chapter 1 :-

- and = "$\cap$" Intersection
- or = "$\cup$" union
- De Morgan's Law :- [1] $\overline{A \cup B} = \bar{A} \cap \bar{B}$

  [2] $\overline{A \cap B} = \bar{A} \cup \bar{B}$

- Probability :- [1] Apriori $P(A) = \dfrac{\text{number of sample outcomes in } A}{\text{number of sample outcomes in } S}$

[2] A posterior $p(A) = \lim\limits_{n \to \infty} \dfrac{\text{number of times } A \text{ occur}}{\text{number of trials}}$

[3] Subjective

[4] Axiomatic $\rightarrow P(A) \geq 0$

$\rightarrow P(S) = 1$

$\rightarrow$ If two events are disjoint (mutually exelusive) if $A \cap B = \emptyset$

then $P(A \cup B) = p(A) + p(B) - p(A \cap B)$

$\rightarrow$ Three events are disjoint $\Rightarrow P(A \cup B \cup C) = p(A) + p(B) + p(C)$

$\Rightarrow P(\bar{A}) = 1 - P(A)$

$\Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$\Rightarrow P(B/A) = \dfrac{P(B \cap A)}{P(A)}$ (given), $P(A \cap B) = -P(A)\,p(B/A)$

$- P(B)\,p(A/B)$

$\Rightarrow$ two events are said to be statistically independent :- $P(A \cap B) = P(A)p(B)$

$P(A \cap B \cap C) = P(A) \cdot p(B) \cdot p(C)$

$\Rightarrow$ Counting techniques :-

* Multiplication rule (independent)
* permutation $\rightarrow$ without replacement (repetition is not allowed, order is important $\rightarrow P_k^n = \dfrac{n!}{(n-k)!}$
* Combination $\rightarrow$ // // (order is not important) $P_k^n = \dfrac{n!}{(n-k)!\,k!}$

Chapter 2 :-

- Probability Mass Function (PMF)

$* P(X=x) \geq 0$

$* \sum_{-\infty}^{\infty} P(X=x) = 1$

- Comulative distribution function (CDF)

$* F_X(x) = P(X \leq x)$

$* F_X(-\infty) = 0$ ← تبدأ

$* F_X(\infty) = 1$ ← تنتهي

$* P(X=x) = F(x) - F_X(x^-)$

- Probability density function (PdF)

$* f_X(x) \geq 0$

$* \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

⇒ Mean and Expected value

$E\{x\} = \mu_x$

$$\overset{D}{=} \sum_{x=-\infty}^{\infty} x\, p(X=x)$$

$$\overset{C}{=} \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

⇒ Variance of X

$\sigma_x^2 = Var\{x\}$

$E\{(X-\mu_x)^2\}$

$$\overset{D}{=} \sum_{x=-\infty}^{\infty} (x-\mu_x)^2 p(X=x)$$

$$\overset{C}{=} \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)\, dx$$

⇒ Standard deviation of X ⇒ $\sigma_x = \sqrt{\sigma_x^2}$

⇒ * Mode of the distribution = $\frac{d f_X(x)}{dx} = 0$, then check the end of the interval

⇒ Median of distribution $p(X \leq x_{median}) = p(x \geq x_{median}) = \frac{1}{2}$

⇒ $\int_{-\infty}^{x_o} f_X(x)\,dx = 1/2$

⇒ - $E\{ax\} = a E\{x\}$

- $E\{b\} = b$
- $E\{x^2\} = \sigma_x^2 + \mu_x^2$
- $\sigma_x^2 = E\{x^2\} - \mu_x^2$

Common discrete distribution

[1] Binomial distribution

- repeated for n times.
- only two outcomes.
- trials are independent.

X = number of success in the n trials

$$\rightarrow p(X=x) = \begin{cases} \binom{n}{x} p^x [1-p]^{n-x} & , x = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow \mu_X = np \quad \rightarrow \sigma_X^2 = np[1-p]$

[2] Geometric distribution

- repeated and only two outcomes
- trials are independent
- X = number of trials to the first success

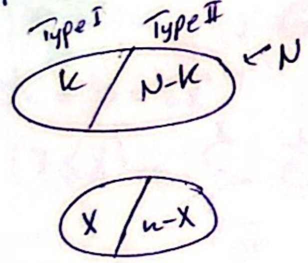
← اول حدوث ال success

$\rightarrow p(X=x) = p(F)^{x-1} p(S)$

$\rightarrow \mu_X = \frac{1}{p(S)} \quad , \quad \sigma_X^2 = \frac{p(F)}{p(S)^2}$

