CH11 Parametric Equations and Polar Coordinates.

11.1 Parametrizations of plane curves.

- Parametric Equations

Df: If x and y are given functions

$$x = f(t), \quad y = g(t), \quad t \in I,$$

where I is an interval. then the set of points $(x,y) = (f(t), g(t))$ is called a parametric curve. The equations are parametric equations for the curve:

- The variable t is a parameter for the curve.

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- I is the parameter interval.

- If $I = [a,b]$ is a closed interval, then the point $(f(a), g(a))$ is the initial point of the curve and

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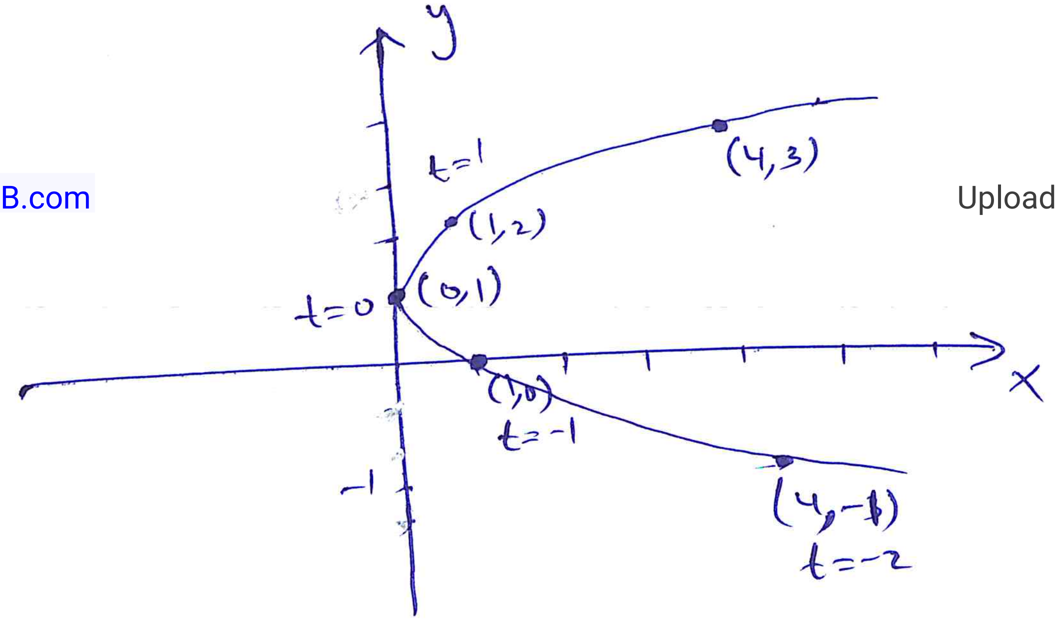
$(f(b), g(b))$ is the terminal point.

Ex: sketch the curve

$x = t^2, \quad y = t+1, \quad -\infty < t < \infty.$

Sol: Values of $x = t^2$ & $y = t+1$ for selected values of t.

t	x	y	(x,y)
-3	9	-2	$(9,-2)$
-2	4	-1	$(4,-1)$
-1	1	0	$(1,0)$
0	0	1	$(0,1)$
1	1	2	$(1,2)$
2	4	3	$(4,3)$
3	9	4	$(9,4)$



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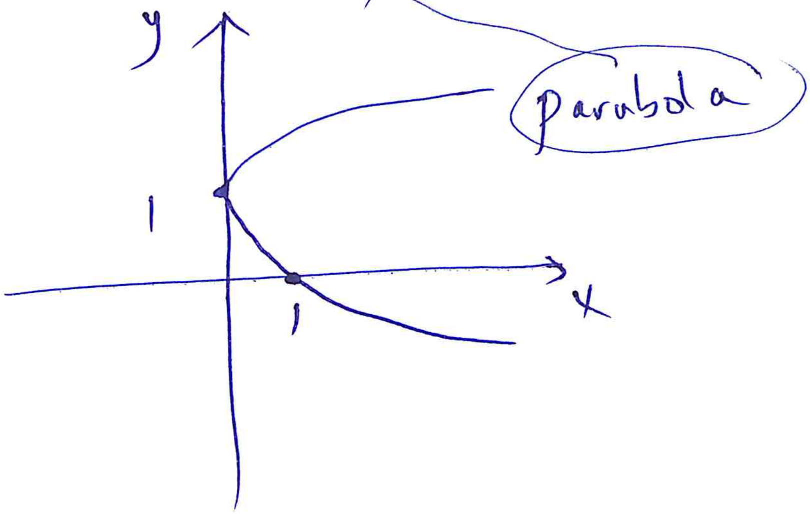
<u>Another method</u> Convert the parametric curve into Cartesian curve if possible. (eliminate t).

$x = t^2, \; y = t+1, \; -\infty < t < \infty.$

- $t = y-1 \Rightarrow x = (y-1)^2, \; x \geq 0$

<u>x-intercept</u> $y=0$
$\Rightarrow x = 1$
$(1,0)$

<u>y-intercept</u> $x=0$
$\Rightarrow y = 1$
$(0,1)$



<u>Ex.</u> sketch the parametric curves. Identify. Locate the directions. (<u>Ex1 — Ex6</u>)

① $x = \cos t, \; y = \sin t, \quad 0 \leq t \leq 2\pi.$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

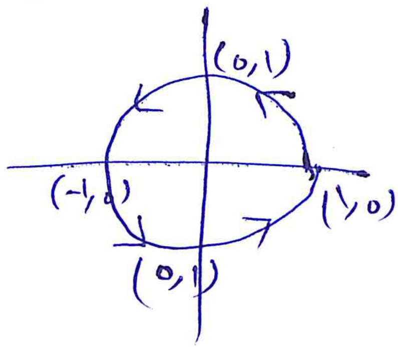
$\therefore x^2 + y^2 = 1$ circle centered at $(0,0)$, radius $= 1$.

$t = 0 \Rightarrow (1,0)$

$t = \pi/2 \Rightarrow (0,1)$

$t = \pi \Rightarrow (-1,0)$

$\vdots$



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Notice that as t increases from 0 to 2π, the point $(x,y) = (\cos t, \sin t)$ starts at $(1,0)$ and traces the entire circle once counterclockwise.

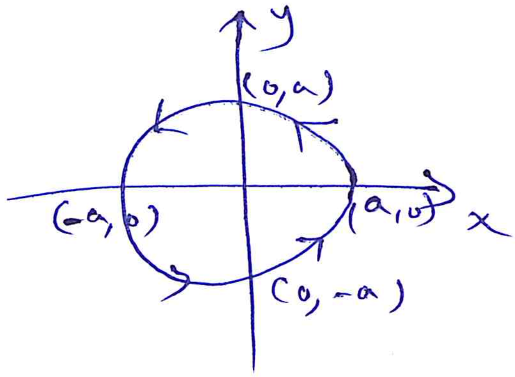
② $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$.

$x^2 = a^2\cos^2 t$, $y^2 = a^2\sin^2 t$

$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t$

$= a^2(\cos^2 t + \sin^2 t) = a^2 \cdot 1 = a^2$

$\therefore \boxed{x^2 + y^2 = a^2}$ circle centered at $(0,0)$ radius $= a$.



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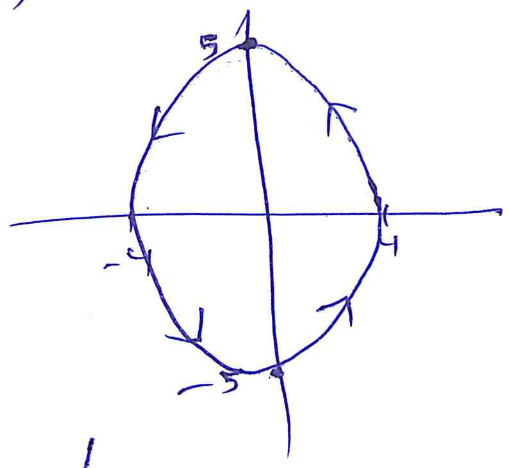
③ $x = 4\sin t$, $y = 5\cos t$, $0 \le t \le 2\pi$.

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \sin^2 t + \cos^2 t = 1$

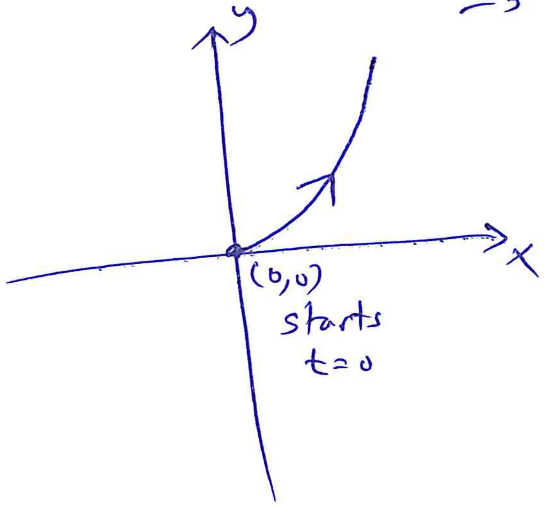
$\Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$ ellipse.



(4) $x = \sqrt{t}$, $y = t$, $t \geq 0$

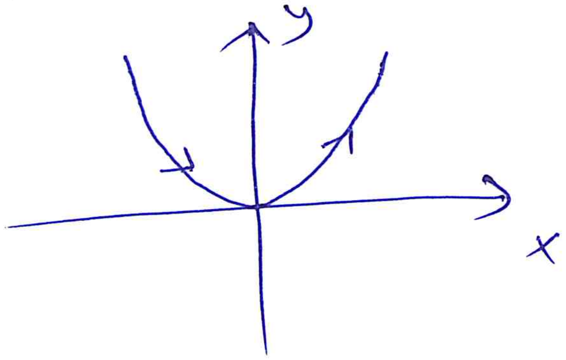
Sol. $y = t = (\sqrt{t})^2 = x^2$

$y = x^2$, $x \geq 0$



(5) $x = t$, $y = t^2$, $-\infty < t < \infty$.

Sol. $y = t^2 = x^2$



(6) $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$

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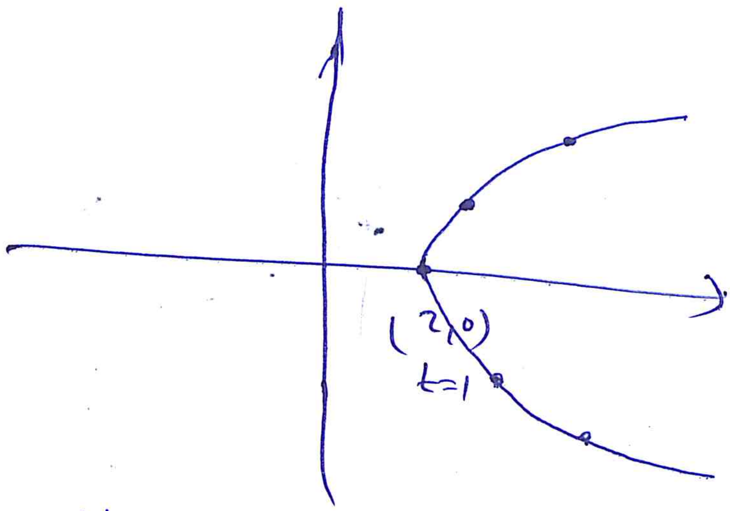
Sol

$x - y = \cancel{t} + \frac{1}{t} - \cancel{t} + \frac{1}{t} = \frac{2}{t}$

$x + y = t + \cancel{\frac{1}{t}} + t - \cancel{\frac{1}{t}} = 2t$

$\Rightarrow (x-y)(x+y) = \left(\frac{2}{t}\right)(2t) = 4$

$\therefore x^2 - y^2 = 4$ hyperbola.



Notice that the parameter domain is $(0,\infty)$ and there is no starting point and no terminal point for the path.

Ex (7) Find a parametrization for the line through the point (a,b) having slope m.

sol. A Cartesian equation of the line is $\boxed{y-b=m(x-a)}$. Set $x-a=t$ we find $x=a+t$ and $y-b=mt$.

$$\Rightarrow \boxed{x=a+t,\ y=b+mt,\ -\infty<t<\infty}$$

parametrizes the line.

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ex. (8) parametrize the line through the points $(1,2)$ and $(-4,5)$.

Sol. $m = \dfrac{\Delta y}{\Delta x} = \dfrac{5-2}{-4-1} = \dfrac{-3}{5}$

$x = 1+t, \quad y = 2 - \frac{3}{5}t, \quad -\infty < t < \infty.$

Ex. (9) Parametrize $x^2 - y^2 = 4$.

set $y = t \Rightarrow x^2 = t^2+4 \Rightarrow x = \sqrt{4+t^2}$

$\Rightarrow x = \sqrt{4+t^2}, \quad y = t, \quad -\infty < t < \infty.$

Another parametrization $x = 2\sec t, \ y = 2\tan t,$
$-\pi/2 < t < \pi/2$

Also, $x = t + \frac{1}{t}, \ y = t - \frac{1}{t}, \ t > 0.$

Ex. (10) Parametrize the motion of a particle that starts at $(a,0)$ and traces the circle $x^2+y^2=a^2$ twice clockwise.

Sol. $x = a\cos t, \ y = -a\sin t, \quad 0 \le t \le 4\pi$

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11.2 Calculus with Parametric Curves

• Tangents and Areas

• A parametrized curve $x = f(t)$, $y = g(t)$ is diff'ble at t if f and g are diff'ble at t.

Parametric Formula for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$ if all three derivatives exist and $\frac{dx}{dt} \neq 0$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Parametric Formula for $\frac{d^2y}{dx^2}$

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If $x = f(t)$, $y = g(t)$ define y as a twice diff'ble function of x, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$.

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$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}}$$

Ex. ① Find the tangent to the curve

$$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at $t = \frac{\pi}{4}$.

Sol. The slope of the curve at $t = \frac{\pi}{4}$ is

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\frac{\pi}{4}} = \left.\frac{\sec^2 t}{\cancel{\sec t}\tan t}\right|_{t=\frac{\pi}{4}}$$

$$= \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

The point at $t = \frac{\pi}{4}$ is $(x,y) = \left(\sec\frac{\pi}{4}, \tan\frac{\pi}{4}\right)$

$$= (\sqrt{2}, 1).$$

The tangent line is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 1 = \sqrt{2}(x - \sqrt{2})$$

$$y = \sqrt{2}x - 2 + 1$$

$$y = \sqrt{2}x - 1.$$

Ex 2. Find $\frac{d^2y}{dx^2}$ as a function of t

if $x = t - t^2, \quad y = t - t^3$.

