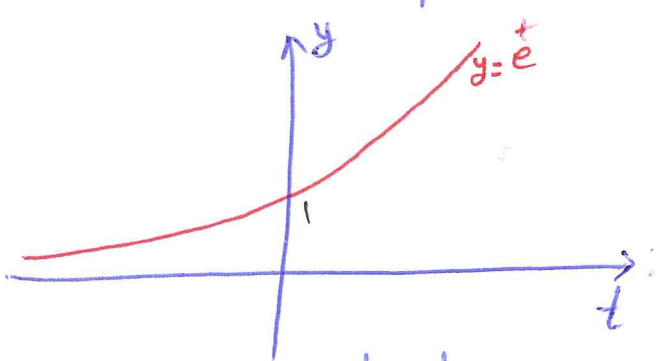
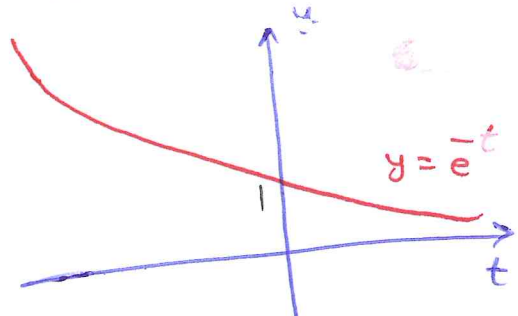


# 7.4 Exponential change and Separable Differential Equations

Recall that the exponential function is



$e^t$  describes growth natural



$e^{-t}$  describes decay natural

Recall that Differential Equations are equations with derivatives (rates) (relations)

DE's that describes growth or decay may have the form

The changes in the amount of  $y(t)$  with time is proportional to the amount present

$$\frac{dy}{dt} = k y \quad \text{--- *}^1$$

DE's may have initial condition:

$$y(0) = y_0 \quad \text{--- *}^2$$

IVP is a DE together with initial condition

$$\frac{dy}{dt} = k y, \quad y(0) = y_0 \quad \text{--- *}$$

\* How to solve the IVP \*? (12)

we use the method of calculus to find the solution  $y(t)$  of \*:

$$\frac{1}{y} \frac{dy}{dt} = k \Leftrightarrow \int \frac{y'}{y} = \int k$$

$$\ln|y| = kt + c \Leftrightarrow |y| = e^{kt+c}$$

$$|y| = e^c e^{kt} \Leftrightarrow y = \pm e^c e^{kt}$$

$$y(t) = D e^{kt} \quad \text{where } D = \pm e^c$$

To find  $D$ , we use the initial condition:

$$y(0) = D e^{k(0)} = y_0 \Leftrightarrow D = y_0$$

Thus, the solution becomes

$$\boxed{y(t) = y_0 e^{kt}} \quad \text{--- *<sup>3</sup>$$

\* Note that if  $k > 0$ , the solution grows exponentially  
if  $k < 0$ , the solution decays exponentially

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The growth equation is  $y(t) = y_0 e^{kt}$  / The decay eq. is  $y(t) = y_0 e^{-kt}$

\* Check \*<sup>3</sup> is a solution for \*

\* Note that Half-life =  $\frac{\ln 2}{k}$  because  $\frac{y_0}{2} = y_0 e^{-kt}$

$$\frac{1}{2} = e^{-kt} \Leftrightarrow -\ln 2 = -kt \Leftrightarrow \boxed{t = \frac{\ln 2}{k}}$$

# Separable Differential Equation

(13)

Example: Solve the DE  $2\sqrt{xy} \frac{dy}{dx} = 1$ ,  $x, y > 0$ .