[3] Hyper-Geometric

- The probability of obtaining (x) items in a selection of (n) times without replacement



$$\rightarrow p(X=x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} & x = 0, \dots \min(n, K) \\ 0 & \text{o.w} \end{cases}$$

$\rightarrow p = \frac{K}{N} \; , \; \mu_X = np \; , \; \sigma_X^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$

[4] Poison distribution

$$P(X=x) = \begin{cases} \dfrac{e^{-b} b^x}{x!} & x = 0, 1, \dots \\ 0 & \text{o.w} \end{cases}$$

← تعتمد على width للفترة الزمنية

← تعتمد على الفترة الزمنية

$\rightarrow b = \lambda * T$ (λ: rate, T: period)

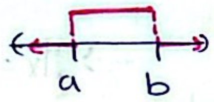
$\rightarrow \mu_X = b$

$\sigma_X^2 = b$

Common Continuous distribution

[1] uniform distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{o.w} \end{cases}$$



$\rightarrow \mu_X = \frac{a+b}{2}$, $\sigma_X^2 = \frac{(b-a)^2}{12}$

[2] Exponential distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{o.w} \end{cases}$$

$\rightarrow \lambda > 0$ $\rightarrow \mu_X = \frac{1}{\lambda}$, $\sigma_X^2 = \frac{1}{\lambda^2}$

[3] Normal (Gaussian) distribution

$\rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$ $\quad -\infty < x < \infty$

$\rightarrow$ Mean = mode = median

$\Rightarrow$ standard normal random variable
- $\rightarrow$ Mean = 0 , $\sigma_X^2 = 1$
- $\rightarrow$ from the table $\phi(x)$
- $\rightarrow$ $\phi(-a) = 1 - \phi(a)$

ex $\Rightarrow$ $P(X \leq 3.14) = \phi(3.14)$

ex. $P(X > 2.5) = 1 - \phi(2.5)$

$\Rightarrow$ not standard gaussian $\rightarrow$ standard gaussian

ex) $P(X \leq 7) \Rightarrow \phi\left(\frac{7-\mu_X}{\sqrt{\sigma_X^2}}\right)$

Approximation

Hyper-Geometric $\longrightarrow$ Binomial

$$P(X=x)=\begin{cases}\dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} & x=0,1\ldots\min(n,k)\\ 0 & \text{o.w}\end{cases} \xrightarrow[P=\frac{k}{N}]{} P(X=x)=\begin{cases}\binom{n}{x}P^{x}(1-P)^{n-x} & x=0,1\ldots\\ 0 & \text{o.w}\end{cases}$$

Binomial $\longrightarrow$ Poisson

$$P(X=x)=\lim_{n\to\infty}\binom{n}{x}(1-p)^{n-x} \longrightarrow P(X=x)=\frac{e^{-b}(b)^{x}}{x!}\ ,\ b=np$$

Binomial $\longrightarrow$ Gaussian

$$F_X(x)=P(X\leq x)$$

$$\sum_{x=0}^{x}\binom{n}{x}p^{x}(1-p)^{n-x} \longrightarrow \mu_X=np\ ,\ \sigma_X^2=np(1-p)$$

$$P(X=x)=\phi\left(\frac{x-\mu_X}{\sqrt{\sigma_X^2}}\right)$$

poisson $\longrightarrow$ Gaussian

"$\mu_X=b=\sigma_X^2$"

$$P(X\leq x)=\sum_{x=0}^{x}\frac{e^{-b}b^{x}}{x!} \longrightarrow P(X=x)=\phi\left(\frac{x-b}{\sqrt{b}}\right)$$

Transformation of R.V

$\longrightarrow$ Discrete :-

$x - P(X=x) - y - P(Y=y)$

$\longrightarrow$ Continuous :-

1- $f_X(x)$

2- write x in order of y $\rightarrow x=g(y)$

3- Find $\dfrac{dy}{dx}$

4- $f_Y(y)=\left.\dfrac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=g(y)}$

Chapter 3 :- two random variables

⇒ Discrete : $\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} P(X=x, Y=y) = 1$

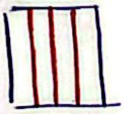
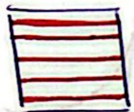
⇒ Continuous :- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = 1$

