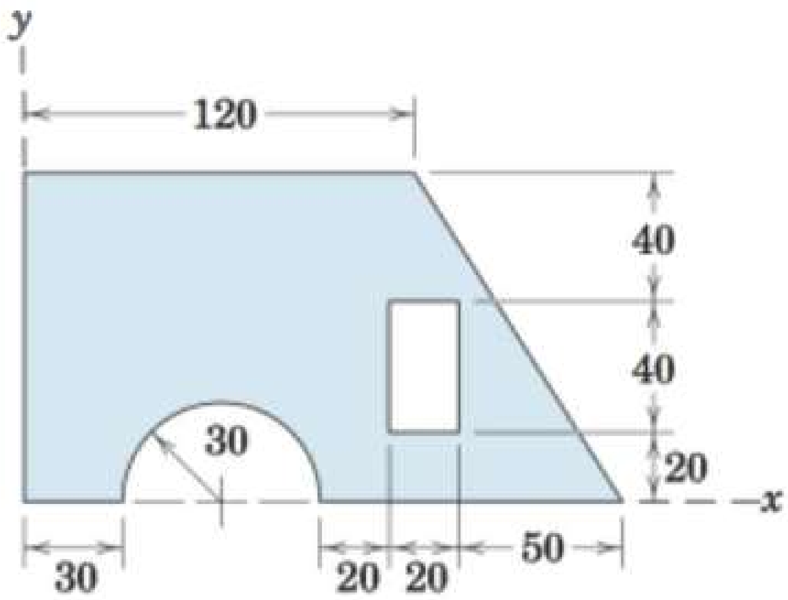


	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	−	+	−
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	−	−

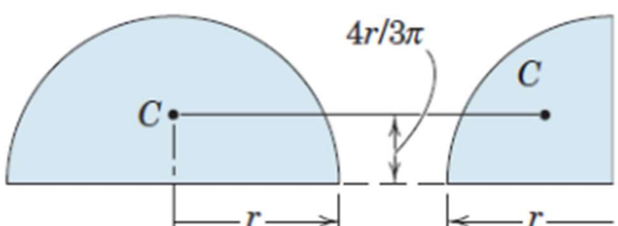


Example - Composite areas

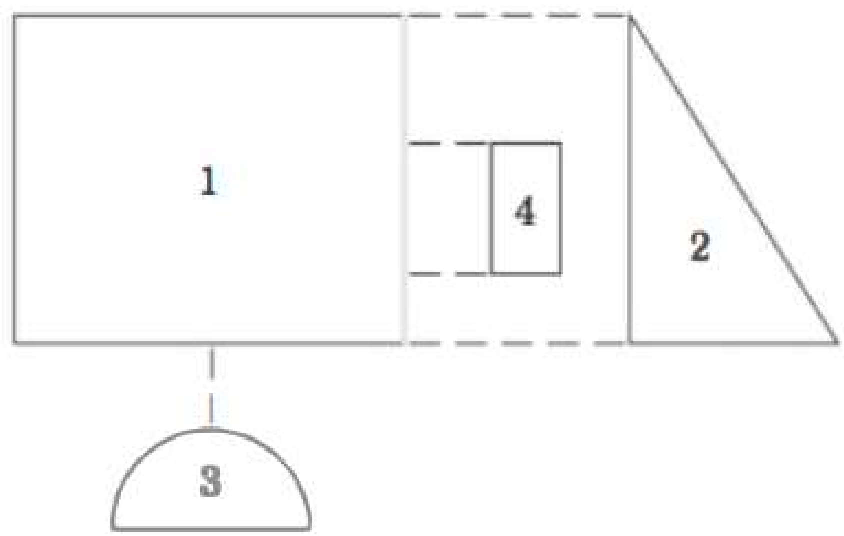
Locate the centroid o f the shaded area.



Dimensions in millimeters



- The composite area is divided into the four elementary shapes shown in figure below. The centroid locations of all these shapes may be obtained from Tables. Note that the areas of the "holes" (parts 3 and 4) shall be taken as negative in the calculations.



Example - Composite areas

- Create a table to facilitate the calculation which shows for each part the are, and the coordinates of the part centroid, then apply the shown equations.