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Sol:

- $y' = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1-3t^2}{1-2t}$

- $\dfrac{dy'}{dt} = \dfrac{d}{dt}\left(\dfrac{1-3t^2}{1-2t}\right) = \dfrac{(1-2t)(-6t)-(1-3t^2)(-2}{(1-2t)^2}$

$$= \frac{-6t+12t^2+2-6t^2}{(1-2t)^2} = \frac{2-6t+6t^2}{(1-2t)^2}$$

- $\dfrac{d^2y}{dx^2} = \dfrac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{\left(\frac{2-6t+6t^2}{(1-2t)^2}\right)}{1-2t}$

$$= \frac{2-6t+6t^2}{(1-2t)^3}.$$

Ex3. Find the area enclosed by the astroid $x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi.$

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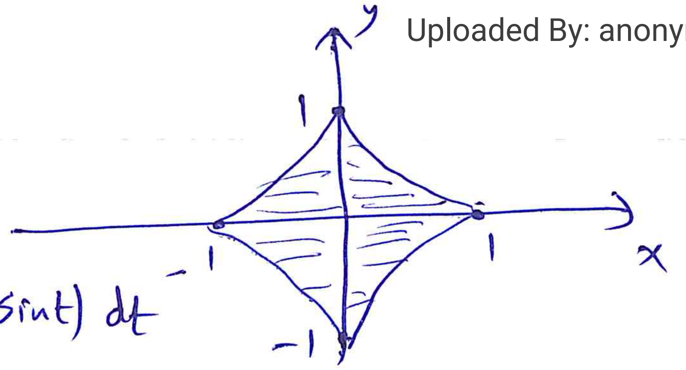
$$A = 4\int_0^1 y\,dx$$



$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t\,(-\sin t)\,dt$$

$$= 12\int_0^{\pi/2} \sin^4 t\,\cos^2 t\,dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{12}{8}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 2t + \cos^2 2t\right)\left(1+\cos 2t\right) dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} \left(1 - \cos 2t - \cos^2 2t + \cos^3 2t\right) dt.$$

$$= \frac{3}{2}\Big[\underbrace{\int_0^{\frac{\pi}{2}} (1-\cos 2t)\, dt}_{I_1} - \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 2t\, dt}_{I_2} + \underbrace{\int_0^{\frac{\pi}{2}} \cos^3 2t\, dt}_{I_3}$$

For $I_1 = \int_0^{\frac{\pi}{2}} (1-\cos 2t)\, dt = t - \frac{\sin 2t}{2}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$

$$I_2 = \int_0^{\frac{\pi}{2}} \cos^2 2t\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 4t)\, dt$$

$$= \frac{1}{2}\left[t + \frac{\sin 4t}{4} \right|_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0\right] = \frac{\pi}{4}$$

$$I_3 = \int_0^{\frac{\pi}{2}} \cos^3 2t\, dt = \int_0^{\frac{\pi}{2}} \cos^2 2t \,\cos 2t\, dt$$

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$$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 2t) \cos 2t \, dt \,.$$

let $u = \sin 2t \Rightarrow du = 2 \cos 2t \, dt$

$x = 0 \Rightarrow u = 0 \quad , \quad x = \pi/2 \Rightarrow u = 0$

$$\therefore I_3 = \int_0^0 (1 - u^2) \frac{du}{2} = 0 \,.$$

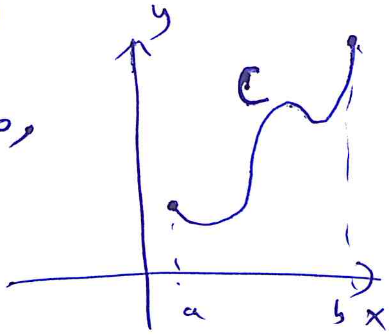
$$\therefore A = \frac{3}{2} \left[\frac{\pi}{2} - \frac{\pi}{4} + 0 \right] = \frac{3}{2} \cdot \frac{\pi}{4} = \frac{3\pi}{8} \,.$$

- <u>length of a parametrically Defined Curve</u>

let C be a curve given by

$x = f(t) \,, \; y = g(t) \,, \quad a \leq t \leq b,$

where f' and g'



are continuous and not simultaneously zero on $[a, b]$ and C is traversed onces as t increases from $t = a$ to $t = b$, then the <u>length of C</u> is

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} \, dt \,.$$

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Ex 4. find the length of the circle of radius r defined by

$$x = r\cos t, \quad y = r\sin t, \qquad 0 \le t \le 2\pi.$$

Sol:
$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\, dt = r\int_0^{2\pi} dt = 2\pi r.$$

Ex. 5 find the length of the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

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Sol:
$$\frac{dx}{dt} = -3\cos^2 t \sin t, \quad \frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cdot \cos^2 t$$

$$= 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)$$

$$= 9\cos^2 t \sin^2 t$$

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2 t \sin^2 t}\; dt$$

$$= 4\int_0^{\frac{\pi}{2}} |3\cos t \overset{\geq 0}{\sin t}|\; dt$$

$$= 4\int_0^{\frac{\pi}{2}} 3\cos t \sin t\, dt \quad (\text{since } \cos t, \sin t \geq 0 \text{ in first-quadrant})$$

$$= 6\int_0^{\frac{\pi}{2}} \sin 2t\, dt = 6\left(-\frac{\cos 2t}{2}\right)\Big|_0^{\pi/2}$$

$$= 6\left(\frac{1}{2} + \frac{1}{2}\right) = 6.$$

- **Area of Surfaces of Revolution**

Let $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ be a smooth curve is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes as follows.

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1. Revolution about the <u>x-axis ($y \geq 0$)</u>:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

2. Revolution about the <u>y-axis ($x \geq 0$)</u>:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Ex. 7</u> Find the area of the surface generated by revolving the circle about the <u>x-axis</u>

$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi.$

<u>Sol.</u> $$L = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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$$= \int_0^{2\pi} 2\pi(1+\sin t) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt.$$

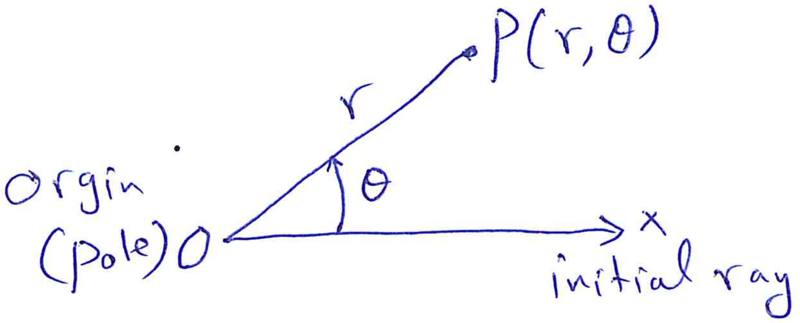
$$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt = 2\pi(t - \cos t)\Big|_0^{2\pi}$$

$$= 2\pi(2\pi - 1 - 0 + 1)$$

$$= 4\pi^2.$$

11.3 Polar Coordinates

• Polar Coordinates

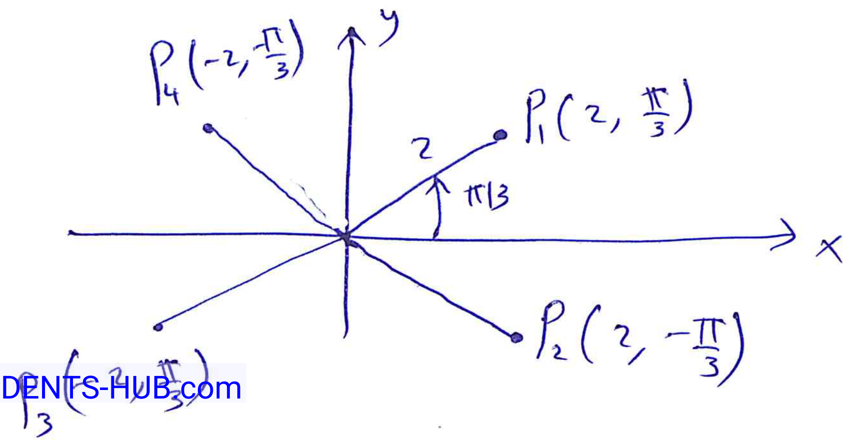


r: directed distance from O to P.

θ: directed angle from initial ray to OP.

Ex. Plot $P_1(2, \frac{\pi}{3})$, $P_2(2, -\frac{\pi}{3})$, $P_3(-2, \frac{\pi}{3})$,

$P_4(-2, -\frac{\pi}{3})$.



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Ex. Find all polar coordinates of the point

$P(2, \frac{\pi}{6})$.

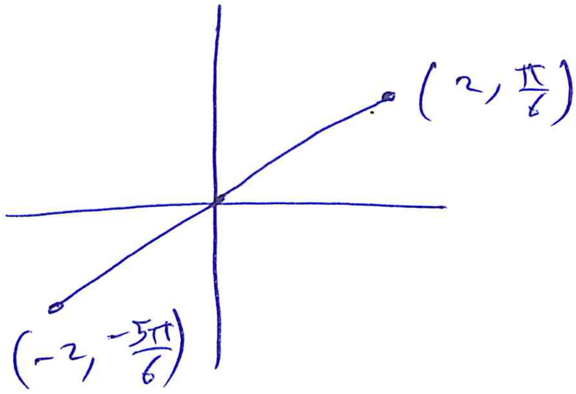
Sol. Case 1. $r = 2 > 0$

$$\left(2, \frac{\pi}{6} + 2\pi k\right), \quad k = 0, \pm 1, \pm 2, \ldots$$

Case 2. $r = -2 < 0$

$$\left(-2, -\frac{5\pi}{6} + 2\pi k\right),$$

$k = 0, \pm 1, \pm 2, \ldots$



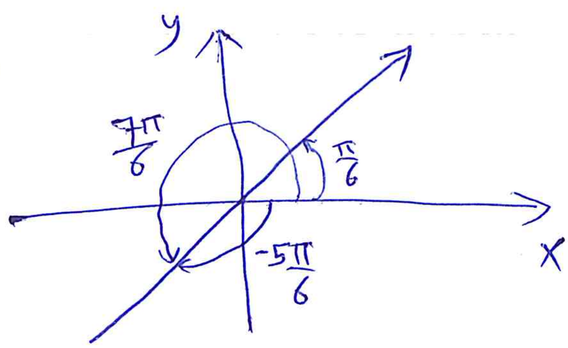
Polar Equations and Graphs

$\boxed{r = a}$: circle of radius $|a|$ centered at O.

$\boxed{\theta = \theta_0}$ line through O making an angle θ_0 with the initial ray.

Ex. a) $r = 9$ and $r = -9$ are equations for the circle of radius 3 centered at O.

b) $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$ and $\theta = \frac{-5\pi}{6}$ are equations for the line.



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Ex. Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$.

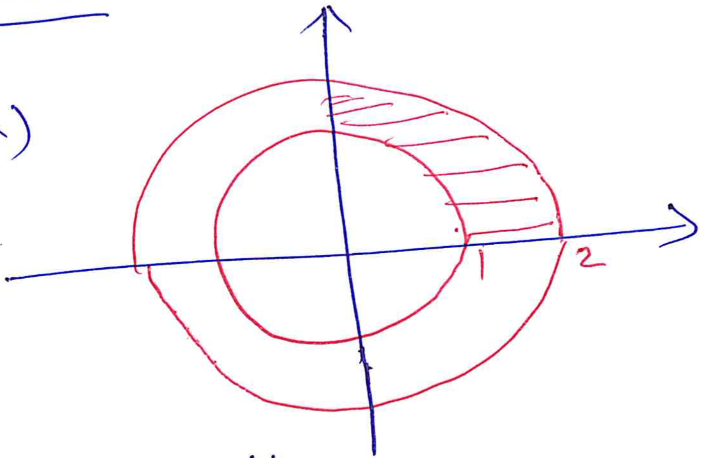
(b) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

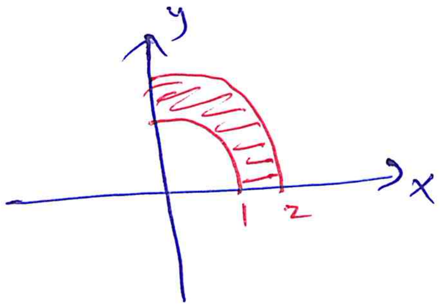
(c) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r).

(d) $r > 1$ (no restriction on θ).

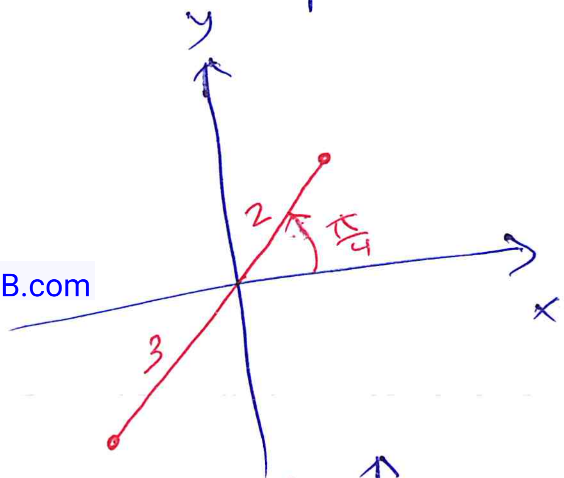
Solution

(a)





(b)



