9. Use the Laplace transform and Table 15.1.1 to solve the integral equation

$$y(t) = 1 - \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

10. Use the third and fifth entries in Table 15.1.1 to derive the sixth entry.

11. Show that $\int_a^b e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}[\operatorname{erf}(b) - \operatorname{erf}(a)].$

12. Show that $\int_{-a}^a e^{-u^2}\, du = \sqrt{\pi}\operatorname{erf}(a).$

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13. The functions erf(x) and erfc(x) are defined for $x < 0$. Use a CAS to superimpose the graphs of erf(x) and erfc(x) on the same axes for $-10 \le x \le 10$. Do the graphs possess any symmetry? What are $\lim_{x\to-\infty}$ erf(x) and $\lim_{x\to-\infty}$ erfc(x)?

15.2 Applications of the Laplace Transform

≡ **Introduction** In Chapter 4 we defined the Laplace transform of a function $f(t)$, $t \ge 0$, to be

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

whenever the improper integral converges. This integral transforms a function f(t) into another function F of the transform parameter s, that is, $\mathscr{L}\{f(t)\} = F(s)$. The main application of the Laplace transform in Chapter 4 was to the solution of certain types of initial-value problems involving linear ordinary differential equations with constant coefficients. Recall, the Laplace transform of such an equation reduces the ODE to an algebraic equation. In this section we are going to apply the Laplace transform to linear partial differential equations. We will see that this transform reduces a PDE to an ODE.

Transform of Partial Derivatives The boundary-value problems considered in this section will involve either the one-dimensional wave and heat equations or slight variations of these equations. These PDEs involve an unknown function of two independent variables $u(x, t)$, where the variable t represents time $t > 0$. We define the Laplace transform of $u(x, t)$ with respect to t by

$$\mathscr{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)\, dt = U(x,s),$$

where x is treated as a parameter. Throughout this section we shall assume that all the operational properties of Sections 4.2, 4.3, and 4.4 apply to functions of two variables. For example, by Theorem 4.2.2, the transform of the partial derivative $\partial u/\partial t$ is

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = s\mathscr{L}\{u(x,t)\} - u(x,0);$$

that is,

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0). \tag{1}$$

Similarly,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x, s) - su(x, 0) - u_t(x, 0). \tag{2}$$

Since we are transforming with respect to t, we further suppose that it is legitimate to interchange integration and differentiation in the transform of $\partial^2u/\partial x^2$:

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial x^2}\,dt = \int_0^\infty \frac{\partial^2}{\partial x^2}[e^{-st}u(x, t)]\,dt$$

$$= \frac{d^2}{dx^2}\int_0^\infty e^{-st}u(x, t)\,dt = \frac{d^2}{dx^2}\mathscr{L}\{u(x, t)\};$$

that is,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2}. \tag{3}$$

In view of (1) and (2) we see that the Laplace transform is suited to problems with initial conditions—namely, those problems associated with the heat equation or the wave equation.

EXAMPLE 1 Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $t > 0$.

SOLUTION From (2) and (3),

$$\mathscr{L}\left\{a^2\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

becomes

$$a^2\frac{d^2}{dx^2}\mathscr{L}\{u(x, t)\} = s^2\mathscr{L}\{u(x, t)\} - su(x, 0) - u_t(x, 0)$$

or

$$a^2\frac{d^2U}{dx^2} - s^2U = -su(x, 0) - u_t(x, 0). \tag{4}$$

≡

The Laplace transform with respect to t of either the wave equation or the heat equation eliminates that variable, and for the one-dimensional equations the transformed equations are then *ordinary differential equations* in the spatial variable x. In solving a transformed equation, we treat s as a parameter.

EXAMPLE 2 Using the Laplace Transform to Solve a BVP

Solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \; 0 < x < 1, \; t > 0$$

subject to

$$u(0, t) = 0, \quad u(1, t) = 0, \; t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x, \; 0 < x < 1.$$

SOLUTION The partial differential equation is recognized as the wave equation with $a = 1$. From (4) and the given initial conditions, the transformed equation is

$$\frac{d^2 U}{dx^2} - s^2 U = -\sin \pi x, \tag{5}$$

where $U(x, s) = \mathscr{L}\{u(x, t)\}$. Since the boundary conditions are functions of t, we must also find their Laplace transforms:

$$\mathscr{L}\{u(0, t)\} = U(0, s) = 0 \qquad \text{and} \qquad \mathscr{L}\{u(1, t)\} = U(1, s) = 0. \tag{6}$$