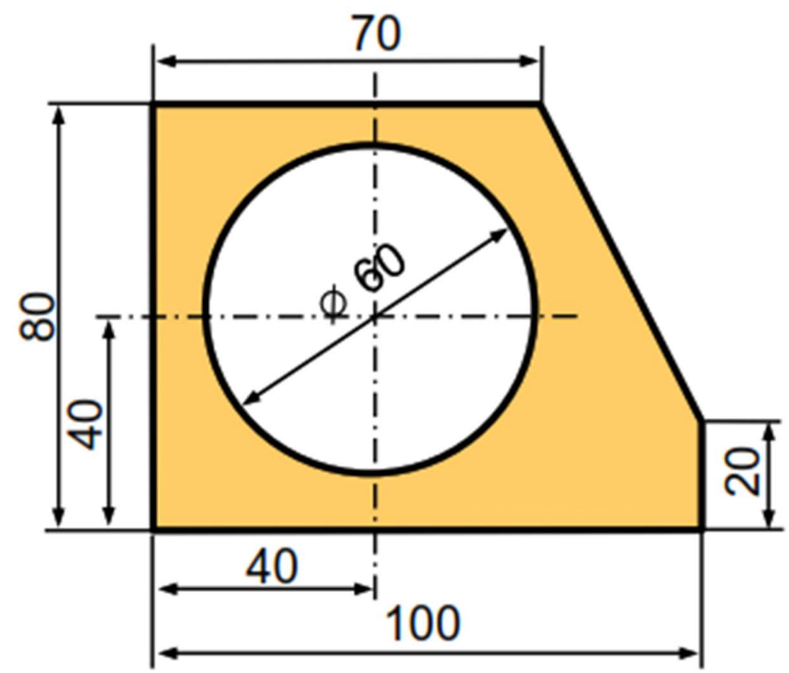
PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

$$\bar{X} = \frac{\sum_1^4 A_i x_i}{\sum_1^4 A_i} = \frac{959{,}000}{12{,}790} = 75.0\ mm$$

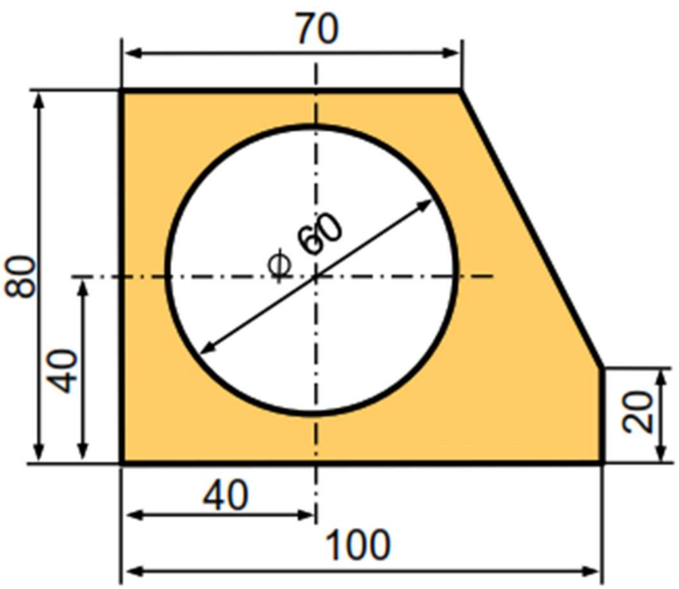
$$\bar{Y} = \frac{\sum_1^4 A_i y_i}{\sum_1^4 A_i} = \frac{650{,}000}{12{,}790} = 50.8\ mm$$