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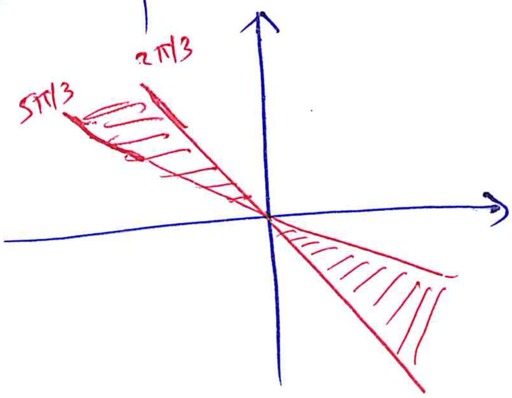
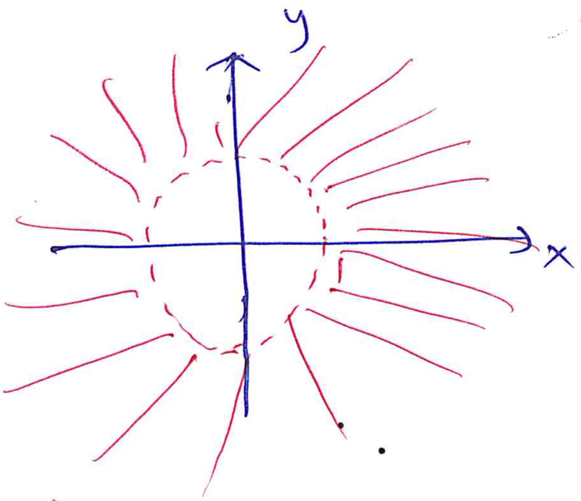
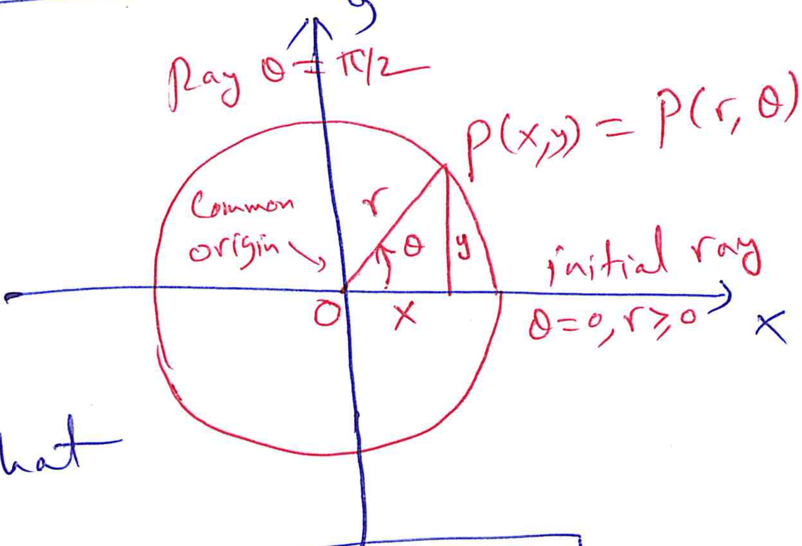
(c)



(d)



Relating Polar and Cartesian coordinates



Notice that

- $\sin\theta = \dfrac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$
- $\cos\theta = \dfrac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$
- $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r$

$\Rightarrow \boxed{r^2 = x^2 + y^2}$

- $\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \boxed{\tan\theta = \dfrac{y}{x}}$

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Ex. Find the Cartesian Coordinates of $\left(\overset{r}{-\sqrt{2}}, \overset{\theta}{\frac{\pi}{4}}\right)$

Sol. $x = r\cos\theta = -\sqrt{2}\cos\frac{\pi}{4} = -\sqrt{2}\cdot\frac{1}{\sqrt{2}} = -1$

$y = r\sin\theta = -\sqrt{2}\sin\frac{\pi}{4} = -\sqrt{2}\cdot\frac{1}{\sqrt{2}} = -1$

$\therefore$ The Cartesian coordinates of $\left(-\sqrt{2}, \frac{\pi}{4}\right)$ is $(-1,-1)$.

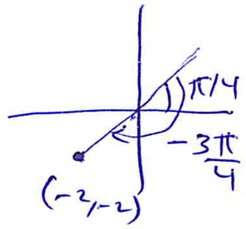
Ex. Find the Polar Coordinates of the point $(\overset{x}{-2}, \overset{y}{-2})$ if $-\pi \le \theta \le \pi$, $r \ge 0$.

Sol. $r^2 = x^2 + y^2 = (-2)^2 + (-2)^2 = 8$.

$\Rightarrow r = \sqrt{8} = 2\sqrt{2} \quad (r \ge 0)$.

$\tan\theta = \frac{y}{x} = \frac{-2}{-2} \Rightarrow \tan\theta = +1$

$\theta = -\frac{3\pi}{4}$.



$\therefore$ the polar coordinates of $(-2,-2)$ is

$(r,\theta) = \left(2\sqrt{2}, -\frac{3\pi}{4}\right)$.

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Ex. Find a polar equation for the following curves:

① $x^2 + (y-3)^2 = 9$

Sol. $\underbrace{x^2+y^2}_{} - 6y + 9 = 9$

$\Rightarrow r^2 - 6r\sin\theta = 0 \Rightarrow r(r - 6\sin\theta) = 0$

$\Rightarrow \boxed{r=0}$ or $\boxed{r = 6\sin\theta}$

$\Rightarrow r = 6\sin\theta$ (Includes both possibilities).

Q61

(2) $y^2 = 4x$

Sol. $r^2\sin^2\theta = 4r\cos\theta$

$\Rightarrow r(r\sin^2\theta - 4\cos\theta) = 0$

$r = 0$ or $r\sin^2\theta = 4\cos\theta$

$\Rightarrow \boxed{r\sin^2\theta = 4\cos\theta}$ (Includes $r=0$).

Ex. Convert the following polar equations to Cartesian equations and identify their graphs.

(1) $r\cos\theta = -4$

Sol. $x = -4$ (vertical line through $x=-4$ on the x-axis).

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(2) $r^2 = 4r\cos\theta$

The Cartesian equation $r^2 = 4r\cos\theta$

$x^2+y^2 = 4x$

$x^2-4x+y^2=0$

$x^2-4x+4+y^2=4$

$(x-2)^2+y^2=4$

(121)

The graph: circle, radius 2, center $(h,k)=(3,0)$

(3) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4$$

or $\boxed{y = 2x-4}$

Graph: line, slope $m=2$, y-intercept $b=-4$

$\boxed{Q_{32})}$ (4) $r^2 \sin(2\theta) = 2$

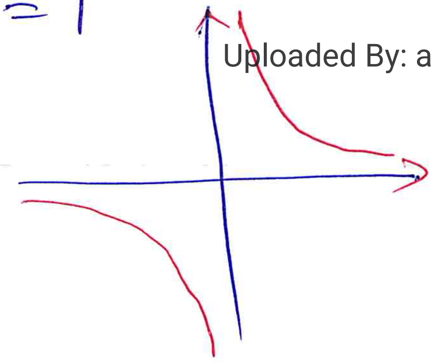
Cartesian equation

$$r^2 \cdot 2\sin\theta\cos\theta = 2$$

$$\Rightarrow (r\sin\theta)(r\cos\theta) = 1$$

$$\Rightarrow yx = 1$$

$$y = \frac{1}{x}$$



Graph: hyperbola

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Q 40 (5) $r = 4\tan\theta \sec\theta$

Cartesian equation: $r = 4\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$

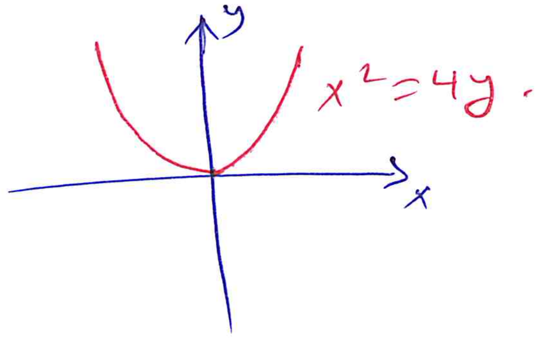
$\Rightarrow r\cos^2\theta = 4\sin\theta$

$r^2\cos^2\theta = 4r\sin\theta$

$(r\cos\theta)^2 = 4(r\sin\theta)$

$x^2 = 4y$.

Graph: Parabola



Q 51 (6) $r\sin(\theta + \frac{\pi}{6}) = 2$

Cartesian equation $r\left[\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right] = 2$

$\Rightarrow r\left[\sin\theta\cdot\frac{\sqrt{3}}{2} + \cos\theta\cdot\frac{1}{2}\right] = 2$

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$\Rightarrow \frac{\sqrt{3}}{2}(r\sin\theta) + \frac{1}{2}(r\cos\theta) = 2$

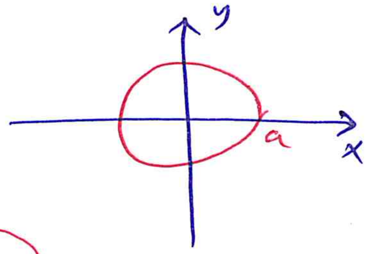
$\frac{\sqrt{3}}{2}y + \frac{1}{2}x = 2$

$\Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$

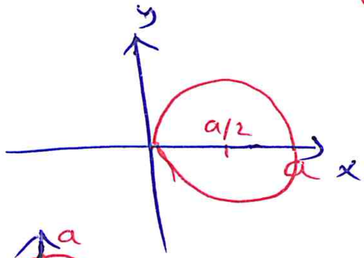
Graph: line, slope $m = -\frac{1}{\sqrt{3}}$, y-intercept $b = \frac{4}{\sqrt{3}}$

11.4 Graphing in Polar coordinates

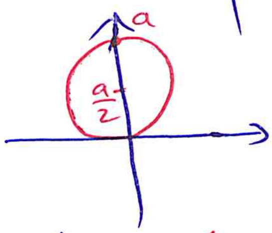
Recall, ① $r = a$, $a \neq 0$



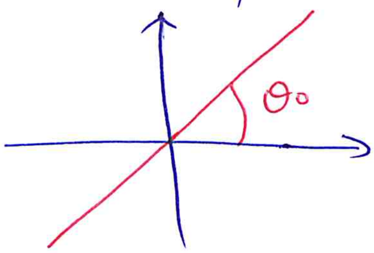
② $r = a\cos\theta$



③ $r = a\sin\theta$



④ $\theta = \theta_0$



Symmetry

① symmetric about the x-axis

$(r,\theta) \longleftrightarrow (r,-\theta)$ or $(-r, \pi-\theta)$.

② symmetric about the y-axis

STUDENTS-HUB.com $(r,\theta) \longleftrightarrow (r, \pi-\theta)$ or $(-r,-\theta)$ Uploaded By: anonymous

③ symmetric about orgin.

$(r,\theta) \longleftrightarrow (-r, \theta)$ or $(r, \pi+\theta)$.

ex. sketch $r = 1-\cos\theta$.

Sol. Symmetry : x-axis

$(r,\theta) \in \text{graph} \Rightarrow r = 1-\cos\theta$.

$(r,-\theta)$: $r = 1-\cos(-\theta) = 1-\cos\theta$

$\Rightarrow (r,-\theta) \in \text{graph}$.

$\therefore$ Symmetric about x-axis.

θ	$r = 1-\cos\theta$	(r,θ)
0	0	$(0,0)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$(\frac{1}{2}, \frac{\pi}{3})$
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{2\pi}{3}$	$\frac{3}{2}$	$(\frac{3}{2}, \frac{2\pi}{3})$
π	2	$(2, \pi)$
$\frac{3\pi}{2}$	1	$(1, \frac{3\pi}{2})$

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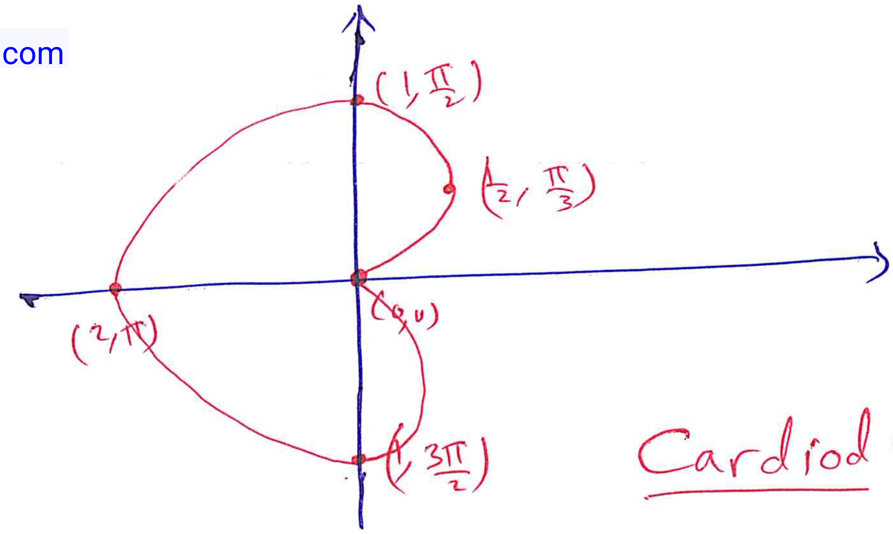
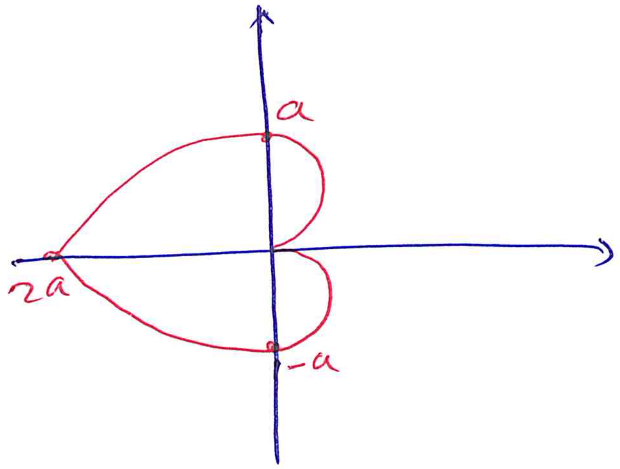


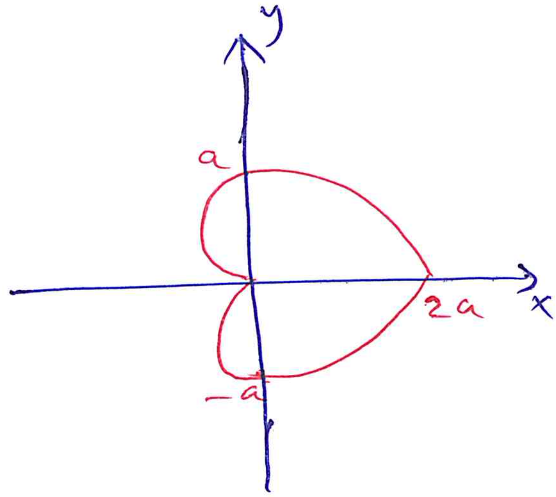
Cardiod.