The results in (6) are boundary conditions for the ordinary differential equation (5). Since (5) is defined over a finite interval, its complementary function is

$$U_c(x, s) = c_1 \cosh sx + c_2 \sinh sx.$$

The method of undetermined coefficients yields a particular solution

$$U_p(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x.$$

Hence

$$U(x, s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

But the conditions $U(0, s) = 0$ and $U(1, s) = 0$ yield, in turn, $c_1 = 0$ and $c_2 = 0$. We conclude that

$$U(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x$$

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + \pi^2} \sin \pi x\right\} = \frac{1}{\pi} \sin \pi x \, \mathscr{L}^{-1}\left\{\frac{\pi}{s^2 + \pi^2}\right\}.$$

Therefore

$$u(x, t) = \frac{1}{\pi} \sin \pi x \sin \pi t.$$

≡

EXAMPLE 3 Using the Laplace Transform to Solve a BVP

A very long string is initially at rest on the nonnegative x-axis. The string is secured at $x = 0$, and its distant right end slides down a frictionless vertical support. The string is set in motion by letting it fall under its own weight. Find the displacement $u(x, t)$.

SOLUTION Since the force of gravity is taken into consideration, it can be shown that the wave equation has the form

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \ t > 0,$$

where g is the acceleration due to gravity. The boundary and initial conditions are, respectively,

$$u(0, t) = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ x > 0.$$

The second boundary condition $\lim_{x \to -\infty} \partial u/\partial x = 0$ indicates that the string is horizontal at a great distance from the left end. Now from (2) and (3),

$$\mathcal{L}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} - \mathcal{L}\{g\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

becomes

$$a^2 \frac{d^2 U}{dx^2} - \frac{g}{s} = s^2 U - su(x, 0) - u_t(x, 0)$$

or, in view of the initial conditions,

$$\frac{d^2 U}{dx^2} - \frac{s^2}{a^2} U = \frac{g}{a^2 s}.$$

The transforms of the boundary conditions are

$$\mathcal{L}\{u(0, t)\} = U(0, s) = 0 \qquad \text{and} \qquad \mathcal{L}\left\{\lim_{x \to \infty} \frac{\partial u}{\partial x}\right\} = \lim_{x \to \infty} \frac{dU}{dx} = 0.$$

With the aid of undetermined coefficients, the general solution of the transformed equation is found to be

$$U(x, s) = c_1 e^{-(x/a)s} + c_2 e^{(x/a)s} - \frac{g}{s^3}.$$

The boundary condition $\lim_{x \to -\infty} dU/dx = 0$ implies $c_2 = 0$, and $U(0, s) = 0$ gives $c_1 = g/s^3$. Therefore

$$U(x, s) = \frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}.$$

Now by the second translation theorem we have

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}\right\} = \frac{1}{2} g\left(t - \frac{x}{a}\right)^2 \mathscr{U}\left(t - \frac{x}{a}\right) - \frac{1}{2} gt^2$$

or

$$u(x, t) = \begin{cases} -\dfrac{1}{2} gt^2, & 0 \le t < \dfrac{x}{a} \\ -\dfrac{g}{2a^2}(2axt - x^2), & t \ge \dfrac{x}{a}. \end{cases}$$

To interpret the solution, let us suppose $t > 0$ is fixed. For $0 \le x \le at$, the string is the shape of a parabola passing through the points $(0, 0)$ and $(at, -\frac{1}{2}gt^2)$. For $x > at$, the string is described by the horizontal line $u = -\frac{1}{2}gt^2$. See **FIGURE 15.2.1**.

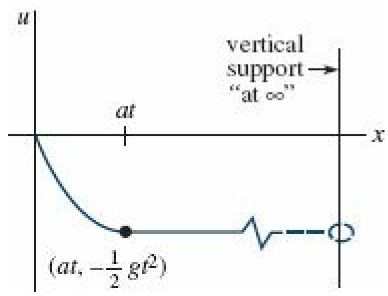

FIGURE 15.2.1 A long string falling under its own weight in Example 3

≡

Observe that the problem in the next example could be solved by the procedure in Section 13.6. The Laplace transform provides an alternative solution.