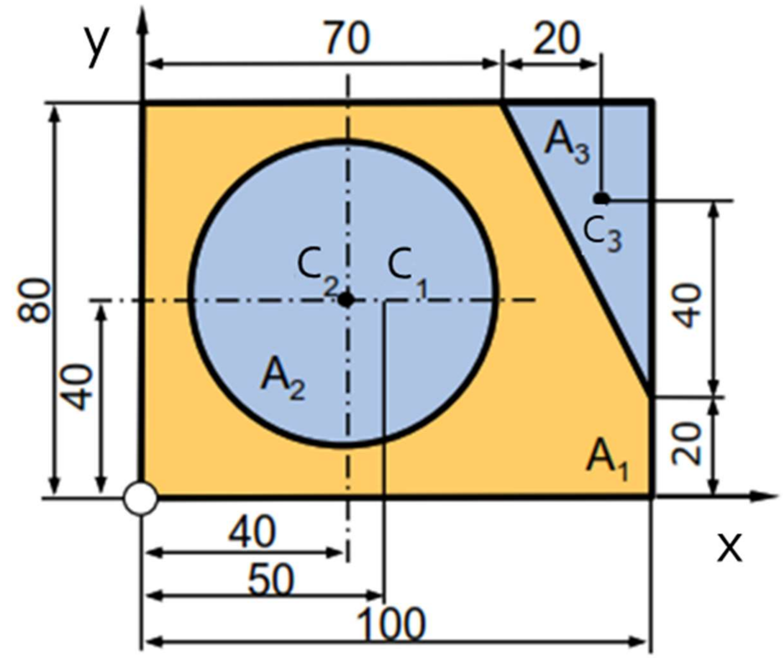
Example2 - Composite areas

Locate the centroid o f the shaded area in reference to the left corner of the section. All Dimensions are in mm



Example2 - Composite areas





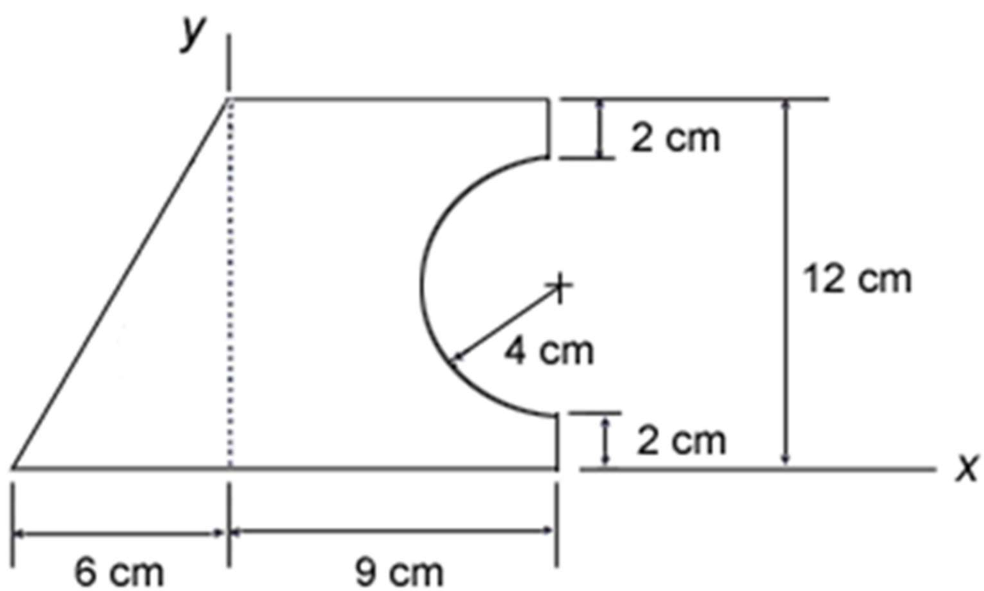
$$\begin{aligned} A &= A_1 - A_2 - A_3 \\ &= (100 \cdot 80 - \pi \cdot 30^2 - 0{,}5 \cdot 30 \cdot 60) mm^2 \\ &= 4273 mm^2 \end{aligned}$$

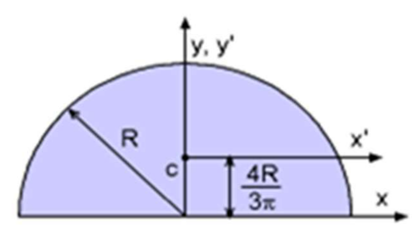
$$\bar{x} = \frac{1}{A} \cdot \sum_{i=1}^{3} \bar{x}_i A_i = \frac{50 \cdot 100 \cdot 80 - 40 \cdot \pi \cdot 30^2 - 90 \cdot 0{,}5 \cdot 30 \cdot 60}{4273} mm \Rightarrow \underline{\underline{\bar{x} = 48{,}19\, mm}}$$

$$\bar{y} = \frac{1}{A} \cdot \sum_{i=1}^{3} \bar{y}_i A_i = \frac{40 \cdot 100 \cdot 80 - 40 \cdot \pi \cdot 30^2 - 60 \cdot 0{,}5 \cdot 30 \cdot 60}{4273} mm \Rightarrow \underline{\underline{y = 35{,}78\, mm}}$$