Graphs

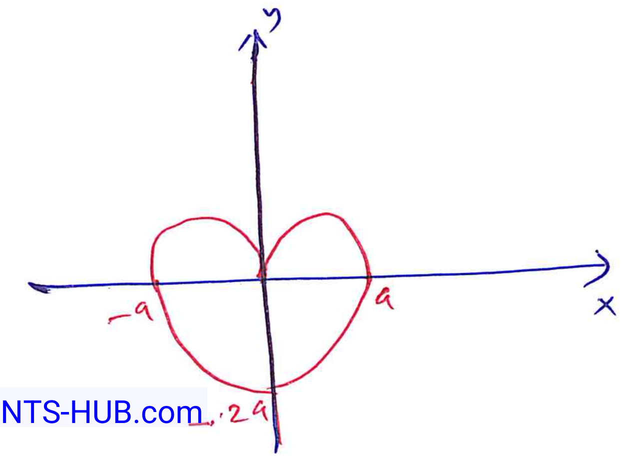
① $r = a(1-\cos\theta)$



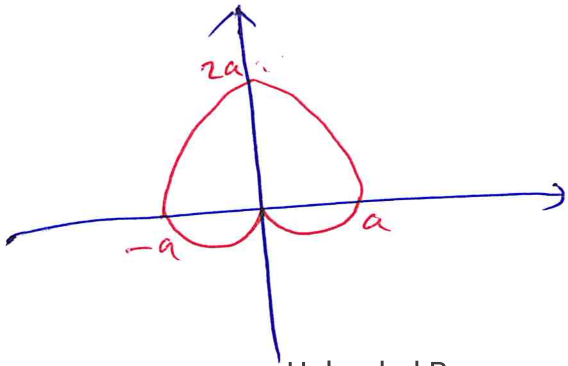
② $r = a(1+\cos\theta)$



③ $r = a(1-\sin\theta)$



④ $r = a(1+\sin\theta)$



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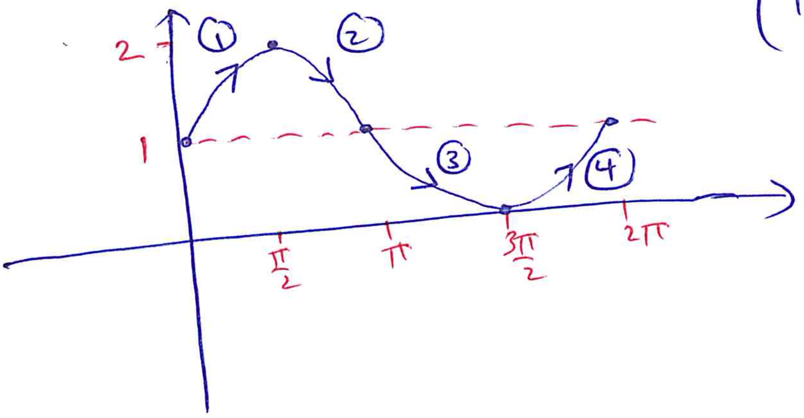
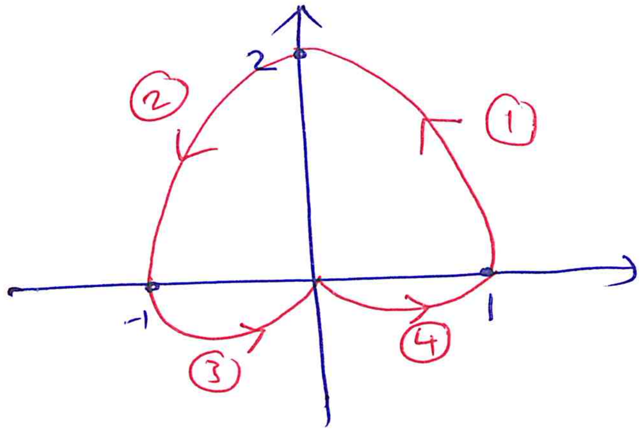
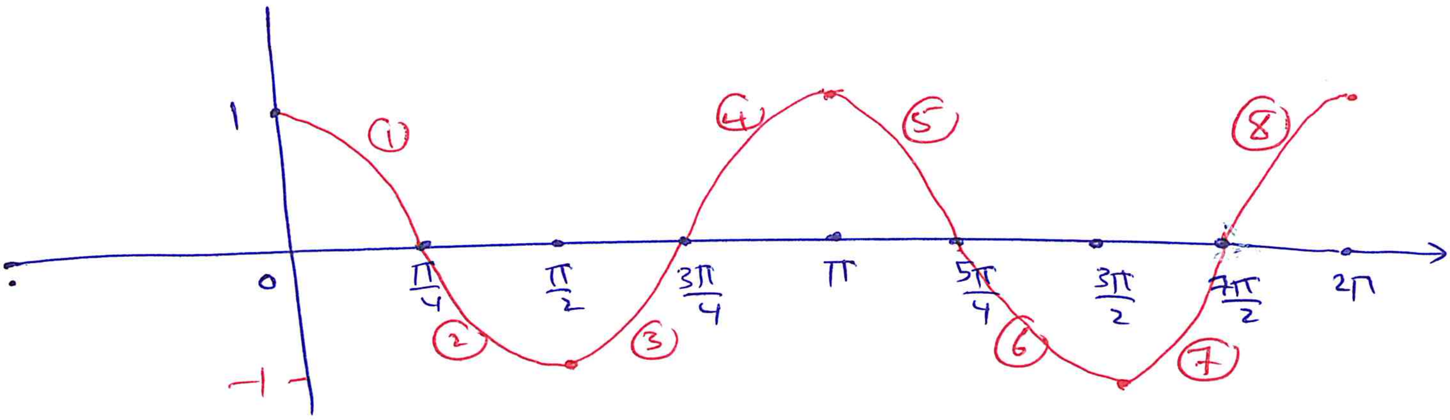
Ex. Sketch $r = 1+\sin\theta$.

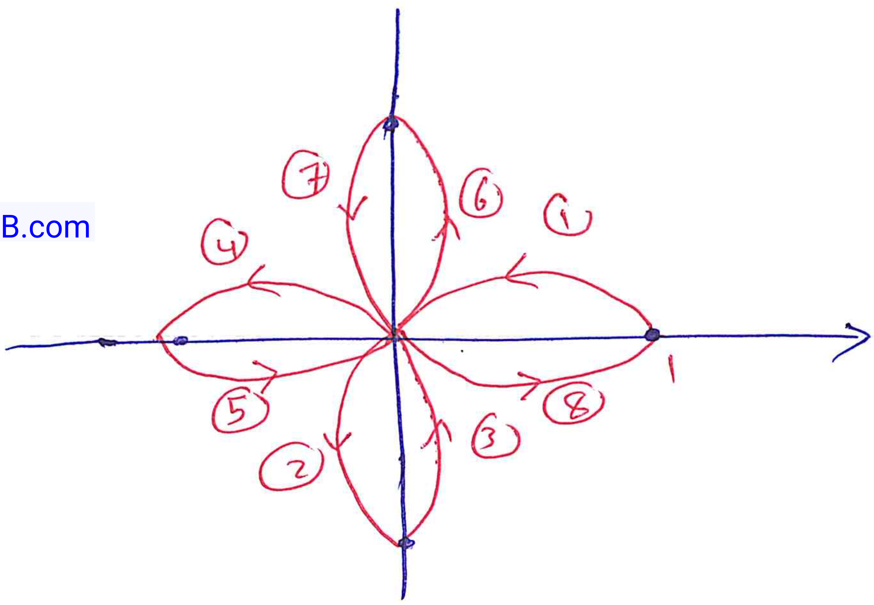
Solution. First sketch as Cartesian.





Ex. sketch $r = \cos 2\theta$





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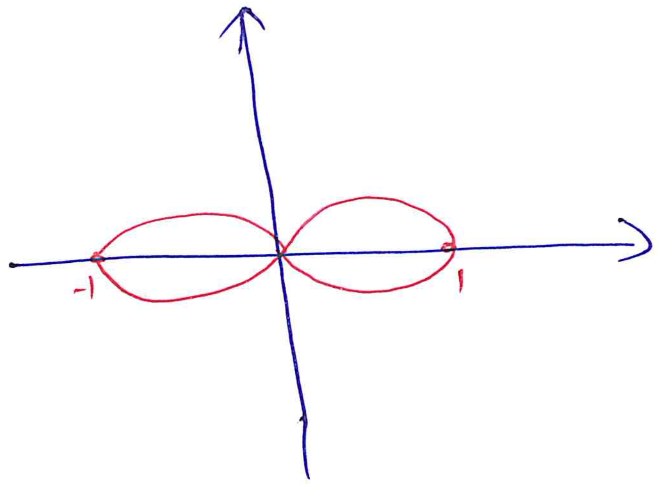
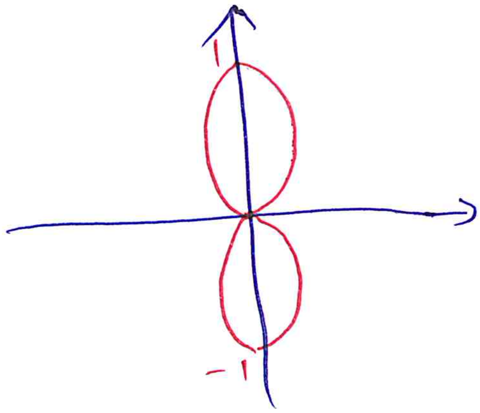
Four-leaved Rose.

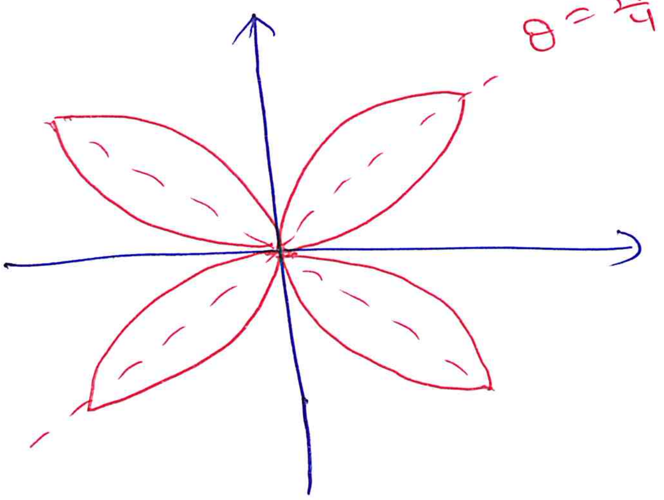
ex. $r^2 = \cos(2\theta)$



$r^2 = -\cos 2\theta$



ex. sketch $r = \sin 2\theta$



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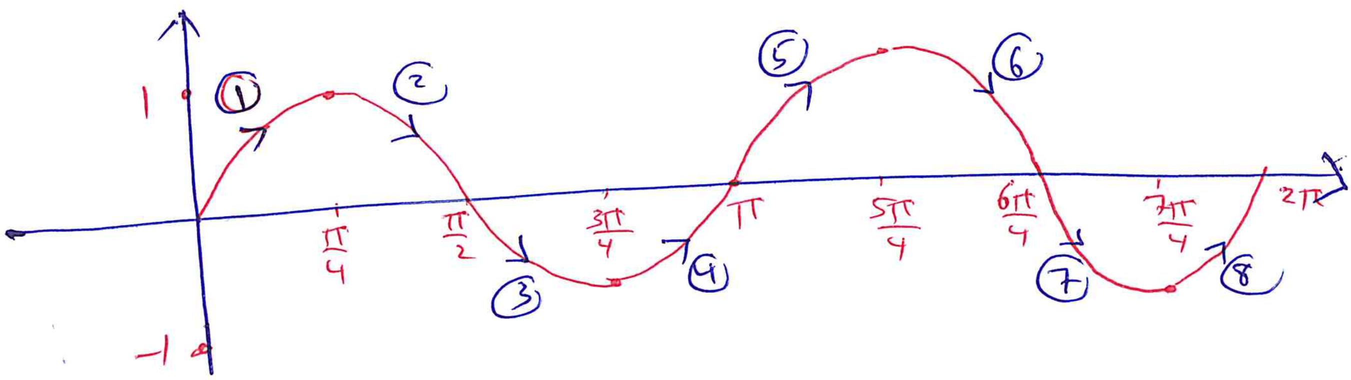
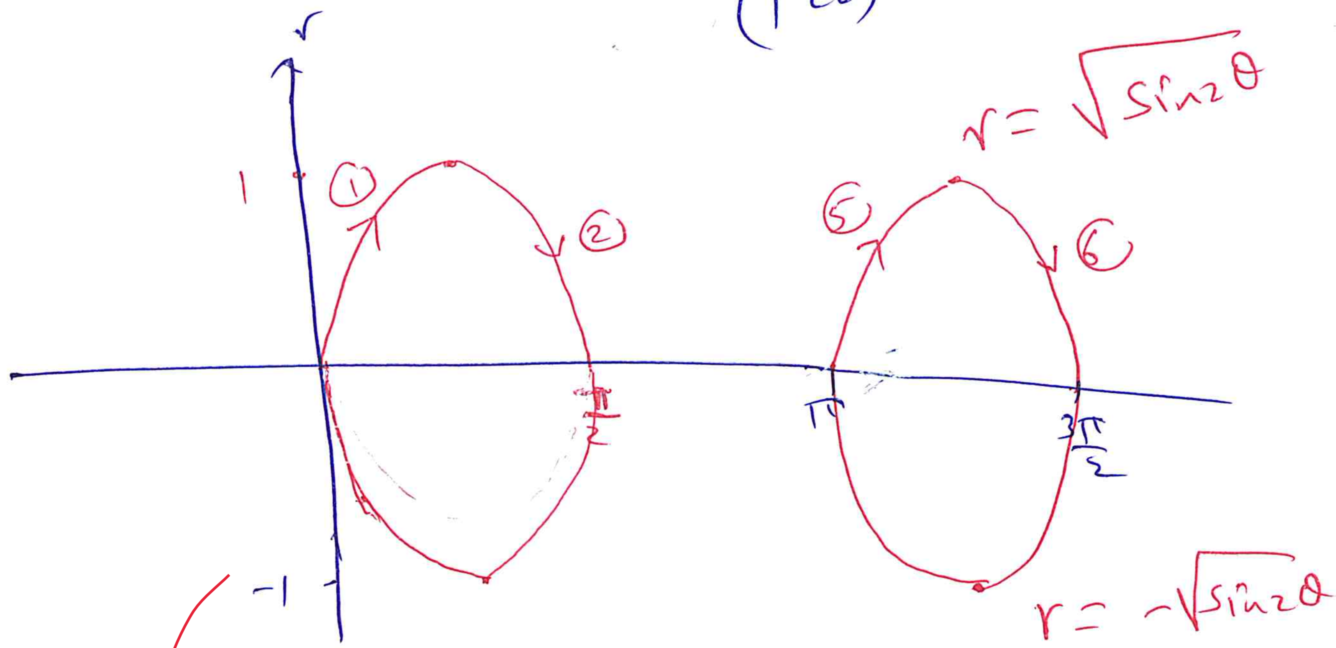
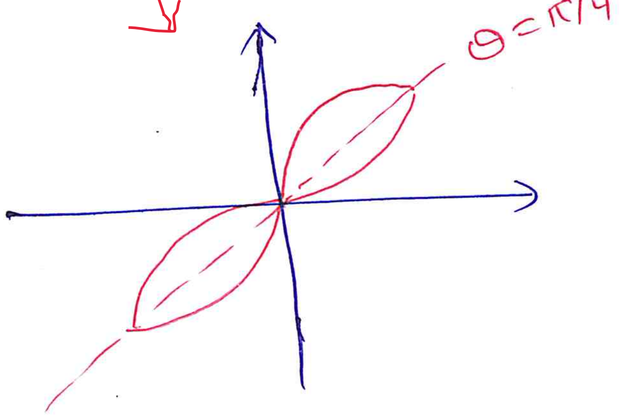
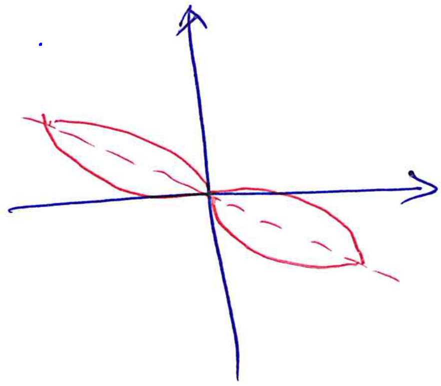
ex. $r^2 = \sin 2\theta$





$r^2 = \sin 2\theta$ ex. $\boxed{r^2 = -\sin 2\theta}$





Slope of the curve $r = f(\theta)$

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$y = r\sin\theta, \quad x = r\cos\theta$

$$\text{slope}\Big|_{(r,\theta)} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Provided $\frac{dx}{d\theta} \neq 0$

- If $r = f(\theta)$ passes through the orgin $(0, \theta_0)$. then

$$\text{slope}\Big|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0 .$$

Ex. Find the slope of the polar curve $r = -2 + 3\cos\theta$ at $\theta = \frac{\pi}{2}$.