EXAMPLE 4 A Solution in Terms of erf(x)

Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \ t > 0$$

subject to

$$u(0, t) = 0, \qquad u(1, t) = u_0, \ t > 0$$

$$u(x, 0) = 0, \qquad 0 < x < 1.$$

SOLUTION From (1) and (3) and the given initial condition,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{L}\left\{\frac{\partial u}{\partial t}\right\}$$

becomes

$$\frac{d^2U}{dx^2} - sU = 0. \tag{7}$$

The transforms of the boundary conditions are

$$U(0, s) = 0 \quad \text{and} \quad U(1, s) = \frac{u_0}{s}. \tag{8}$$

Since we are concerned with a finite interval on the x-axis, we choose to write the general solution of (7) as

$$U(x, s) = c_1 \cosh(\sqrt{s}x) + c_2 \sinh(\sqrt{s}x).$$

Applying the two boundary conditions in (8) yields, respectively, $c_1 = 0$ and $c_2 = u_0/(s \sinh \sqrt{s})$. Thus

$$U(x, s) = u_0 \frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}}.$$

Now the inverse transform of the latter function cannot be found in most tables. However, by writing

$$\frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}} = \frac{e^{\sqrt{s}x} - e^{-\sqrt{s}x}}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})} = \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1 - e^{-2\sqrt{s}})}$$

and using the geometric series

$$\frac{1}{1 - e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}}$$

we find

$$\frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}} = \sum_{n=0}^{\infty} \left[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s} \right].$$

If we assume that the inverse Laplace transform can be done term by term, it follows from entry 3 of Table 15.1.1 that

- Also see Problem 8 in Exercises 15.1

$$\begin{aligned} u(x, t) &= u_0 \mathscr{L}^{-1}\left\{ \frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}} \right\} \\ &= u_0 \sum_{n=0}^{\infty} \left[\mathscr{L}^{-1}\left\{ \frac{e^{-(2n+1-x)\sqrt{s}}}{s} \right\} - \mathscr{L}^{-1}\left\{ \frac{e^{-(2n+1+x)\sqrt{s}}}{s} \right\} \right] \\ &= u_0 \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{2n + 1 - x}{2\sqrt{t}} \right) - \operatorname{erfc}\left(\frac{2n + 1 + x}{2\sqrt{t}} \right) \right]. \end{aligned} \tag{9}$$

The solution (9) can be rewritten in terms of the error function using $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$:

$$u(x, t) = u_0 \sum_{n=0}^{\infty} \left[\operatorname{erf}\left(\frac{2n + 1 + x}{2\sqrt{t}} \right) - \operatorname{erf}\left(\frac{2n + 1 - x}{2\sqrt{t}} \right) \right]. \tag{10}$$

FIGURE 15.2.2(a), obtained with the aid of the 3D plot function in a CAS, shows the surface over the rectangular region $0 \le x \le 1$, $0 \le t \le 6$ defined by the partial sum $S_{10}(x, t)$ of the solution (10). It is apparent from the surface and the accompanying two-dimensional graphs that at a fixed value of x (the curve of intersection of a plane slicing the surface perpendicular to the x-axis on the interval [0, 1], the temperature $u(x, t)$ increases rapidly to a constant value as time increases. See **Figure 15.2.2(b)** and **15.2.2(c)**. For a fixed time (the curve of intersection of a plane slicing the surface perpendicular to the t-axis), the temperature $u(x, t)$ naturally increases from 0 to 100. See **Figure 15.2.2(d)** and **15.2.2(e)**.

