ch 2 Solutions of Nonlinear Equations $f(x) = 0$.

2.1 To solve $f(x) = 0$, we solve $g(x) = x$ where we assume that $f(x) = g(x) - x = 0$.

Def: (Fixed point): A real number p is called a fixed point of $g(x)$ iff $g(p) = p$.

Note: Finding the roots of $f(x) = 0$ is equivelent to finding the fixed point of $g(x)$

Example: Find the fixed points of:

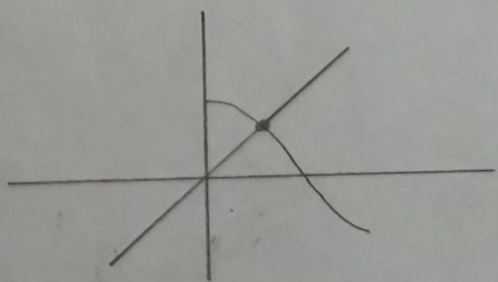
1) $g(x) = x^3 \Rightarrow x^3 = x \Rightarrow x(x^2-1) = 0 \Rightarrow x = \{0, 1, -1\}$

2) $g(x) = x^2 + 1 \Rightarrow$ NO fixed pont

3) $g(x) = x \Rightarrow$ In finitly many fixed points

4) what about $\cos x = x$, we can't solve it

so we sketch:



so we conclude that $\exists$ fixed point

Fixed point Iteration:

How to solve $g(x) = x$?

We start with P_0, then we define:

$$P_1 = g(P_0)$$

$$P_2 = g(P_1)$$

$$\vdots$$

$$P_n = g(P_{n-1})$$

$$P_{n+1} = g(P_n) \quad , n = 0, 1, 2, \ldots$$

we call this process fixed point Iteration.

Thm: If g is continuous and the sequence generated by the fixed point Iteration converges to P, then the number P is the fixed point of g $(g(P) = P)$

proof: $$P = \lim_{n\to\infty} P_{n+1} = \lim_{n\to\infty} g(P_n) \underset{\substack{\downarrow \\ \text{Continuity}}}{=} g\left(\lim_{n\to\infty} P_n\right) = g(P)$$

If $\boxed{P_0 = 3} \Rightarrow g(3) = -3$ & $g(-3) = -3$

Example: $x^2 - x - 12 = 0$, the roots are 4 and -3

① If we assume $\boxed{g(x) = x^2 - 12}$, the $f(x) = 0$ is equivalent to $f(x) = g(x) - x = 0 \Leftrightarrow g(x) = x$.

around 4

If $\boxed{P_0 = 5}$, then $P_1 = 13$, $P_2 = 157$, $P_3 = 24637$

so the sequence is diverge, (bad choose of $g(x)$)

(2) $x = \sqrt{x+12}$

If $\boxed{P_0 = 5}$, then $P_1 = \sqrt{17} \approx 4.1231$

$P_2 = 4.0154$

$\vdots$

$P_9 = 4.000000007 \longrightarrow$ Converg to 4

Since the sequence converges to 4, then 4 is fixed point

(3) $x(x-1) = 12 \implies x = \dfrac{12}{x-1}$

If $\boxed{P_0 = 5}$, then $P_1 = 3$, $P_2 = 6$

$P_3 = 2.4$, $P_4 = 8.57$

$P_5 =$. , $P_6 = 20.5$

$P_7 = 0.6$, $P_8 = -30$

$\vdots$

$P_{26} = -3.0317$

$\vdots$

$P_{36} = -3.00178 \longrightarrow$ conver to -3

$\implies$ -3 is a fixed point.

<u>Note:</u> 1) F.P.I can't solve all Questions

2) F.P.I needs alot of Iteration terms.

We always ask:

Is there a solution? does this solution is unique?

does the F.P.I converges to the fixed point?

To answer these questions we have the following thm.