Example3 - Composite areas - HW

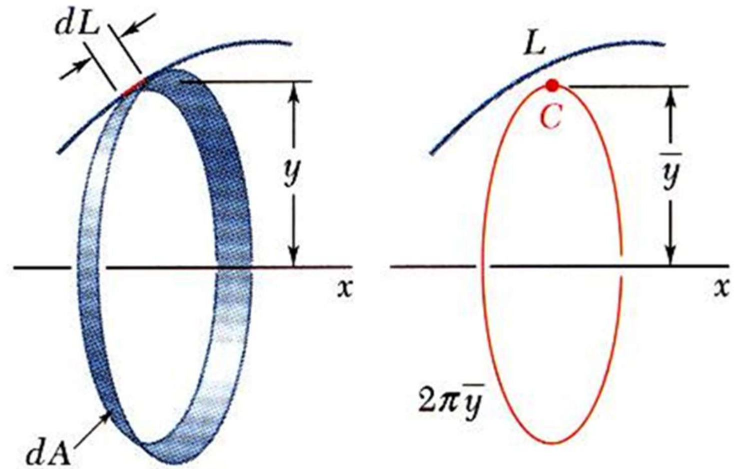
Determine the centroids of the area shown in the figure in reference to the X & Y axis shown. (All dimensions in cm.). Tabulate your calculations.





5.2B Theorems Of Pappus-Guldinus

- **Theory I**: If a plane curve is rotated about a fixed axis the curve will generate a surface. The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.



- To prove the theory an element dL rotating around x, produce an area dA equal to $\boldsymbol{dA = 2\pi y dL}$

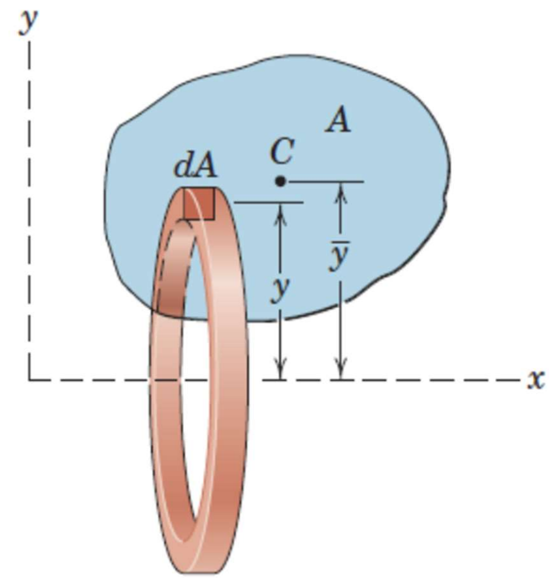
$$\rightarrow A = 2\pi \int y dL$$

But from previous equations $\bar{y}L = \int y dL$

$\rightarrow \boldsymbol{A = 2\pi \bar{y} L}$ which is what the theorem stated.

5.2B Theorems Of Pappus-Guldinus

- **Theory II:** If a plane area is rotated about a fixed axis the area will generate a body. The volume of the body of revolution is equal to the generating area times the distance traveled by its centroid through the rotation.