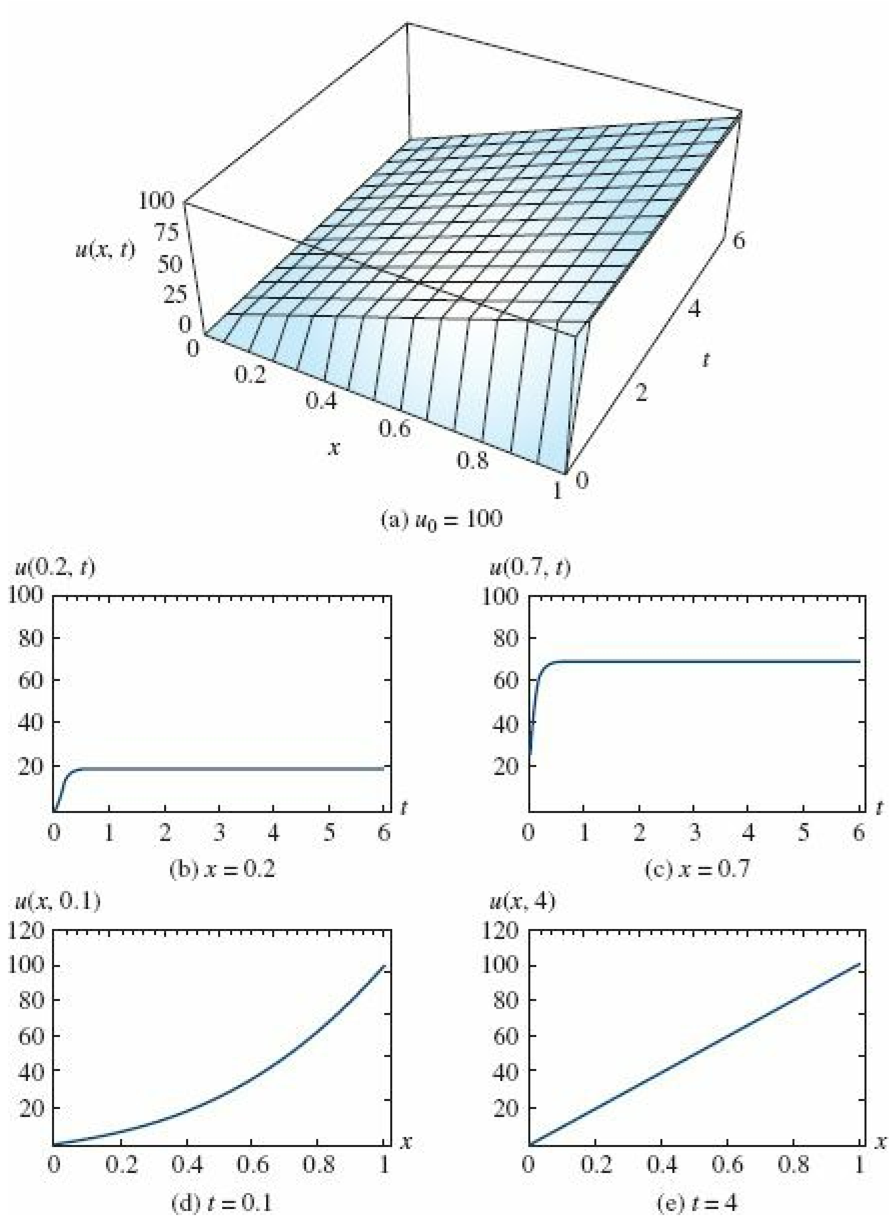

FIGURE 15.2.2 Graph of solution given in (10). In (b) and (c), x is held constant. In (d) and (e), t is held constant.

15.2 Exercises Answers to selected odd-numbered problems begin on page ANS-34.

In the following problems use tables as necessary.

1. A string is secured to the x-axis at $(0, 0)$ and $(L, 0)$. Find the displacement $u(x, t)$ if the string starts from rest in the initial position $A \sin(\pi x/L)$.

2. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \ t > 0$$

$$u(0, t) = 0, \quad u(1, t) = 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 2 \sin \pi x + 4 \sin 3\pi x.$$

Solve for $u(x, t)$.

3. The displacement of a semi-infinite elastic string is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \ t > 0$$

$$u(0, t) = f(t), \quad \lim_{x \to \infty} u(x, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ x > 0.$$

Solve for $u(x, t)$.

4. Solve the boundary-value problem in Problem 3 when

$$f(t) = \begin{cases} \sin \pi t, & 0 \le t \le 1 \\ 0, & t > 1. \end{cases}$$

Sketch the displacement $u(x, t)$ for $t > 1$.

5. In Example 3, find the displacement $u(x, t)$ when the left end of the string at $x = 0$ is given an oscillatory motion described by $f(t) = A \sin \omega t$.

6. The displacement $u(x, t)$ of a string that is driven by an external force is determined from

$$\frac{\partial^2 u}{\partial x^2} + \sin \pi x \sin \omega t = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \ t > 0$$

$$u(0, t) = 0, \quad u(1, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ 0 < x < 1.$$

Solve for $u(x, t)$.