⇒ marginal PMF of X $\longrightarrow \sum_{y=-\infty}^{\infty} P(X=x, Y=y)$

⇒ marginal PMF of Y $\longrightarrow \sum_{x=-\infty}^{\infty} P(X=x, Y=y)$

⇒ X and Y are said to be ~~statically~~ statistically independent

$$P(X=x, Y=y) = P(X=x)P(Y=y)$$

⇒ double integration ⇒ dydx  ⇒ dxdy 

$\longrightarrow \int_{-\infty}^{\infty} k\,dy = \text{area} \longrightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\,dy\,dx = \text{area} * dx$ = volume slice

⇒ $\rho_{XY} = \dfrac{E\{(X-\mu_X)(Y-\mu_Y)\}}{\sigma_X\sigma_Y} = \dfrac{E(XY)-\mu_X\mu_Y}{\sigma_X\sigma_Y}$

⇒ Notes :-

$\boxed{1}$ $E\{g(x,y)\} \overset{D}{=} \sum_{y=-\infty}^{\infty}\sum_{x=-\infty}^{\infty} g(x,y)\,P(X=x, Y=y)$

$$\overset{C}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dy\,dx$$

$\boxed{2}$ $E\{aX+bY\} = aE\{X\} + bE\{Y\}$

R.V

$\boxed{3}$ $E\{aXY\} = \sum_{y=-\infty}^{\infty}\sum_{x=-\infty}^{\infty} axy\,P(X=x, Y=y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} axy\, f_{XY}(x,y)\,dy\,dx$$

$\boxed{4}$ If X and Y are st. In.

$E\{aXY\} = aE\{X\}E\{Y\}$

$\boxed{5}$ Covariance μ_{XY}

$\mu_{XY} = E\{(X-\mu_X)(Y-\mu_Y)\}$

$\mu_{XX} = E\{(X-\mu_X)(X-\mu_X)\}$

$= E\{(X-\mu_X)^2\} = \sigma_X^2$

$\boxed{6}$ correlation coefficient

$\rho_{XY} = \dfrac{\mu_{XY}}{\sigma_X\sigma_Y}$, $-1 \leq \rho_{XY} \leq 1$

$\rho_{XY} = 0 \longrightarrow$ uncorrelated

$\rho_{XY} = \mp 1 \longrightarrow$ are fully correlated

⇒ If X and Y are S.I then X and Y are uncorrelated and if X and Y are uncorrelated then they may or may not be S.I

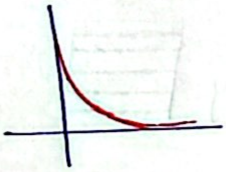
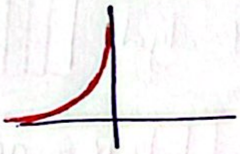
⇒ marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

⇒ // // of Y $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

⇒ X and Y are said to be st.I if $f_{XY}(x,y) = f_X(x) f_Y(y)$

⇒ <u>Note</u>: let $R = aX + bY$

- $\mu_R = E\{R\} = E\{a\mu_X + b\mu_Y\}$
- $\sigma_X^2 = Var\{R\} = a^2\sigma_X^2 + b^2\sigma_Y^2 + \underbrace{2ab\sigma_X\sigma_Y \rho_{XY}}_{2ab\,\mu_{XY}}$

⇒ e^{-x}  e^{x} 

<u>Chapter 4 and 5</u> :-

- $\bar{x}, \hat{\mu}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$
- $\hat{\sigma}_x^2, s_x^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu_x)^2$ (μ_x: real mean)
- $\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$ ($\hat{\mu}_x$: sample)
- $s_x, \hat{\sigma}_x = \sqrt{s_x^2}$
- coveriance $\hat{\mu}_{xy}, C_{xy}$

$$C_{xy} = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \hat{\mu}_x)(y_i - \hat{\mu}_y)$$

- Sample correlation coefficient

$$r_{xy} = \frac{C_{xy}}{s_x s_y}$$

⇒ * $s_x^2 = \frac{1}{n(n-1)}\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]$

⇒ * $C_{xy} = \frac{1}{n(n-1)}\left[n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right]$

⇒ mean square error $= \frac{1}{n}\sum_{i=1}^{n} (y_i - g(x_i))^2$, $g(x_i) = \alpha x_i + \beta$