Sol. $f(\theta) = -2 + 3\cos\theta$, $f'(\theta) = -3\sin\theta$

$$\text{slope}\Big|_{(r,\theta)} = \frac{(-3\sin\theta)(\sin\theta) + (-2+3\cos\theta)\cos\theta}{(-3\sin\theta)(\cos\theta) - (-2+3\cos\theta)\sin\theta}$$

at $\theta = \pi/2$, $\text{slope} = \frac{-3(1)(1) + (-2+0)(0)}{-3(1)(0) - (-2+0)(1)}$

$$= \frac{-3}{+2} = \frac{-3}{2}.$$

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ex. Find the slope of $r = -2 + 3\cos\theta$ at $(0, \frac{\pi}{3})$.

Sol. $\text{slope} = \tan\frac{\pi}{3} = \sqrt{3}$ (the curve passes through the orgin $(0, \frac{\pi}{3})$.

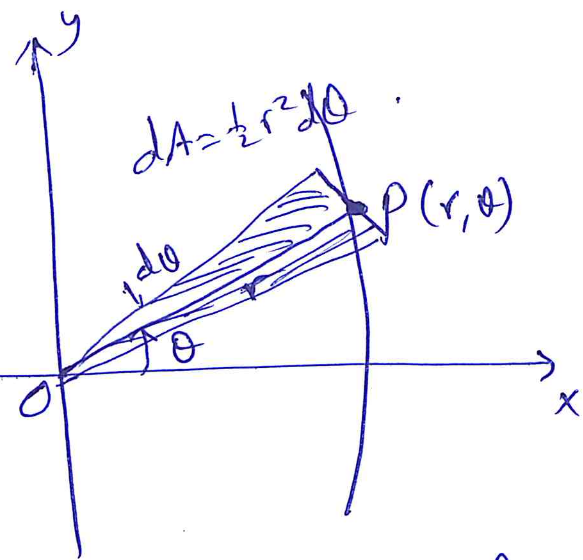
11.5 Areas and lengths in Polar coordinates

Area in the plane

Area of the Fan-Shaped Region between the orgin and the curve $r=f(\theta)$, $\alpha \le \theta \le \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta}$$



The area differential dA for the curve $r=f(\theta)$.

Ex1. Find the area of the region in the plane enclosed by the Cardiod $r = 2(1+\cos\theta)$.

Sol.

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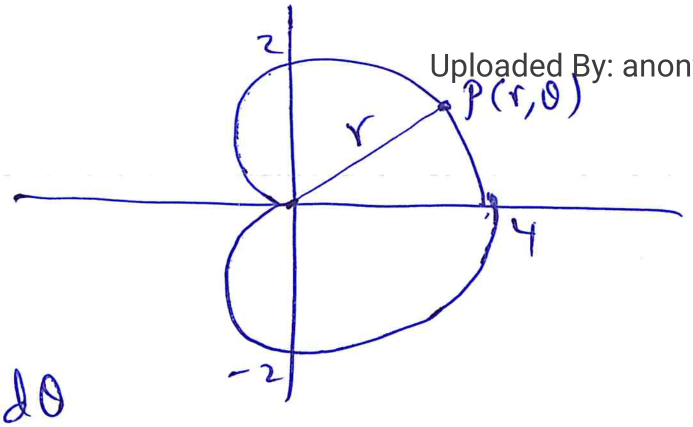
$$\text{Area} = \int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{2} \cdot 4(1+\cos\theta)^2 \, d\theta$$

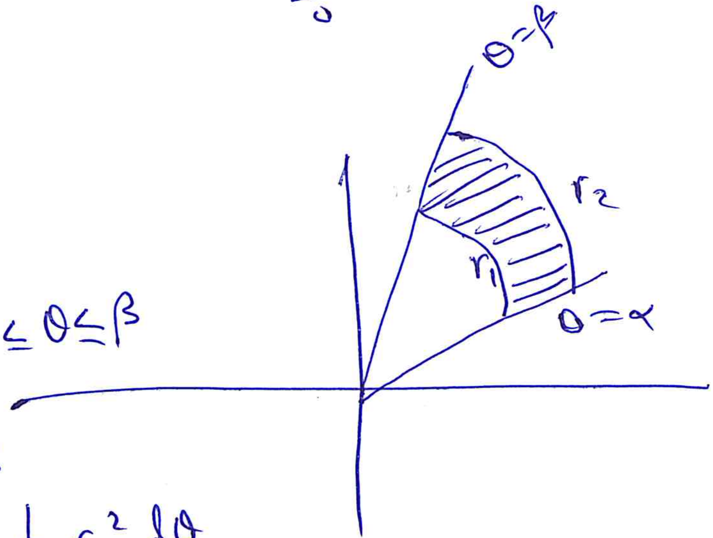


$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[2 + 4\cos\theta + 2\cdot\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= \int_0^{2\pi} \left(3 + 4\cos\theta + \cos 2\theta\right) d\theta$$

$$= 3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\Big]_0^{2\pi} = 6\pi - 0 = 6\pi.$$



Area of the region

$0 \le r_1(\theta) \le r \le r_2(\theta), \ \alpha \le \theta \le \beta$

is $A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta$

$$\boxed{A = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta}$$

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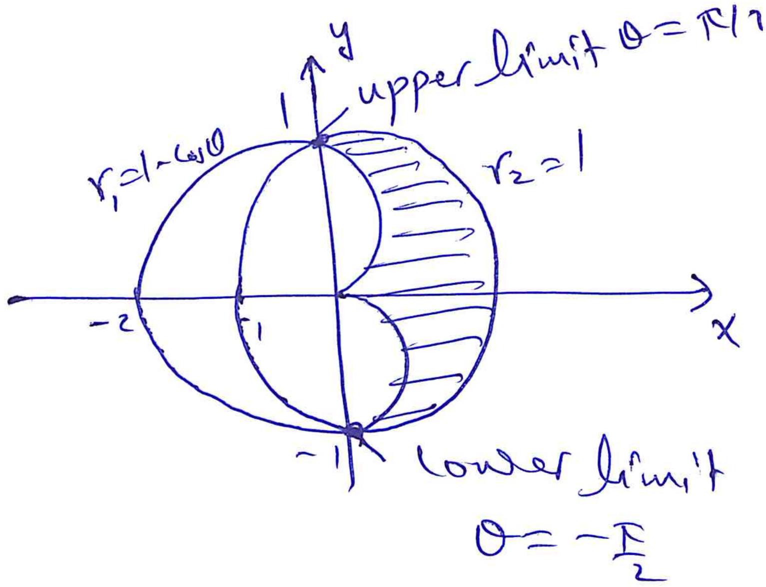
Ex 2. Find the area of the region that lies inside the circle $r=1$ and outside the cardiod $r=1-\cos\theta$.

Sol.



$$\text{Area}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2-r_1^2)\,d\theta$$

$$=2\int_0^{\pi/2} \frac{1}{2}(r_2^2-r_1^2)\,d\theta$$

$$=\int_0^{\pi/2}\left[(1)^2-(1-\cos\theta)^2\right]d\theta$$

$$=\int_0^{\pi/2}\left(\cancel{1}-\cancel{1}+2\cos\theta-\cos^2\theta\right)d\theta$$

$$=\int_0^{\pi/2}\left(2\cos\theta-\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=2\sin\theta-\frac{\theta}{2}-\frac{\sin 2\theta}{4}\Big]_0^{\pi/2}$$

$$=\left(2-\frac{\pi}{4}-0\right)-(0-0-0)=2-\frac{\pi}{4}.$$

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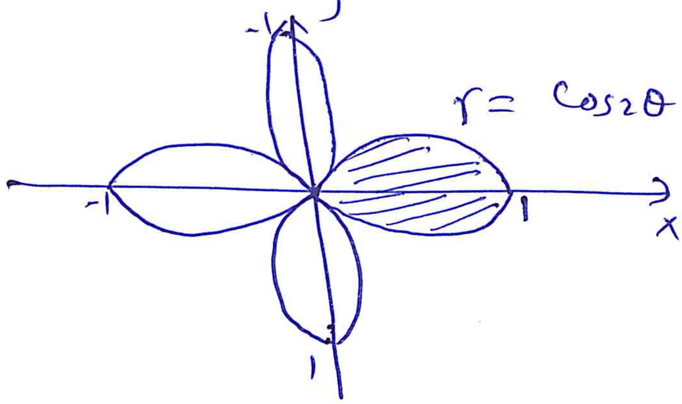
Lecture problems 5, 18.

[5] Find the area of the region that lies inside one leaf of the four-leaved rose $r = \cos 2\theta$.



Solution.

$$\text{Area} = 2\int_0^{\pi/4} \tfrac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\cos 2\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}.$$

[18] Find the area of the region that lies inside the circle $r = 4\sin\theta$ and below the horizontal line $r = 3\csc\theta$.

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Sol. $r = 4\sin\theta \Rightarrow r^2 = 4r\sin\theta$

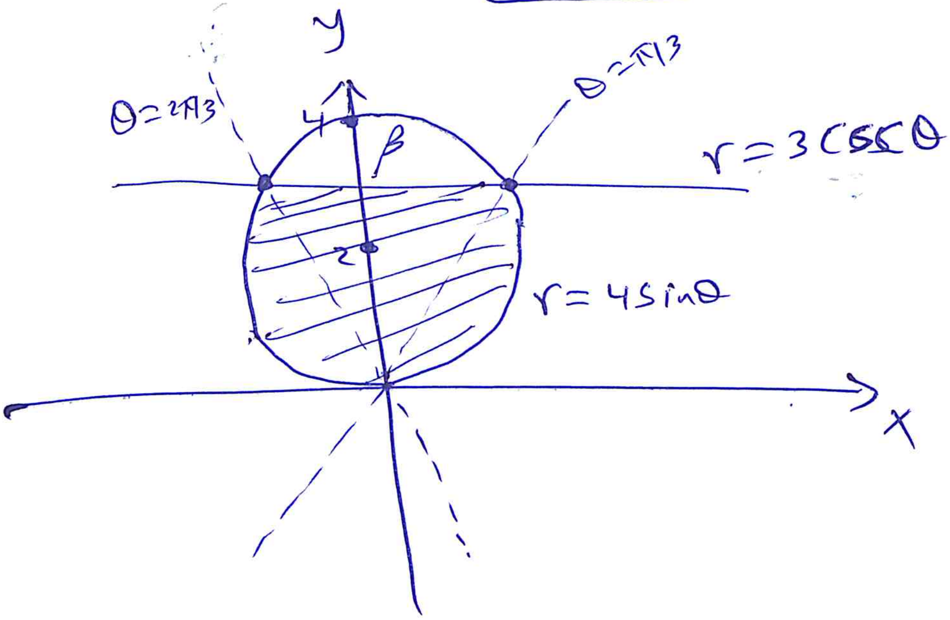
$$x^2 + y^2 = 4y$$

$$\boxed{x^2 + (y-2)^2 = 4}$$ circle with center $(0,2)$ radius $= 2$

(134)

$r = 3\csc\theta \Rightarrow r\sin\theta = 3$

$\boxed{y = 3}$ horizontal line



<u>Intersections</u> $r = 4\sin\theta$ and $r = 3\csc\theta$

$\Rightarrow 4\sin\theta = 3\csc\theta$

$\Rightarrow \sin^2\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$ ($\sin\theta > 0$ in First and second quadrant).

$\theta = \frac{\pi}{3}$ or $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

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$\therefore$ Area $= \pi(2)^2 - \beta$

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$= 4\pi - 2\int_{\pi/3}^{\pi/2}\left[\frac{1}{2}(4\sin\theta)^2 - \frac{1}{2}(3\csc\theta)^2\right]d\theta$

$= 4\pi - \int_{\pi/3}^{\pi/2}\left(16\sin^2\theta - 9\csc^2\theta\right)d\theta$

$$= 4\pi - \int_{\pi/3}^{\pi/2} \left[16 \left(\frac{1 - \cos 2\theta}{2} \right) - 9 \csc^2\theta \right] d\theta$$

$$= 4\pi - \left[8\theta - \frac{8 \sin 2\theta}{2} + 9 \cot\theta \right]_{\pi/3}^{\pi/2}$$

$$= 4\pi - \left[(4\pi - 0 + 0) - \left(\frac{8\pi}{3} - 4\left(\frac{\sqrt{3}}{2}\right) + 9 \frac{1}{\sqrt{3}} \right) \right]$$

$$= \cancel{4\pi} - \cancel{4\pi} + \frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3}$$

$$= \frac{8\pi}{3} + \sqrt{3}.$$

Length of a polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

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Q23) Find the length of the Cardiod $r = 1 + \cos\theta$.