Thm: If $g \in C[a,b]$ and $g(x) \in [a,b]$, $\forall x \in [a,b]$

then g has at least one fixed point in $[a,b]$

Furthermore: If $g'(x)$ is defined on (a,b)

& a positive constant $k < 1$ exists with

$$|g'(x)| \leq k < 1 \quad , \forall x \in (a,b)$$

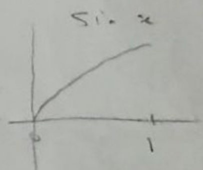
Then g has a unique fixed point P in $[a,b]$.

Example: $\cos x = x$ on $[0,1]$.

1) $\cos x$ is continuous on $[0,1]$.
2) $\cos x \in [0,1]$, $\forall x \in [0,1]$

$\Rightarrow \exists$ fixed point in $[0,1]$.

for uniqueness:



$$|\cos' x| = |\sin x| < 1 \quad , \forall x \in (0,1)$$

$\sin x$ is Increasing, then the $\max\limits_{(0,1)} |\sin x| = \sin 1 = 0.84147$

$\Rightarrow$ there exist unique fixed point on $(0,1)$

$\Rightarrow$ $f(x) = \cos x - x$ has a unique soln in $[0,1]$

proof: If $g(a)=a$ or $g(b)=b$, then g has a fixed point at the end point.

If Not, then $g(a) > a$ & $g(b) < b$

Let $h(x) = g(x) - x$, $h(x)$ is cont. on $[a,b]$

then $h(a) = g(a) - a > 0$

& $h(b) = g(b) - b < 0$

so h satisfies Bolzano's thm, therefore there exist c in $[a,b]$ such that $h(c) = 0$

$\Rightarrow$ $h(c) = g(c) - c = 0$

$\Rightarrow$ $g(c) = c$, so $\exists$ fixed point in $[a,b]$

Now: for Uniqueness:

Suppose there exists two different fixed points P_1 and P_2

w.l.o.G assume $P_1 < P_2$.

g is Continuous on $[P_1, P_2] \subset [a,b]$.

g is differentiable on (P_1, P_2)

$\Rightarrow$ g satisfies the Conditions of Mean value thm, therefor

$\exists$ c such that $g'(c) = \dfrac{g(P_2) - g(P_1)}{P_2 - P_1} = \dfrac{P_2 - P_1}{P_2 - P_1}$

$\Rightarrow$ $g'(c) = 1$ which is contradiction of the assumption that $|g'(x)| \leq k < 1$, $\forall x \in [a,b]$

Example: show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on $[-1,1]$.

Sol: (1) g is cont. on $[-1,1]$

(2) $g(x) \in [-1,1]$?? we need to show that

It's enough to show that max & min of $g(x)$ are in $[-1,1]$.

The maximum & minimum occurs either ~~at~~ the end points of the Interval or when $g'(x)=0$

$g'(x)=0 \Rightarrow \frac{2x}{3}=0 \Rightarrow x=0$

$$\left.\begin{array}{l} g(-1) = 0 \\ g(1) = 0 \\ g(0) = -\frac{1}{3} \end{array}\right\} \Rightarrow g(x) \in [-1,1] \Rightarrow \exists \text{ a fixed point}$$

Now $|g'(x)| = \left|\frac{2}{3}x\right| \le \frac{2}{3} < 1$, $\forall x \in (-1,1)$.

so g has a unique fixed point in $[-1,1]$

Thm: Fixed point thm:

Assume that (i) g and $g' \in C[a,b]$

(ii) $K > 0$ positive constant

(iii) $P_0 \in (a,b)$

(iv) $g(x) \in [a,b]$, $\forall x \in [a,b]$.

1) If $|g'(x)| \leq k < 1$, $\forall x \in [a,b]$, then

for any P_0 in (a,b), the sequence defined by

$P_{n} = g(P_{n-1})$, $n \geq 0$ converges to the Unique

fixed point P in $[a,b]$ Convergenc...