- To prove the theory an element dA is rotated around x, then the volume produced $dV = 2\pi y dA$

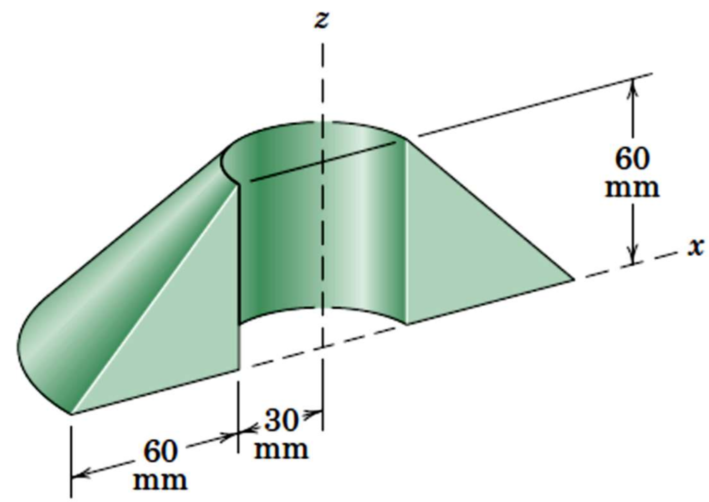
$$\rightarrow V = 2\pi \int y dA$$

But we know that $\bar{y}A = \int y dA$

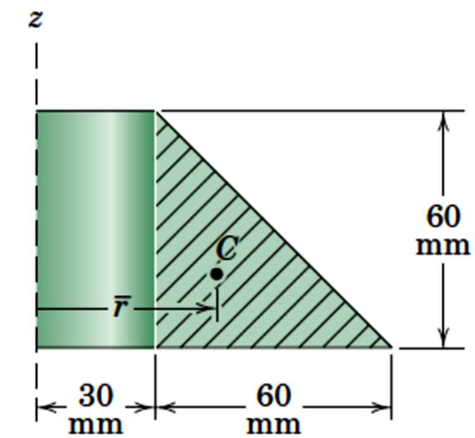
$$\rightarrow V = 2\pi \bar{y} A$$

Example - Theorems Of Pappus-Guldinus

- Calculate the volume V of the solid generated by revolving the 60-mm right triangular area through 180° about the z-axis. If this body were constructed of steel, what would be its mass m?



- <u>With the angle of revolution $\theta = 180° = \pi$</u>



$$V = \theta \bar{r} A = \pi[30 + \tfrac{1}{3}(60)][\tfrac{1}{2}(60)(60)] = 2.83(10^5)\ \text{mm}^3$$

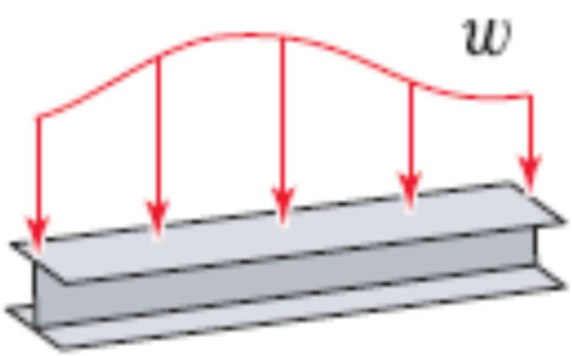
- The mass of the body is then

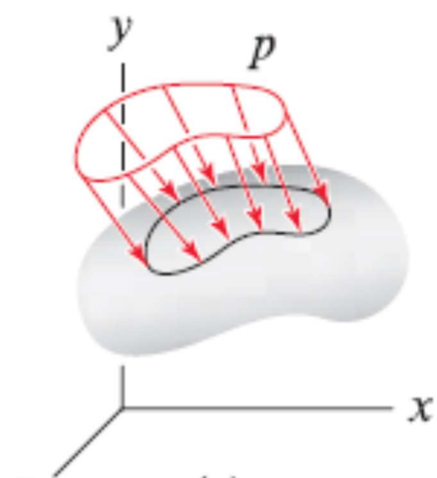
$$m = \rho V = \left[7830\ \frac{\text{kg}}{\text{m}^3}\right][2.83(10^5)\ \text{mm}^3]\left[\frac{1\ \text{m}}{1000\ \text{mm}}\right]^3$$

$$= 2.21\ \text{kg}$$

5.3A Distributed Loads on Beams

- Distributed forces are forces that are distributed along a line, over a surface, or throughout a volume.
- There are many examples in engineering analysis of distributed loads. It is convenient in some cases to represent such loads as a concentrated force located at the centroid of the distributed load





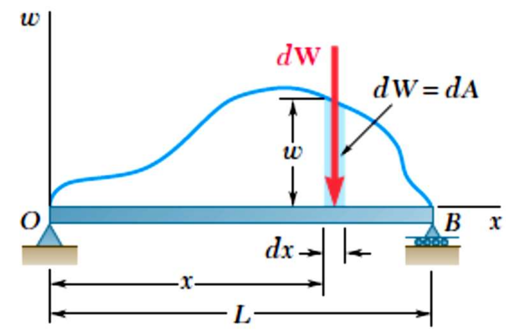
(a)

5.3A Distributed Loads on Beams

- Consider a beam supporting a distributed load as shown in figure (a).