Solution. $r = 1 + \cos\theta$, $\frac{dr}{d\theta} = -\sin\theta$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$$, we use $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

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$$= 2\int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta$$

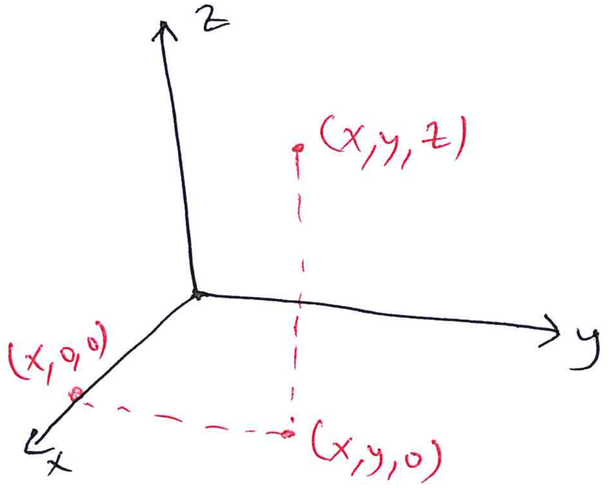
$$= 4\int_0^{\pi} \left|\cos\frac{\theta}{2}\right| d\theta$$

$$= 4\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$, since $\cos\frac{\theta}{2} \geq 0$, for $0 \leq \theta \leq \pi$

$$= 8\sin\frac{\theta}{2}\Big]_0^{\pi} = 8(1-0) = 8.$$

CH12 vectors

12.1 Three dimensional coordinates systems



- 8 octants
- First octant : $x \geq 0, y \geq 0, z \geq 0$
- xy-plane $(z = 0)$
- xz-plane $(y = 0)$
- yz-plane $(x = 0)$

Ex. Describe in space.

① $z \geq 0$: The half space consisting of all points above xy plane

② $x = -3$: plane perpendicular to x-axis at $x = -3$, parallel to yz-plane.

③ $z = 0, x \leq 0, y \geq 0$: second quadrant in xy plane

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④ $-1 \leq y \leq 1$: the slab between the planes $y = -1$ and $y = 1$.

⑤ $y = -2, z = 2$: The line in which the planes $y = -2$ and $z = 2$ intersect

⑥ $x^2 + y^2 = 4, z = 3$: Circle in the plane $z = 3$

· The distance between two points in space

$P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$

distance between P_1 and P_2 is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Ex. If $P_1(2,3,4)$, $P_2(-1,5,0)$, find $|P_1P_2|$.

Sol. $|P_1P_2| = \sqrt{(-1-2)^2+(5-3)^2+(0-4)^2}$

$$= \sqrt{9+4+16} = \sqrt{29}.$$

· The standard equation for the sphere with radius a and center (x_0, y_0, z_0) is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2$$

Ex. Find the center and the radius of the sphere $3x^2+3y^2+3z^2+2y-2z=9$.

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Sol. $x^2+y^2+\frac{2}{3}y+\frac{1}{9}+z^2-\frac{2}{3}z+\frac{1}{9} = 3+\frac{1}{9}+\frac{1}{9}$

$$(x-0)^2+\left(y+\frac{1}{3}\right)^2+\left(z-\frac{1}{3}\right)^2 = \frac{29}{9}$$

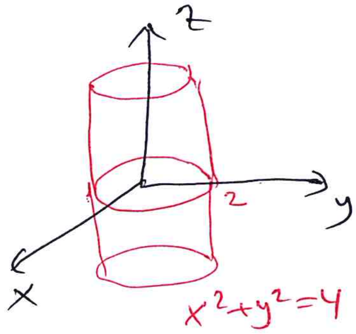
center $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ radius $= \frac{\sqrt{29}}{3}$

12.1 (Exercises)

Discussion Problems (13, 32, 34)

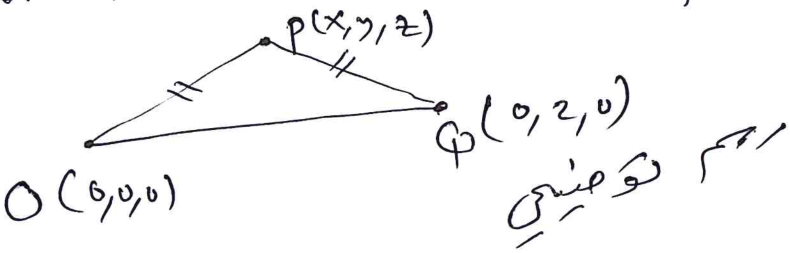
$\boxed{13}$ $x^2+y^2=4$, $z=y$

The ellipse formed by the intersection of the cylinder $x^2+y^2=4$ and the plane $z=y$.



$\boxed{32}$ The set of points in space equidistant from the orgin and the point $(0,2,0)$

Sol.



$|\overline{PO}| = |\overline{PQ}|$

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{(x-0)^2+(y-2)^2+(z-0)^2}$

$\Rightarrow \sqrt{x^2+y^2+z^2} = \sqrt{x^2+(y-2)^2+z^2}$

$\Rightarrow x^2+y^2+z^2 = x^2+y^2-4y+4+z^2$

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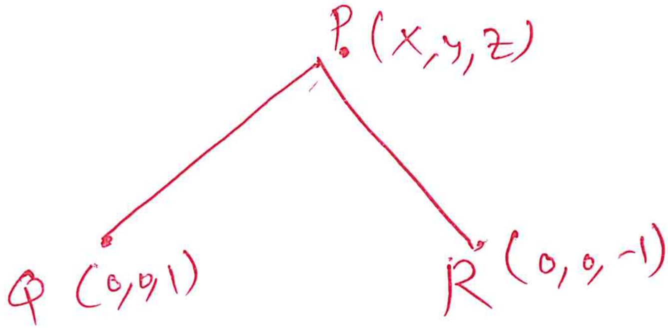
$\Rightarrow -4y+4=0 \Rightarrow y=1$

(34) The set of points in space that lie 2 units from the point $(0,0,1)$ and at the same time, 2 units from the point $(0,0,-1)$.

Sol.



$|\overline{PQ}| = 2$ and $|\overline{PR}| = 2$

$\sqrt{(x-0)^2+(y-0)^2+(z-1)^2} = 2$ and

$\sqrt{(x-0)^2+(y-0)^2+(z+1)^2} = 2$

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$\Rightarrow x^2+y^2+(z-1)^2=4$ and $x^2+y^2+(z+1)^2=4$

$\Rightarrow x^2+y^2+(z-1)^2 = x^2+y^2+(z+1)^2$

$\Rightarrow x^2+y^2+z^2-2z+1 = x^2+y^2+z^2+2z+1$

$\Rightarrow \boxed{z=0} \Rightarrow x^2+y^2=3$

$\therefore x^2+y^2=3, \quad z=0$.

Discussion Problems 8, 12, 14, 20, 26, 30, 36, 43, 56, 64

$\boxed{8}$ $y^2+z^2=1, x=0$

Note that $x=0$ is yz-plane.

$\therefore$ $y^2+z^2=1, x=0$ means the circle $y^2+z^2=1$ in the yz-plane.

$\boxed{12}$ $x^2+(y-1)^2+z^2=4, y=0$

Since $y=0$, then $x^2+(y-1)^2+z^2=4$ becomes $x^2+(0-1)^2+z^2=4$

$\Rightarrow \boxed{x^2+z^2=3}, y=0$

$\therefore$ Answer: The circle $x^2+z^2=3$ in the xz-plane.

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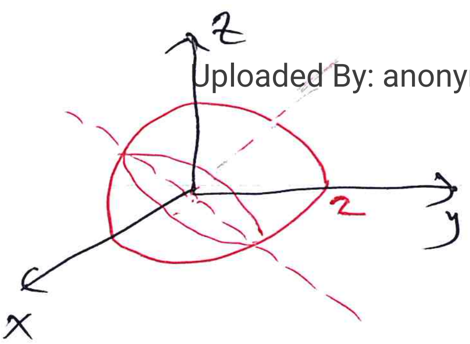
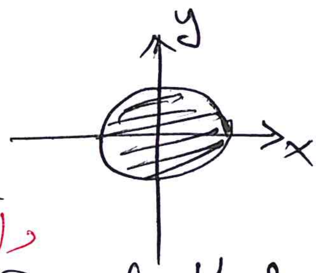
$\boxed{14}$ $x^2+y^2+z^2=4, y=x$



The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$

$\boxed{20}$ a) $x^2+y^2 \le 1$, $z=0$ ↓ xy-plane



محيط / داخل

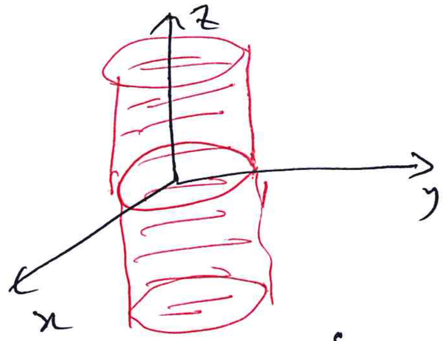
The circumference and interior of the circle $x^2+y^2=1$ in the xy-plane

b) $x^2+y^2 \le 1$, $z=3$

the circumference and interior of the circle $x^2+y^2=1$ in the plane $z=3$.

c) $x^2+y^2 \le 1$, no restriction on z

A solid cylindrical column of radius 1 whose axis is the z-axis.



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(26) The plane through the point $(3,-1,2)$ perpendicular to a) x-axis b) y-axis c) z-axis.

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Ans. (a) $x=3$ (b) $y=-1$ (c) $z=2$

$\boxed{30}$ The circle of radius 1 centered at $(-3,4,1)$ and lying in a plane parallel to the a) xy-plane b) yz-plane c) xz-plane

Sol. (a) $(x+3)^2+(y-4)^2=1$, $z=1$

(c) $(x+3)^2+(z-1)^2=1$, $y=4$.

(b) $(y-4)^2+(z-1)^2=1$, $x=-3$

$\boxed{36}$ $0\le x\le 2$, $0\le y\le 2$, $0\le z\le 2$

(The solid cube in the first octant bounded by the coordinate planes and the planes $x=2$, $y=2$, $z=2$

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$\boxed{43}$ $P_1(1,4,5)$, $P_2(4,-2,7)$

$$|P_1P_2| = \sqrt{(4-1)^2+(-2-4)^2+(7-5)^2}$$

$$= \sqrt{9+36+4} = \sqrt{49} = 7.$$

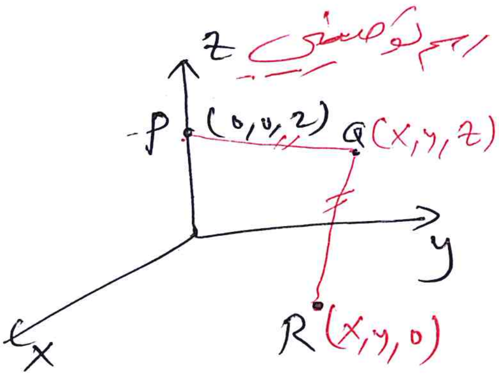
$\boxed{56}$ $x^2+y^2+z^2-6y+8z=0$

Sol. $x^2+y^2-6y+\left(\frac{-6}{2}\right)^2+z^2+8z+\left(\frac{8}{2}\right)^2=0+9+16$

$(x-0)^2+(y-3)^2+(z+4)^2=25$

Center $(0,3,-4)$ radius $=\sqrt{25}=5$

$\boxed{64}$ Find an eq. for the set of all points equidistant from the point $(0,0,2)$ and the xy-plane



Sol. $|\overline{PQ}|=|\overline{QR}|$

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2+(z-2)^2}=\sqrt{(x-x)^2+(y-y)^2+(z-0)^2}$

$\Rightarrow \sqrt{x^2+y^2+(z-2)^2}=\sqrt{z^2}$

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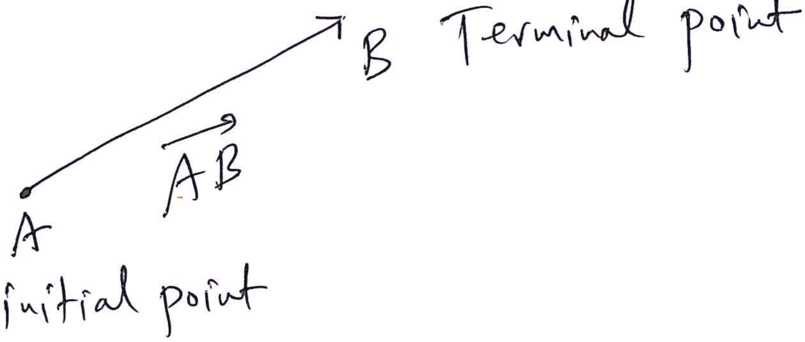
$\Rightarrow x^2+y^2+\cancel{z^2}-4z+4=\cancel{z^2}$

$x^2+y^2-4z+4=0$

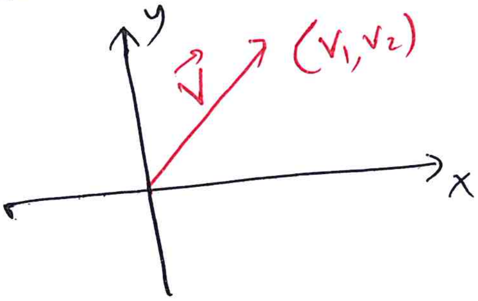
$\Rightarrow z=\frac{1}{4}x^2+\frac{1}{4}y^2+1.$

12.2 vectors



- length of $\vec{AB}$ or magnitude of $\vec{AB}$ is denoted by $|\vec{AB}|$
- Two vectors are equal if they have the same direction and length.

Two dimension (standard)



$$\vec{V} = \langle V_1, V_2 \rangle$$

$$\vec{V} = \langle V_1, V_2, V_3 \rangle$$

Three dimension

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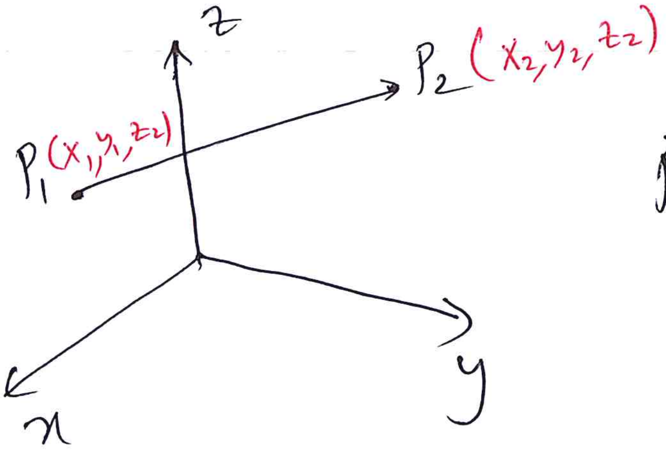

$$\vec{P_1P_2} = P_2 - P_1$$

$$= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Components (direction)

$$\text{length} = |\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Ex. Find (a) the component (b) the length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

Sol. (a) $\overrightarrow{PQ} = Q - P = \langle -5+3,\ 2-4,\ 2-1 \rangle$
$= \langle -2, -2, 1 \rangle$

(b) $\text{length} = |\overrightarrow{PQ}| = \sqrt{4+4+1} = 3$.