In this case P is said to be attractive fixed point

2) If $|g'(x)| > 1$, $\forall x \in [a,b]$, then

the sequence $P_{n} = g(P_{n-1})$ will not converge to P ~~or diverge~~.

In this case P is said to be Repulsive. fixed pt.

3) If $|g'(x)| = 1$, then the FPT Failed.

Re..

proof: ① from previous thm, we can conclude that

$\exists!$ fixed point p in $[a,b]$ with $g(p) = p$.

Now since $g(x) \in [a,b]$, $\forall x \in [a,b]$ then the

sequence $\{p_n\}_{n=0}^{\infty}$ is defined $\forall n$ & $p_n \in [a,b]$

Now using the fact $|g'(x)| \leq K<1$, $\forall x \in [a,b]$

and using Mean value thm: on $[p_{n-1}, p]$ we have: $g'(c) = \frac{|p_n - p|}{|p_{n-1} - p|}$

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c)|\,|p_{n-1} - p|$$

$$\leq K\,|p_{n-1} - p| \quad , \text{where } c \in (a,b)$$

by Induction:

$$|p_n - p| \leq K\,|p_{n-1} - p| \leq K^2\,|p_{n-2} - p| \leq \cdots \leq K^n\,|p_0 - p|$$

Since $0 < K < 1$ then $\lim_{n\to\infty} K^n = 0$.

$$\Rightarrow 0 \leq \lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} K^n\,|p_0 - p| = 0$$

$$\Rightarrow \lim_{n\to\infty} p_n = p$$

② Exercise:

(Q#2))(15 Points) Consider the fixed point iteration $p_{n+1} = \sqrt[4]{p_n + 1} = g(p_n)$.

a- Show that $g(x)$ has a unique fixed point in I = [1,2].

b- Show that if $p_0 \in I$, then the fixed point iteration converges.

c- Estimate the fixed point p starting with $p_0 = 1.5$, (do only 4iterations)

(a) $g(x) = \sqrt[4]{x+1}$ continuous on $[1,2]$

$g(x)$ is increasing 2 , $f'(x) > 0$ on $[1,2]$ then f is Increasing

$\Rightarrow$ $g(1) \leq g(x) \leq g(2)$ $1 \leq g(1) \leq g(2) \leq 2$

$g(x) \in [1,2]$

$1 \leq \boxed{1.18207} \leq g(x) \leq \boxed{1.31607} \leq 2$

1 1 1

So $g(x)$ has a fixed pt in $[1,2]$

(b) $g'(x) = \frac{1}{4}(x+1)^{-3/4} = \frac{1}{4\sqrt[4]{(x+1)^3}}$. 1

$|g'(x)| = g'(x)$ is decreasing 2

$\Rightarrow$ $|g'(x)| \leq \frac{1}{4\sqrt[4]{(1+1)^3}} = \boxed{0.148650889} < 1$ 2

So ~~F~~PI of $g(x)$ converges

(c) $p_0 = 1.5$

$p_1 = 1.25743343$

$p_2 = 1.22575518 2$

$p_3 = 1.22143215 3$

$p_4 = 1.22083863 2$

5

$(\frac{2}{2})(30)$

Corollary: (1) $|p_n - p| \leq K^n |p_0 - p|$, $\forall n \geq 1$

(2) $|p_n - p| \leq K^n \dfrac{|p_1 - p_0|}{1-K}$, $\forall n \geq 1$

These two formulas give the error estimate for F.P.I

proof: Home work. (1 credit for tomorrow)

Example: Investigate the nature of F.P and show your answer by examples for $g(x) = 1 + x - \frac{x^2}{4}$

Sol: $g(x) = x \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ (F.P)

When $\boxed{x = 2}$, $g'(x) = 1 - \frac{x}{2} \Rightarrow |g'(2)| = 0 < 1$.