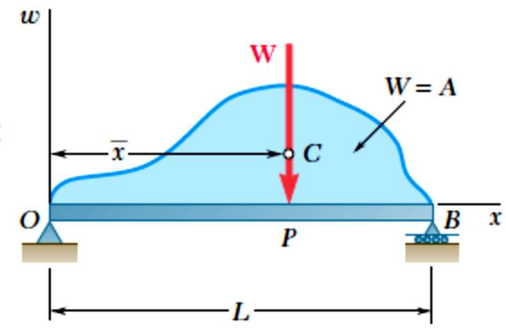
- The magnitude of the force exerted on an element of the beam with length dx is $dw = w\,dx$, accordingly the total load supported by the beam can be calculated as

$$W = \int_0^L w dx$$ = to Area under load curve



(a)

=



(b)

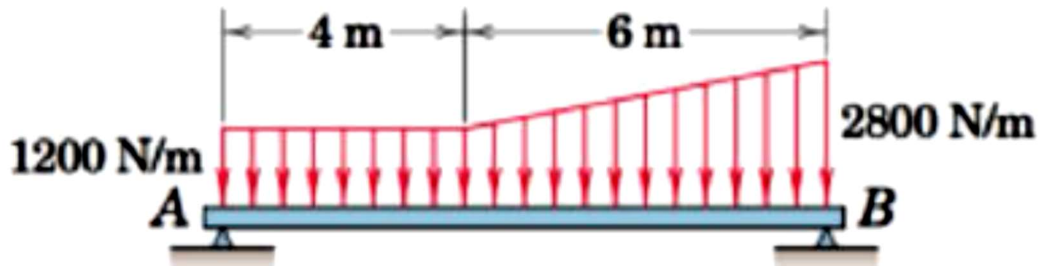
- Point of application P of the load W can be calculated using the equivalent force concept as

$$(op)W = \int_0^L x dw\,, \qquad or \quad \bar{X}W = \int_0^L xwdx\,, \qquad then \quad \bar{X} = \frac{\int_0^L xwdx}{W}$$

- This implies that line of action of equivalent load W passes through the centroid of the area under the load curve.