Vectors Algebra operations

Let $\vec{V} = \langle V_1, V_2, V_3 \rangle$, $\vec{u} = \langle u_1, u_2, u_3 \rangle$. Then

1. $\vec{u} + \vec{V} = \langle u_1+V_1,\ u_2+V_2,\ u_3+V_3 \rangle$
2. $\vec{u} - \vec{V} = \langle u_1-V_1,\ u_2-V_2,\ u_3-V_3 \rangle$
3. $\alpha \vec{u} = \langle \alpha u_1,\ \alpha u_2,\ \alpha u_3 \rangle$, α is scalar.

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Ex. If $\vec{u} = \langle -1, 3, 1 \rangle$, $\vec{V} = \langle 4, 7, 0 \rangle$,
find (a) $2\vec{u} + 3\vec{V}$ (b) $|\frac{1}{2}\vec{V}|$

Sol. (a) $2\vec{u} + 3\vec{V} = 2\langle -1,3,1 \rangle + 3\langle 4,7,0 \rangle$
$= \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle$
$= \langle -2+12,\ 6+21,\ 2+0 \rangle$
$= \langle 10, 27, 2 \rangle$

(b) $\left|\frac{1}{2}\vec{V}\right| = \left|\langle 2, \frac{7}{2}, 0\rangle\right|$

$$= \sqrt{4+\frac{49}{4}+0} = \sqrt{65/4} = \frac{\sqrt{65}}{2}$$

- Properties of vectors operations

1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
3. $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = \vec{0}$
5. $0\vec{u} = \vec{0}$.
6. $1\vec{u} = \vec{u}$
7. $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$, α scalar
8. $\alpha\beta\vec{u} = \alpha(\beta\vec{u})$, α,β scalars
9. $(\alpha+\beta)\vec{u} = \alpha\vec{u}+\beta\vec{u}$, α,β scalars.

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- Unit vectors

A vector of length 1 is called unit vector.

Standard unit vectors

$i = \langle 1,0,0\rangle$, $j = \langle 0,1,0\rangle$, $k = \langle 0,0,1\rangle$

Notice

$$\vec{V} = \langle V_1, V_2, V_3 \rangle$$

$$= \langle V_1, 0, 0\rangle + \langle 0, V_2, 0\rangle + \langle 0, 0, V_3\rangle$$

$$= V_1\langle 1,0,0\rangle + V_2\langle 0,1,0\rangle + V_3\langle 0,0,1\rangle$$

$$= V_1 i + V_2 j + V_3 k.$$

Ex. $\vec{V} = \langle -1, 3, 1\rangle = -i + 3j + k.$

• If $\vec{u} \neq \vec{0}$, then $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$, called the direction of the nonzero vector $\vec{u}$.

Ex. Find a unit vector $\vec{V}$ in the direction of vector $\overrightarrow{P_1P_2}$ where $P_1(1,0,1)$, $P_2(3,2,0)$.

Sol. $\vec{V} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|}$

$$\overrightarrow{P_1P_2} = P_2 - P_1$$
$$= (3-1)i + (2-0)j + (0-1)k$$
$$= 2i + 2j - k$$

$$|\overrightarrow{P_1P_2}| = \sqrt{4+4+1} = 3.$$

$\therefore \vec{V} = \frac{1}{3}(2i + 2j - k)$

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$$= \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Q33 Find a vector $\vec{w}$ of length 7 in the direction of $\vec{V} = 12i - 5k$

Sol. $|\vec{V}| = \sqrt{(12)^2 + (-5)^2} = 13$

$$\vec{w} = 7\frac{\vec{V}}{|\vec{V}|} = \frac{7}{13}(12i - 5k)$$

Ex: If $\vec{V} = 3i - 4j$ is a velocity vector, express $\vec{V}$ as a product of its speed times a unit vector in the direction of motion.

Sol: $|\vec{V}| = \sqrt{9+16} = 5$ (speed = length of $\vec{V}$).

. the unit vector $\frac{\vec{V}}{|\vec{V}|}$ has the same direction of $\vec{V}$

$$\frac{\vec{V}}{|\vec{V}|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$\therefore \vec{V} = 3i - 4j = |\vec{V}| \frac{\vec{V}}{|\vec{V}|}$$

$$= \underset{\substack{\downarrow \\ \text{length} \\ \text{(speed)}}}{(5)} \underbrace{\left(\frac{3}{5}i - \frac{4}{5}j\right)}_{\text{Direction of motion.}}$$

Summary. If $\vec{V} \neq \vec{0}$, then

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(1) $\frac{\vec{V}}{|\vec{V}|}$ is a unit vector in the direction of $\vec{V}$.

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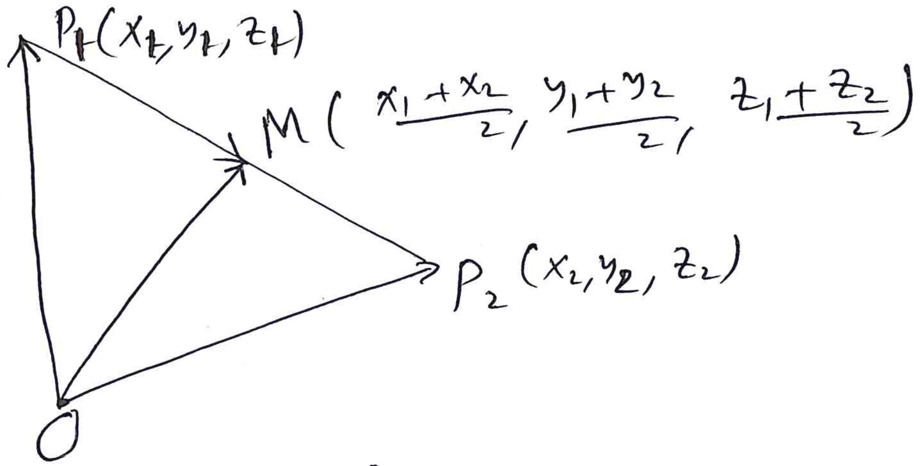
(2) the equation $\vec{V} = |\vec{V}| \frac{\vec{V}}{|\vec{V}|}$ expresses $\vec{V}$ as its length times its direction.

• Midpoint of a line segment

The Midpoint M of the line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



$$\vec{OM} = \vec{OP_1} + \frac{1}{2}(\vec{P_1P_2})$$
$$= \vec{OP_1} + \frac{1}{2}(\vec{OP_2} - \vec{OP_1})$$
$$= \frac{1}{2}(\vec{OP_1} + \vec{OP_2})$$
$$= \frac{1}{2}(x_1 i + y_1 j + z_1 k + x_2 i + y_2 j + z_2 k)$$
$$= \frac{x_1+x_2}{2} i + \frac{y_1+y_2}{2} j + \frac{z_1+z_2}{2} k$$

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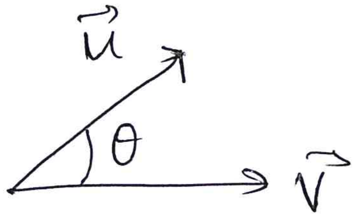
Ex. The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$M\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5, 1, 2).$$

12.3 The Dot Product

Df: the dot product ($\vec{u}$ dot $\vec{v}$) of vectors $\vec{u} = u_1 i + u_2 j + u_3 k$ and $\vec{v} = v_1 i + v_2 j + v_3 k$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$= |\vec{u}|\,|\vec{v}| \cos\theta$$



The angle between $\vec{u}$ and $\vec{v}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \right)$$

Ex. If $\vec{u} = \langle 1, -2, -1 \rangle$, $\vec{v} = \langle -6, 2, -3 \rangle$, then

$$\vec{u} \cdot \vec{v} = (1)(-6) + (-2)(2) + (-1)(-3)$$
$$= -6 - 4 + 3 = -7.$$

Ex. Find the angle between $\vec{u} = i - 2j - k$ and $\vec{v} = 6i + 3i + 2k$

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Sol: $|\vec{u}| = \sqrt{1+4+1} = \sqrt{6}$, $|\vec{v}| = \sqrt{36+9+4} = 7$

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$$\vec{u} \cdot \vec{v} = (1)(6) + (-2)(3) + (-1)(2) = -2$$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \right) = \cos^{-1}\left(\frac{-2}{7\sqrt{6}} \right) \approx \ldots$$

- Two vectors are orthogonal (perpendicular) iff $\vec{u} \cdot \vec{v} = 0$

ex. $\vec{0}$ is orthogonal to every vector $\vec{u}$

since $\vec{0}\cdot\vec{u} = \langle 0,0,0\rangle \cdot \langle u_1, u_2, u_3\rangle$

$= (0)(u_1) + (0)(u_2) + (0)(u_3)$

$= 0.$

ex. $\vec{u} = 3i - 2j + k$ and $\vec{v} = 2j + 4k$ are orthogonal because

$\vec{u}\cdot\vec{v} = (3)(0) - (2)(2) + (1)(4) = -4 + 4 = 0.$

ex. If $\vec{u} = 3i - 2j + k$ and $\vec{v} = 2j + xk$ are orthogonal, find x.

sol: $\vec{u}\cdot\vec{v} = 0 \Rightarrow (3)(0) + (-2)(2) + (1)(x) = 0$

$\Rightarrow -4 + x = 0 \Rightarrow x = 4.$

Dot Product Properties

If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors, c is scalar then

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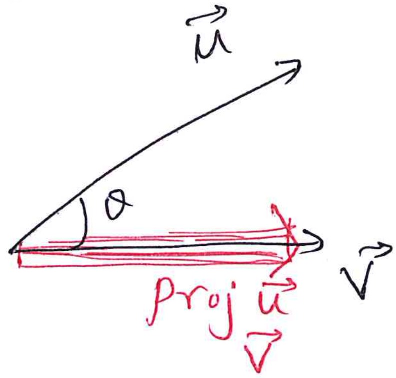
① $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ ② $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

③ $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

④ $\vec{0}\cdot\vec{u} = 0$ ⑤ $\vec{u}\cdot\vec{u} = |\vec{u}|^2.$

Vector projection



- $$\text{Proj}_{\vec{v}} \vec{u} = (\text{length})(\text{direction})$$

$$= \left(|\vec{u}| \cos\theta\right) \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$\therefore$ The vector projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{Proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

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- The scalar component of $\vec{u}$ onto $\vec{v}$ is

$$\text{Comp}_{\vec{v}} \vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

Ex. If $\vec{u} = 6i + 3j + 2k$

$\vec{V} = i - 2j - 2k$, find

(a) $\text{proj}_{\vec{V}} \vec{u}$ (b) $\text{comp}_{\vec{V}} \vec{u}$.

Sol. (a) $\vec{u} \cdot \vec{V} = 6 - 6 - 4 = -4$

$|\vec{V}| = \sqrt{1+4+4} = 3$

$$\text{proj}_{\vec{V}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{V}}{|\vec{V}|^2}\right)\vec{V} = \frac{-4}{9}(i - 2j - 2k)$$

$$= -\frac{4}{9}i + \frac{8}{9}j + \frac{8}{9}k$$

(b) $\text{Comp}_{\vec{V}} \vec{u} = \dfrac{\vec{u} \cdot \vec{V}}{|\vec{V}|} = -\dfrac{4}{3}$.

Ex. If $\vec{u} = 4$, $\vec{V} = 5$, $\theta = \pi/3$ is the angle between $\vec{u}$ and $\vec{V}$, then find $|\vec{u} + \vec{V}|$.

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Sol. $|\vec{u} + \vec{V}|^2 = (\vec{u} + \vec{V}) \cdot (\vec{u} + \vec{V})$

$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{V} + \vec{V} \cdot \vec{u} + \vec{V} \cdot \vec{V}$

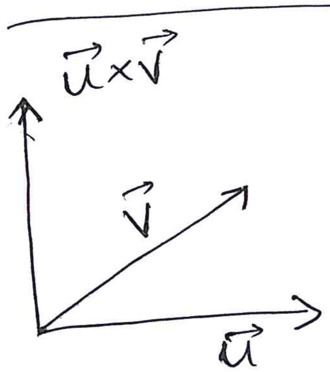
$= |\vec{u}|^2 + 2|\vec{u}||\vec{V}| \cos\frac{\pi}{3} + |\vec{V}|^2$

$= (4)^2 + 2(4)(5)\left(\frac{1}{2}\right) + (5)^2 = 61$

$\Rightarrow |\vec{u} + \vec{V}| = \sqrt{61}$.

12.4 The Cross Product

The cross product of Two vectors in space



- $\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$,

  $\vec{n}$: is the unit vector normal to plane containing $\vec{u}$ and $\vec{v}$.

- Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u}\times\vec{v} = \vec{0}$.

Properties of the cross product

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be any vectors and r, s are scalars. Then

1. $(r\vec{u})\times(s\vec{v}) = (rs)(\vec{u}\times\vec{v})$.
2. $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$
3. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$.
4. $(\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u} + \vec{w}\times\vec{u}$

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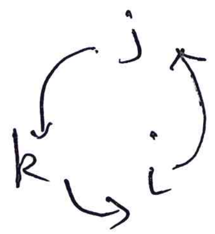
5. $\vec{0}\times\vec{u} = \vec{0}$.
6. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$.



$$\left.\begin{aligned} i\times j &= k \\ j\times k &= i \\ k\times i &= j \end{aligned}\right\} \quad i\times i = j\times j = k\times k = \vec{0}$$

· Calculating the cross product as determinant

If $\vec{u} = u_1 i + u_2 j + u_3 k$, $\vec{v} = v_1 i + v_2 j + v_3 k$,

then

$$\vec{u} \times \vec{v} = (u_1 i + u_2 j + u_3 k) \times (v_1 i + v_2 j + v_3 k)$$

$$= \cancel{(u_1 v_1)\, i \times i}^{\,\vec{0}} + (u_1 v_2)\, i \times j + (u_1 v_3)\, i \times k$$

$$+ (u_2 v_1)\, j \times i + \cancel{(u_2 v_2)\, j \times j}^{\,\vec{0}} + (u_2 v_3)\, j \times k$$

$$+ (u_3 v_1)\, k \times i + (u_3 v_2)\, k \times j + \cancel{(u_3 v_3)\, k \times k}^{\,\vec{0}}$$

$$= (u_1 v_2) k + (u_1 v_3)(-j) + (u_2 v_1)(-k)$$

$$+ (u_2 v_3)\, i + (u_3 v_1)\, j + (u_3 v_2)(-i)$$

$$= (u_2 v_3 - u_3 v_2)\, i + (u_3 v_1 - u_1 v_3)\, j + (u_1 v_2 - u_2 v_1)\, k$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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Ex. If $\vec{u} = 2i + j + k$, $\vec{v} = -4i + 3j + k$

a) Find $\vec{u} \times \vec{v}$.