$\Rightarrow$ $x = 2$ is an attractive ~~fixed~~-point

(i.e) The sequence $\{p_n\}_{n=0}^{\infty}$ converges to 2.

to show that Let $\boxed{p_0 = 1.6}$, then:

- $p_1 = 1.96$, $p_2 = 1.996$, ... , $p_n \longrightarrow 2$.
- If $\boxed{p_0 = 2.5}$, then $p_1 = 1.9375$, $p_2 = 1.9990$

  $p_3 = 1.99999975$, ... , $p_n \longrightarrow 2$.

When $\boxed{x = -2}$, $|g'(-2)| = 2 > 1 \Rightarrow x = -2$ is Repulsive

$p_0 = -2.05$, $p_1 = -2.1$, $p_2 = -2.2$, ...

Not converge to -2.

Example: Estimate (Find) the number of Iterations required for fixed point iteration to Converge to the fixed point of $g(x) = \pi + \frac{1}{2}\sin\frac{x}{2}$. with

(a) 4 digits accuracy (i.e) $< 10^{-4}$

(b) 10 digit " (i.e) $< 10^{-10}$ (Not Solved)

using $\boxed{P_0 = \pi}$

Sol: Using Corollary: $|P_n - p| \leq k^n \frac{|P_1 - P_0|}{1-k} < \epsilon = 10^{-4}$ or 10^{-10}

we need to find K & P_1

$P_1 = g(P_0) = \pi + \frac{1}{2}\sin\frac{\pi}{2} = \boxed{\pi + \frac{1}{2}}$

Need to find K.

$|g'(x)| = \left|\frac{1}{4}\cos\frac{x}{2}\right| \leq \frac{1}{4}$, so assume $\boxed{K = \frac{1}{4}}$

$$\Rightarrow \frac{\left(\frac{1}{4}\right)^n \left|\pi + \frac{1}{2} - \pi\right|}{1 - \frac{1}{4}} < 10^{-4}.$$

$\Rightarrow \left(\frac{1}{4}\right)^n < 10^{-4} \times \left(\frac{3}{4}\right) \times 2 = 1.5 \times 10^{-4}$

$\Rightarrow \ln\left(\frac{1}{4}\right)^n < \ln\left(1.5 \times 10^{-4}\right)$

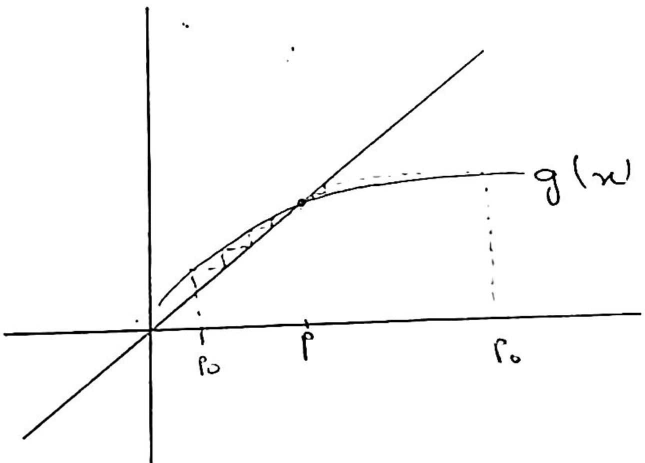
$\Rightarrow -n \ln 4 < -8.804875$

$\Rightarrow n > \frac{+8.804875}{\ln 4} \approx 6.3$

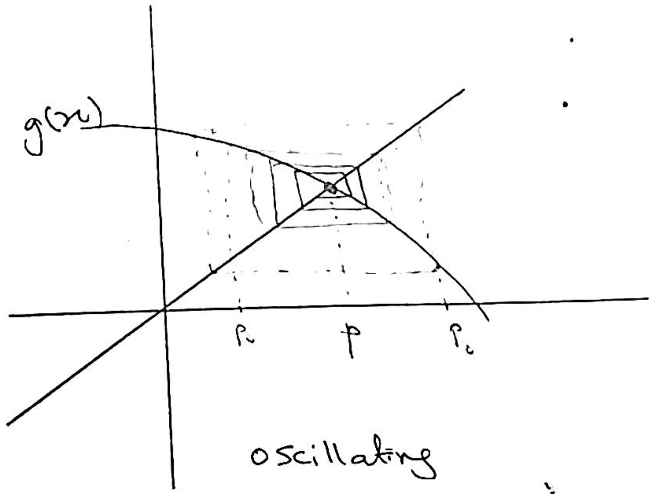
$\Rightarrow$ we need 7 Iterations to solve

Similarly we solve for $\epsilon = 10^{-10}$, we get 17 Iteration.