Example 1-Distributed Loads on Beams

Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



$$\sum M_A = 0$$

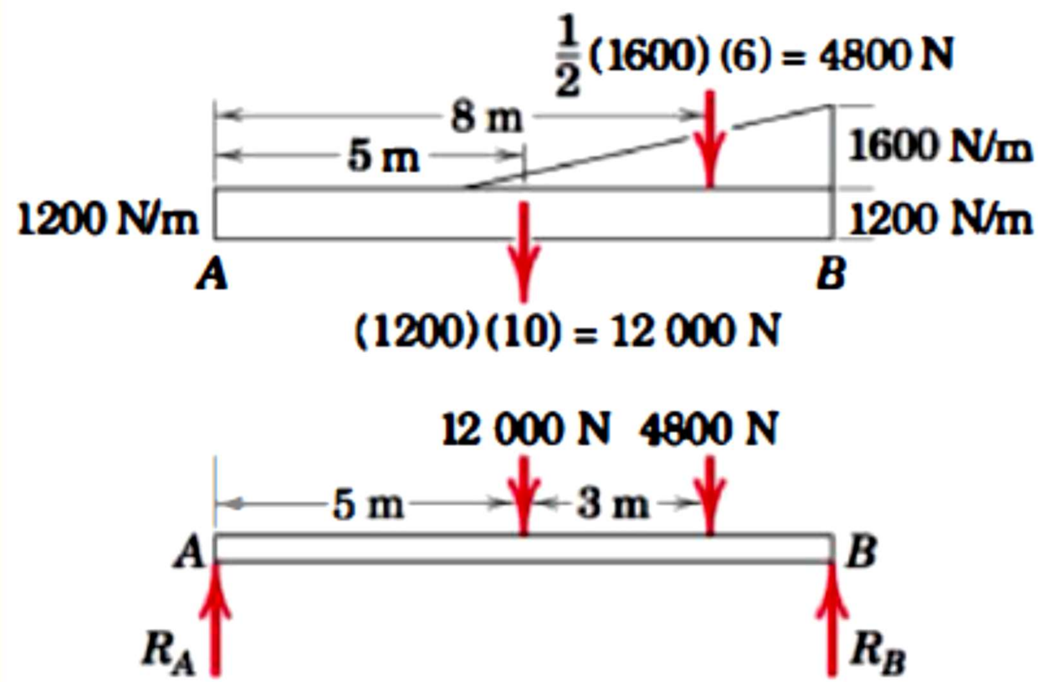
$$12\,000(5) + 4800(8) - R_B(10) = 0$$

$$R_B = 9840 \text{ N or } 9.84 \text{ kN}$$

$$\sum M_B = 0$$

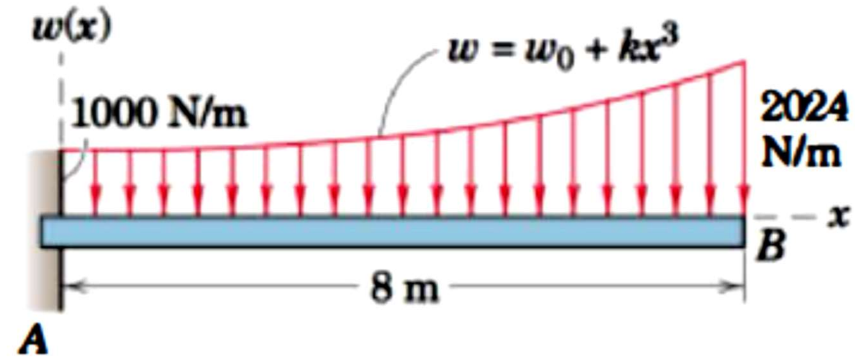
$$R_A(10) - 12\,000(5) - 4800(2) = 0$$

$$R_A = 6960 \text{ N or } 6.96 \text{ kN}$$



Example 2-Distributed Loads on Beams

Example: Determine the reaction at the support A of the loaded cantilever beam.



Solution.

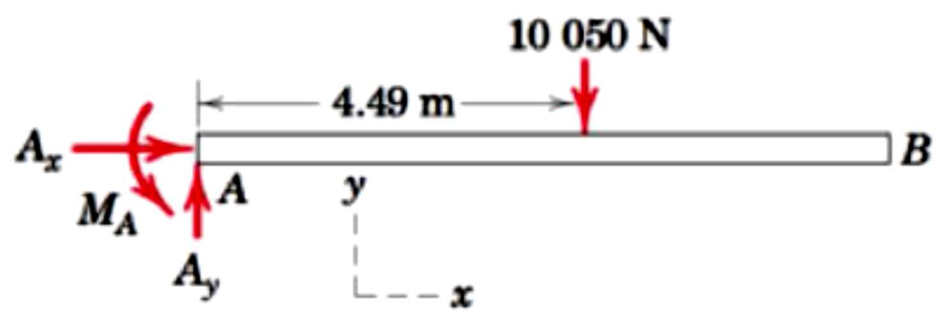
- The constants in the load distribution are found to be $w_0 = 1000\ N/m$ and $k = 2\ N/m^4$
- The load R is then

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Bigg|_0^8 = 10\,050\ \text{N}$$

- The x-coordinate of the centroid of the area is found by

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx$$

$$= \frac{1}{10\,050}\left(500x^2 + \frac{2}{5}x^5\right)\Big|_0^8 = 4.49\ \text{m}$$



$$\sum M_A = 0$$

$$M_A - (10\,050)(4.49) = 0$$

$$M_A = 45\,100\ \text{N}\cdot\text{m}$$

$$\sum F_y = 0, \qquad A_y = 10\,050\ N$$