Sol.

$$\vec{u} \times \vec{v} = \begin{array}{ccc} \oplus & \ominus & \oplus \\ \end{array}$$
$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= (1-3)i - (2+4)j + (6+4)k$$

$$= -2i - 6j + 10k.$$

b) Find a unit vector perpendicular to the plane containing $\vec{u}$ and $\vec{v}$.

Sol.
$$\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{1}{\sqrt{4+36+100}}(-2i - 6j + 10k)$$

$$= \frac{1}{\sqrt{140}}(-2i - 6j + 10k)$$

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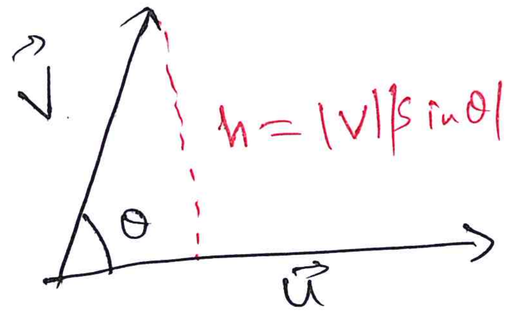
- $\boxed{|\vec{u} \times \vec{v}| \text{ is the area of a parallelogram}}$

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,|\sin\theta|\,|\vec{n}| = |u|\,|v| \sin\theta$$



Area = base . height

$= |\vec{u}|\,|\vec{v}|\,|\sin\theta| = |\vec{u} \times \vec{v}|$

Ex. Find the area of the triangle with vertices $P(+1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Sol. Area $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$, where

$\vec{PQ} = (2-1)i + (1+1)j + (-1-0)k$

$= i + 2j - k.$

$\vec{PR} = (-1-1)i + (1+1)j + (2-0)k$

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$= -2i + 2j + 2k.$

$|\vec{PQ} \times \vec{PR}| = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= (4+2)i - (2-2)j + (2+4)k$$

$$= 6i + 6k$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{6^2+6^2} = 6\sqrt{2}$$

$\therefore$ the area of the triangle $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$$= \frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}.$$

- <u>Triple scalar or Box product</u>

The product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is called the triple scalar product of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

$$\begin{array}{c} \\ (\vec{u} \times \vec{v}) \cdot \vec{w} = \end{array} \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{array} \cdot \vec{w}$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k \right) \cdot \vec{w}$$

$$= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

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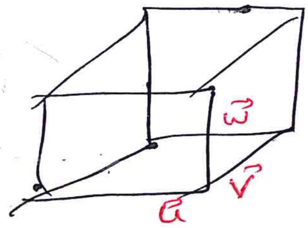
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\therefore (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \text{ where}$$

$\vec{u} = u_1 i + u_2 j + u_3 k$, $\vec{v} = v_1 i + v_2 j + v_3 k$, and $\vec{w} = w_1 i + w_2 j + w_3 k$.



• $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$ = the volume of the parallelepiped (parallelogram-sided box) determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$

See Figure 12.34 page 685.

Ex. Find the volume of the box (parallelepiped) determined by $\vec{u} = i + 2j - k$, $\vec{v} = -2i + 3k$, and $\vec{w} = 7j - 4k$.

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Sol. $(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$

$= 1(0 - 21) - 2(8 - 0) - 1(-14 - 0)$

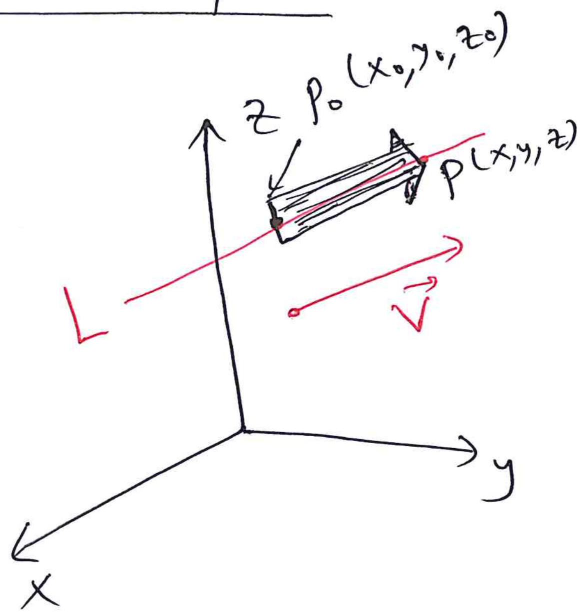
$= -21 - 16 + 14 = -23$.

12.5 Lines and Planes in Space

. Line and line segments in space

In the plane, a line is determined by a point and a number giving the slope of the line

[$y - y_0 = m(x - x_0)$]



In space, a line is determined by a point and a vector giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\vec{v} = v_1 i + v_2 j + v_3 k$

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Then $L = \{ P(x, y, z) : \overrightarrow{P_0P} \parallel \vec{v} \}$

thus $\overrightarrow{P_0P} = t\,\vec{v}$, t scalar parameter

$-\infty < t < \infty$

$\Rightarrow (x-x_0)i+(y-y_0)j+(z-z_0)k = t(v_1 i+v_2 j+v_3 k)$

$\Rightarrow xi+yj+zk = \underbrace{x_0 i+y_0 j+z_0 k}_{} + t(v_1 i+v_2 j+v_3 k)$

$$\boxed{\vec{r}(t) = r_0 + t\vec{V}, \quad -\infty < t < \infty}$$

Also, $x = x_0+tv_1$, $y = y_0+tv_2$, $z = z_0+tv_3$, $-\infty < t < \infty$.

- Vector equation for the line L through $P_0(x_0,y_0,z_0)$ parallel to $\vec{V}$ is

$$\boxed{\vec{r}(t) = r_0 + t\vec{V}, \quad -\infty < t < \infty}$$

where $\vec{r}$ is is the position vector of a point $P(x,y,z)$ on L and r_0 is the position vector of $P_0(x_0,y_0,z_0)$.

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- The standard parametric equations of the line through $P_0(x_0,y_0,z_0)$ parallel to $\vec{V} = v_1 i+v_2 j+v_3 k$ are

$$\boxed{x = x_0+tv_1, \quad y = y_0+tv_2, \quad z = z_0+tv_3, \quad -\infty < t < \infty}$$

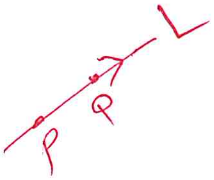
Ex 1. Find parametric equations for the line through $(\overset{x_0}{-2}, \overset{y_0}{0}, \overset{z_0}{4})$ parallel to $\vec{v} = \overset{v_1}{2}i + \overset{v_2}{4}j \overset{v_3}{-2}k$.

Sol. $x = -2 + 2t, \ y = 0 + 4t, \ z = 4 - 2t, \quad -\infty < t < \infty$

or $x = -2 + 2t, \ y = 4t, \ z = 4 - 2t, \quad -\infty < t < \infty$

Ex 2. Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Sol. the vector $\vec{PQ} = (1+3)i + (-1-2)j + (4+3)k$
$= 4i - 3j + 7k$ is



parallel to the line. Take P_0 is P.

$x = -3 + 4t, \ y = 2 - 3t, \ z = -3 + 7t, \quad -\infty < t < \infty$

are the parametric equations for the line

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Rmk. You can take Q as P_0.

Ex 3. Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Sol. $\vec{V} = \vec{PQ} = 4i - 3j + 7k$.

P_0 is $P(-3, 2, -3)$

$\therefore x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t,$

$0 \leq t \leq 1$

Note that $t = 0 \Rightarrow x = -3, y = 2, z = -3$ is P

$t = 1 \Rightarrow x = 1, y = -1, z = 4$ is Q.

Rmk. $r(t) = r_0 + t\vec{V}$ (vector Eq).

$$= \underbrace{r_0}_{\text{initial position}} + \underbrace{t}_{\text{Time}}\underbrace{|\vec{V}|}_{\text{Speed}}\underbrace{\left(\frac{\vec{V}}{|\vec{V}|}\right)}_{\text{Direction}}$$

If we think of a line as the path of a particle starting at P_0 and moving in the direction of vector $\vec{V}$

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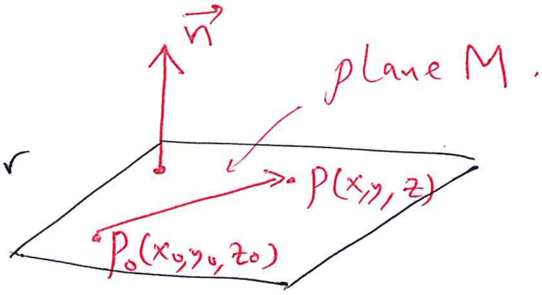
• An Equation for a plane in space

A plane in space is determined by a point on the plane and its orientation (a vector that is perpendicular to the plane $\vec{n}$).



Suppose that plane M passes through $P_0(x_0,y_0,z_0)$ and is normal to the nonzero vector $\vec{n} = Ai + Bj + Ck$. Then

$$M = \{ P(x,y,z) : \overrightarrow{P_0P} \text{ is orthogonal to } \vec{n} \}$$

Thus, $\boxed{\overrightarrow{P_0P} \cdot \vec{n} = 0}$ — (1)

$$\Rightarrow [(x-x_0)i + (y-y_0)j + (z-z_0)k] \cdot (Ai + Bj + Ck) = 0$$

$$\Rightarrow \boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0} \quad (2)$$

$$\Rightarrow \boxed{Ax + By + Cz = D}, \quad (3)$$ where

$$D = Ax_0 + By_0 + Cz_0$$

(1) is called vector equation

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(2) is called component equation

(3) " " " " simplified.

Ex(6) Find an equation for the plane through $P_0(\overset{x_0}{-3}, \overset{y_0}{0}, \overset{z_0}{7})$ perpendicular to $\vec{n} = \overset{A}{5}i + \overset{B}{2}j \overset{C}{-}k$

Sol. the component eq. is

$$5(x+3) + 2(y-0) + (-1)(z-7) = 0$$

Simplifying we obtain $5x + 15 + 2y - z + 7 = 0$

$$\Rightarrow \boxed{5x + 2y - z = -22}$$

Ex. 7 Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Sol. We find a vector normal to the plane

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$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$= (2i - k) \times (3j - k)$$

$$= \begin{array}{ccc} + & - & + \end{array}$$

$$= \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k$$

You can take any point say $A(0,0,1)$.

The eq. is

$$3(x-0)+2(y-0)+6(z-1)=0$$

$$\Rightarrow 3x+2y+6z=6$$

- line of Intersection

- Two planes are parallel iff their normals are parallel. That is
  $L_1 \parallel L_2$ iff $\vec{n}_1 = k\vec{n}_2$, for some scalar k.

- Two planes that are not parallel intersect in a line

Ex 8. Find a vector parallel to the line of intersection of the planes

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$$3x-6y-2z=15 \text{ and } 2x+y-2z=5.$$

Sol: $\vec{n}_1 = 3i-6j-2k$, $\vec{n}_2 = 2i+j-2k$

$\vec{n}_1 \times \vec{n}_2$ is a vector parallel to the planes' line of intersection.

$$\vec{n}_1 \times \vec{n}_2 = \begin{array}{ccc} \oplus & \ominus & + \\ \end{array}$$
$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix} k$$

$$= (12+2)i - (-6+4)j + (3+12)k$$

$$= 14i + 2j + 15k.$$

Any nonzero scalar multiple of $\vec{n}_1 \times \vec{n}_2$ will do as well

Ex 9 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect

sol. $\boxed{\vec{V} = 14i + 2j + 15k}$ (Example 8).

- We need a point P on the line

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$$6(2x + y = 5)$$

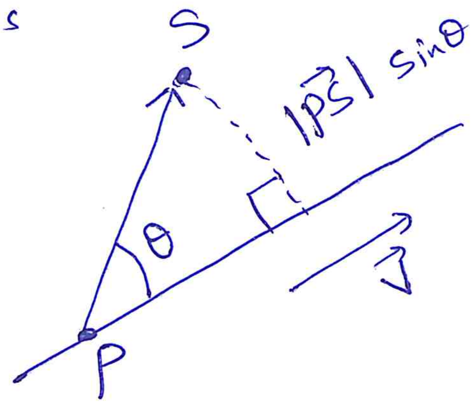
$$15x = 45 \Rightarrow \boxed{x = 3} \Rightarrow \boxed{y = -1}$$

$$\boxed{P(3, -1, 0)}$$

the line is $x = 3 + 14t$, $y = -1 + 2t$, $z = 15t$, $-\infty < t < \infty$.

Distance from a point S to a line through P parallel to $\vec{v}$ is

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$



Ex5. Find the distance from the point $S(1,1,5)$ to the line $L: x = 1+t,\ y = 3-t,\ z = 2t$.

Sol. We see from the eqs for L that L passes through $P(1,3,0)$ parallel to $\vec{v} = i - j + 2k$

$$\vec{PS} = (1-1)i + (1-3)j + (5-0)k = -2j + 5k.$$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i + 5j + 2k$$

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$$\therefore\ d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

The distance from a point to a plane

If P is a point on a plane with normal $\vec{n}$, then the distance from any point S to the plane is $d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$

where $\vec{n} = Ai + Bj + Ck$ is normal to the plane.

Ex 11. Find the distance from $S(1,1,3)$ to the plane $3x + 2y + 6z = 6$.

Sol. $\vec{n} = 3i + 2j + 6k$ is normal to the plane.

$P(0,3,0)$ is a point on the plane.

$\vec{PS} = (1-0)i + (1-3)j + (3-0)k = i - 2j + 3k$

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$|\vec{n}| = \sqrt{9+4+36} = 7.$

The distance from S to the plane is

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| (i - 2j + 3k) \cdot \left(\tfrac{3}{7}i + \tfrac{2}{7}j + \tfrac{6}{7}k \right) \right|$$

$$= \left| \tfrac{3}{7} - \tfrac{4}{7} + \tfrac{18}{7} \right| = \tfrac{17}{7}.$$

Angle Between planes

Ex12. Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

Solution. the vectors

$$\vec{n}_1 = 3i-6j-2k \quad , \vec{n}_2 = 2i+j-2k$$

are normals to the planes. the angle between them is

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{(3)(2)-(6)(1)+(-2)(-2)}{\sqrt{9+36+4}\,\sqrt{4+1+4}}\right)$$

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$$= \cos^{-1}\left(\frac{4}{21}\right)$$

$$\approx 1.38 \text{ radians.} \ (\approx 79^\circ).$$