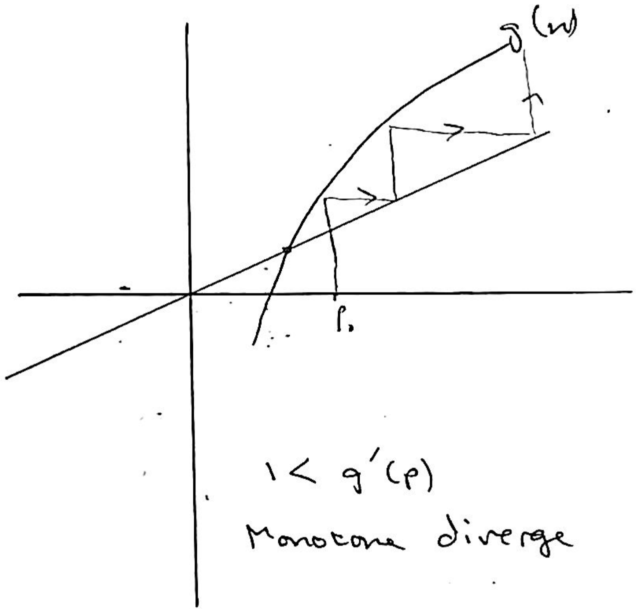
Graphical Interpretation of Fixed Point Iteration.



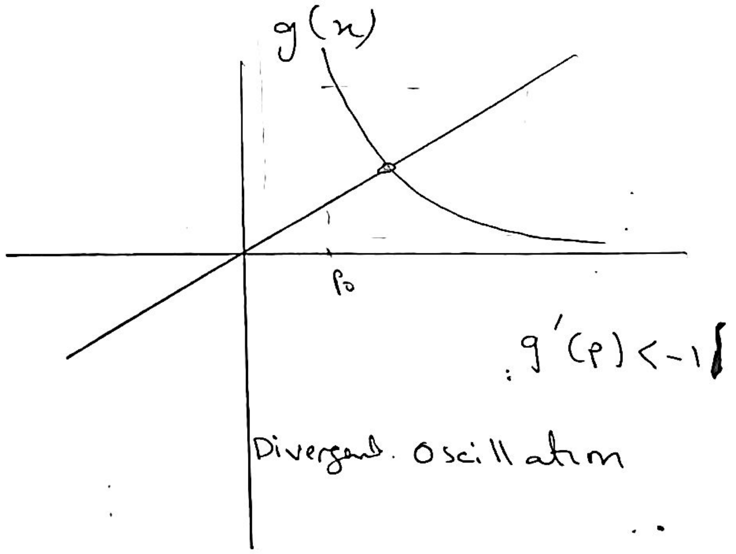
Monotone Convergence $\quad 0 < g'(p) < 1$



Oscillating convergence.
$-1 < g'(p) < 0$



$1 < g'(p)$
Monotone diverge



$g'(p) < -1$

Divergent. Oscillation

2.3 Stopping criteria:

1) $|f(c_n)| < \epsilon \implies f(c_n) \approx 0$

2) $|c_n - c_{n-1}| < \delta$. (i.e) two succesive terms are close
② (more used)

3) $$\frac{2\,|p_n - p_{n-1}|}{|p_n| + |p_{n-1}|} < \delta$$

4) $|p_{n+1} - p_n| / p_n < \varepsilon$