7. A uniform bar is clamped at $x = 0$ and is initially at rest. If a constant force F_0 is applied to the free end at $x = L$, the longitudinal displacement $u(x, t)$ of a cross section of the bar is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$$

$$u(0, t) = 0, \quad E \left.\frac{\partial u}{\partial x}\right|_{x=L} = F_0, \quad E \text{ a constant},\ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: Expand $1/(1 + e^{-2sL/a})$ in a geometric series.]

8. A uniform semi-infinite elastic beam moving along the x-axis with a constant velocity $-v_0$ is brought to a stop by hitting a wall at time $t = 0$. See **FIGURE 15.2.3** The longitudinal displacement $u(x, t)$ is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0,\ t > 0$$

$$u(0, t) = 0, \quad \lim_{x\to\infty} \frac{\partial u}{\partial x} = 0,\ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = -v_0,\ x > 0.$$

Solve for $u(x, t)$.

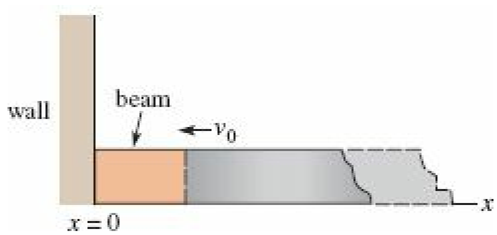

FIGURE 15.2.3 Moving elastic beam in Problem 8

9. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0,\ t > 0$$

$$u(0, t) = 0, \quad \lim_{x\to\infty} u(x, t) = 0,\ t > 0$$

$$u(x, 0) = xe^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ x > 0.$$

10. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0,\ t > 0$$

$$u(0, t) = 1, \quad \lim_{x\to\infty} u(x, t) = 0,\ t > 0$$

$$u(x, 0) = e^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ x > 0.$$

In Problems 11–18, use the Laplace transform to solve the heat equation $u_{xx} = u_t$, $x > 0$, $t > 0$ subject to the given conditions.

11. $u(0, t) = u_0, \quad \lim_{x\to\infty} u(x, t) = u_1, \quad u(x, 0) = u_1$

12. $u(0, t) = u_0, \quad \lim_{x\to\infty} \dfrac{u(x, t)}{x} = u_1, \quad u(x, 0) = u_1 x$

13. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = u(0, t), \quad \lim_{x\to\infty} u(x, t) = u_0, \quad u(x, 0) = u_0$

14. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = u(0, t) - 50, \quad \lim_{x\to\infty} u(x, t) = 0, \quad u(x, 0) = 0$

15. $u(0, t) = f(t), \quad \lim_{x\to\infty} u(x, t) = 0, \quad u(x, 0) = 0$

[*Hint*: Use the convolution theorem.]

16. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = -f(t), \quad \lim_{x\to\infty} u(x, t) = 0, \quad u(x, 0) = 0$

17. $u(0, t) = 60 + 40\,\mathcal{U}(t - 2), \quad \lim_{x\to\infty} u(x, t) = 60,$

$u(x, 0) = 60$

18. $u(0, t) = \begin{cases} 20, & 0 < t < 1 \\ 0, & t \geq 1 \end{cases}, \quad \lim_{x\to\infty} u(x, t) = 100,$

$u(x, 0) = 100$

19. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < 1, \; t > 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 100 - u(1, t), \quad \lim_{x\to-\infty} u(x, t) = 0, \; t > 0$$

$$u(x, 0) = 0, \quad -\infty < x < 1.$$

20. Show that a solution of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}, \qquad x > 0, \; t > 0$$

$$u(0, t) = 0, \qquad \lim_{x\to\infty} \frac{\partial u}{\partial x} = 0, \; t > 0$$

$$u(x, 0) = 0, \; x > 0,$$

where r is a constant, is given by

$$u(x, t) = rt - r\int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{k\tau}}\right) d\tau.$$

21. A rod of length L is held at a constant temperature u_0 at its ends $x = 0$ and $x = L$. If the rod's initial temperature is $u_0 + u_0 \sin(x\pi/L)$, solve the heat equation $u_{xx} = u_t$, $0 < x < L$, $t > 0$ for the temperature $u(x, t)$.