$$2\sqrt{x}\sqrt{y} dy = dx \Leftrightarrow 2y^{\frac{1}{2}} dy = x^{-\frac{1}{2}} dx$$

$$2 \cdot \frac{2}{3} y^{\frac{3}{2}} = \frac{2}{1} x^{\frac{1}{2}} + C \Leftrightarrow \frac{4}{3} y^{\frac{3}{2}} - 2\sqrt{x} = C$$

implicit solution

$$\frac{4}{3} y^{\frac{3}{2}} = 2\sqrt{x} + C \Leftrightarrow y = \left[ \frac{3}{4} (2\sqrt{x} + C) \right]^{\frac{2}{3}}$$

explicit solution

Example: Solve the IVP  $\frac{dy}{dx} = \frac{y \cos x}{1+3y^3}$ ,  $y(0) = 1$

$$(1+3y^3) dy = y \cos x dx$$

$$\left(\frac{1}{y} + 3y^2\right) dy = \cos x dx$$

$$\ln|y| + y^3 = \sin x + C$$

$$\ln 1 + (1)^3 = \sin(0) + C \Leftrightarrow \boxed{1 = C}$$

The solution becomes  $\ln|y| + y^3 = \sin x + 1$  implicit solution

Example (Radioactive) The half-life of the plutonium is 24,360

years. If 10g of plutonium is released into the atmosphere,

how many years will it take for 80% of the plutonium to

decay.  $P_0 = 10g$

Equation for decay is  $P(t) = P_0 e^{-kt}$

$$\text{Half-life} = \frac{\ln 2}{k} \Leftrightarrow k = \frac{\ln 2}{24,360} \approx 0.00002845$$

We need to find the time  $t$  when  $P(t) = 20\% P_0$

$$0.2\% = P_0 e^{-kt} \Leftrightarrow -kt = \ln 0.2 = -1.61 \quad (14)$$

$$\Leftrightarrow t = \frac{1.61}{0.00002845} \approx 56690 \text{ year}$$

### Heat Transfer "Newton's law of Cooling"

- If  $H(t) = H$  is the temperature of the object at time  $t$   
 $H_s$  is the constant surrounding temperature  
 then the DE that describes the heat transfer is

$$\frac{dH}{dt} = -k(H - H_s) \quad \dots *$$

\* To solve \* Let  $y = H - H_s$

$$\frac{dy}{dt} = \frac{dH}{dt} = -k(H - H_s)$$

$$\frac{dy}{dt} = -ky$$

$$\Leftrightarrow y(t) = y_0 e^{-kt}$$

$$\Leftrightarrow H - H_s = (H_0 - H_s) e^{-kt}$$

$$H(t) = H_s + (H_0 - H_s) e^{-kt}$$

Example A boiled egg at  $98^\circ\text{C}$  is put in a sink of  $18^\circ\text{C}$  water.  
 After 5 min the egg's temperature is  $38^\circ\text{C}$ .  
 How long will it take the egg to reach  $20^\circ\text{C}$ ?

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$$t(0) = H_0 = 98^\circ\text{C}, \quad H_s = 18^\circ\text{C}, \quad H(5) = 38^\circ\text{C}$$

We need to find  $t$  such that  $H(t) = 20^\circ\text{C}$ .

$$H(t) = 18 + (98 - 18) e^{-kt}$$

$$H(t) = 18 + 80 e^{-kt}$$

$$H(5) = 18 + 80 e^{-k(5)} = 38$$

$$\Leftrightarrow 80 e^{-k(5)} = 20$$



$$e^{-5k} = \frac{1}{4} \Leftrightarrow -5k = -\ln 4 \Leftrightarrow k = \frac{\ln 4}{5} \approx 0.28$$

(15)

$$H(t) = 18 + 80 e^{-0.28t}$$

Thus,  $5 \quad 20 = 18 + 80 e^{-0.28t}$

$$2 = 80 e^{-0.28t}$$

$$e^{-0.28t} = \frac{1}{40} \Leftrightarrow -0.28t = -\ln 40$$

$$\Leftrightarrow t = \frac{\ln 40}{0.28} \approx 13 \text{ min.}$$

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