→ $\frac{d\varepsilon}{d\alpha} = 0$, $\frac{d\varepsilon}{d\beta} = 0$ → matrix

⇒ $\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$

⇒ β, α (matrix)

$$\beta = \frac{\begin{vmatrix} \sum y_i & \sum x_i \\ \sum x_i y_i & \sum x_i^2 \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}} , \quad \alpha = \frac{\begin{vmatrix} n & \sum y_i \\ \sum x_i & \sum x_i y_i \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}}$$

$\beta = \hat{\mu}_y - \alpha \hat{\mu}_x$, $\alpha = \dfrac{C_{xy}}{S_x^2}$

$Y = \alpha X + \beta$

$Y = \dfrac{C_{xy}}{S_x^2} X + \hat{\mu}_y - \dfrac{C_{xy}}{S_x^2} \hat{\mu}_x$

$\dfrac{Y - \hat{\mu}_y}{S_y} = r_{xy} \left(\dfrac{X - \hat{\mu}_x}{S_x} \right)$

⇒ $\text{Var}\{aX + bY\}$
$= a^2 \sigma_x^2 + b^2 \sigma_y^2 + 2ab \underbrace{\mu_{xy}}_{\sigma_x \sigma_y \rho_{xy}}$

← لما يحكي average $p(\mu_x < x)$
⇒ $\phi\left(\dfrac{x - \hat{\mu}_x}{\hat{\sigma}_x}\right)$

⇒ $MSE = \dfrac{1}{n} \sum_{i=1}^{n} (Y_{measured} - Y_{theo})^2$

⇒ <u>Note</u> :- Linearization

$Y = aX^b$

$\ln[Y] = \ln[a] + b \ln[X]$

$u = \beta + \alpha r$ ⇒ $u = \ln Y$
$r = \ln X$

⇒ بحل matrix → $a = e^{\beta}$
$b = \alpha$

⇒ Properties of point estimator

(1) unbiased :- estimator unbiased if
$E\{\hat{\mu}_x\} = \mu_x$

(2) For an unbiased estimator, we prefer the one with minimum variance.

←

⇒ Likelihood

⇒ 1- بعوض 2- بشتق وبساوي بالصفر

⇒ Central Limit theorem

- $E\{\hat{\mu}_x\} = \mu_x$
- $\text{Var}\{\hat{\mu}_x\} = \dfrac{\sigma_x^2}{n}$

⇒ If X_1 and X_2 are S.I
$Y = X_1 + X_2 \rightarrow E\{Y\} = E\{X_1\} + E\{X_2\}$

$\Rightarrow P(\theta_1 \leq \theta \leq \theta_2) = 1-\alpha$

- θ :- unknown parameter
- $(1-\alpha)$:- confidence coefficient
  α : confidence level
  θ_1, θ_2 :- Lower and upper con. limits

$\Rightarrow$ Finding interval estimator for the mean and the variance

$\boxed{1}$ Confidence interval on the mean (variance known)

$$Z = \frac{\hat{\mu}_x - E\{\hat{\mu}_x\}}{\sqrt{Var\{\hat{\mu}_x\}}} \Rightarrow -Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}$$

standard gaussian random variable

$$P\left(\hat{\mu}_x - Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_x^2}{n}} \leq \mu_x \leq \hat{\mu}_x + Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_x^2}{n}}\right) = 1-\alpha$$

$\boxed{2}$ Confidence interval on the mean (variance unknown)
(T-distribution)

$$P\left(\hat{\mu}_x - t_{\frac{\alpha}{2},n-1}\sqrt{\frac{\hat{\sigma}_x^2}{n}} \leq \mu_x \leq \hat{\mu}_x + t_{\frac{\alpha}{2},n-1}\sqrt{\frac{\hat{\sigma}_x^2}{n}}\right) = 1-\alpha$$

$(n-1) \rightarrow$ degree of freedom

$\boxed{3}$ Confidence interval on the variance (Mean known)
(χ^2 chi-square distribution)

$$P\left(\frac{n\hat{\sigma}_x^2}{\chi^2_{\frac{\alpha}{2},n}} \leq \sigma_x^2 \leq \frac{n\hat{\sigma}_x^2}{\chi^2_{1-\frac{\alpha}{2},n}}\right) = 1-\alpha$$

$\boxed{4}$ Confidence interval for the variance (Mean unknown)

$n \rightarrow n-1$

$$P\left(\frac{(n-1)\hat{\sigma}_x^2}{\chi^2_{\frac{\alpha}{2},n-1}} \leq \sigma_x^2 \leq \frac{(n-1)\hat{\sigma}_x^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right) = 1-\alpha$$