22. If there is a heat transfer from the lateral surface of a thin wire of length L into a medium at constant temperature u_m, then the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0,$$

where h is a constant. Find the temperature $u(x, t)$ if the initial temperature is a constant u_0 throughout and the ends $x = 0$ and $x = L$ are insulated.

23. A rod of unit length is insulated at $x = 0$ and is kept at temperature zero at $x = 1$. If the initial temperature of the rod is a constant u_0, solve $ku_{xx} = u_t$, $0 < x < 1$, $t \geq 0$ for the temperature $u(x, t)$. [*Hint*: Expand $1/(1 + e^{-2\sqrt{s/k}})$ in a geometric series.]

24. An infinite porous slab of unit width is immersed in a solution of constant concentration c_0. A dissolved substance in the solution diffuses into the slab. The concentration $c(x, t)$ in the slab is determined from

$$D\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}, \quad 0 < x < 1,\ t > 0$$

$$c(0, t) = c_0, \quad c(1, t) = c_0,\ t > 0$$

$$c(x, 0) = 0, \quad 0 < x < 1,$$

where D is a constant. Solve for $c(x, t)$.

25. A very long telephone transmission line is initially at a constant potential u_0. If the line is grounded at $x = 0$ and insulated at the distant right end, then the potential $u(x, t)$ at a point x along the line at time t is determined from

$$\frac{\partial^2 u}{\partial x^2} - RC\frac{\partial u}{\partial t} - RGu = 0, \quad x > 0,\ t > 0$$

$$u(0, t) = 0, \quad \lim_{x\to\infty}\frac{\partial u}{\partial x} = 0,\ t > 0$$

$$u(x, 0) = u_0,\ x > 0,$$

where R, C, and G are constants known as resistance, capacitance, and conductance, respectively. Solve for $u(x, t)$. [*Hint*: See Problem 7 in Exercises 15.1.]

26. Starting at $t = 0$, a concentrated load of magnitude F_0 moves with a constant velocity v_0 along a semi-infinite string. In this case the wave equation becomes

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + F_0\delta\left(t - \frac{x}{v_0}\right),$$

where $\delta(t - x/v_0)$ is the Dirac delta function. Solve this PDE subject to

$$u(0, t) = 0, \quad \lim_{x\to\infty} u(x, t) = 0,\ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ x > 0$$

(a) when $v_0 \neq a$, and

(b) when $v_0 = a$.

27. In Problem 9 of Exercises 14.3 you were asked to find the time-dependent temperatures $u(r, t)$ within a unit sphere. The temperatures outside the sphere are described by the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad r > 1,\ t > 0$$

$$u(1, t) = 100, \ t > 0$$

$$\lim_{r\to\infty} u(r, t) = 0$$

$$u(r, 0) = 0, \ r > 1.$$

Use the Laplace transform to find $u(r, t)$. [*Hint*: After transforming the PDE, let $v(r, s) = r\,U(r, s)$, where $\mathscr{L}\{u(r, t)\} = U(r, s)$.]

28. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \quad x > 0,\ t > 0,\ h \text{ constant}$$

$$u(0, t) = u_0, \ \lim_{x\to\infty} u(x, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \quad x > 0$$

is

$$u(x, t) = \frac{u_0 x}{2\sqrt{\pi}}\int_0^t \frac{e^{-h\tau - x^2/4\tau}}{\tau^{3/2}}\,d\tau.$$

≡ Computer Lab Assignment

29. (a) The temperature in a semi-infinite solid is modeled by the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad x > 0,\ t > 0$$

$$u(0, t) = u_0, \quad \lim_{x\to\infty} u(x, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \ x > 0.$$

Solve for $u(x, t)$. Use the solution to determine analytically the value of $\lim_{t\to\infty} u(x, t)$, $x < 0$.

(b) Use a CAS to graph $u(x, t)$ over the rectangular region defined by $0 \le x \le 10$, $0 \le t \le 15$. Assume $u_0 = 100$ and $k = 1$. Indicate the two boundary conditions and initial condition on your graph. Use 2D and 3D plots of $u(x, t)$ to verify your answer to part (a).

30. (a) In Problem 29, if there is a constant flux of heat into the solid at its left-hand boundary, then the boundary condition is $\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A,\ A > 0,\ t > 0$. Solve for $u(x, t)$. Use the solution to determine analytically the value of $\lim_{t\to\infty} u(x, t)$, $x > 0$.

