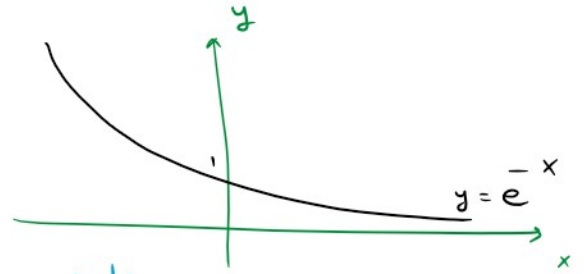
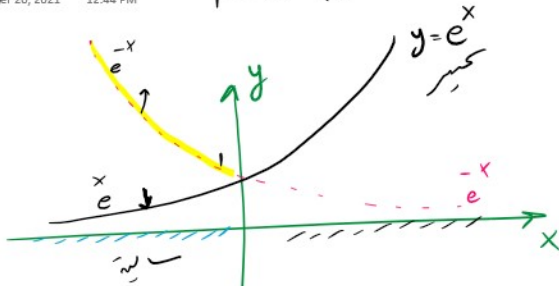


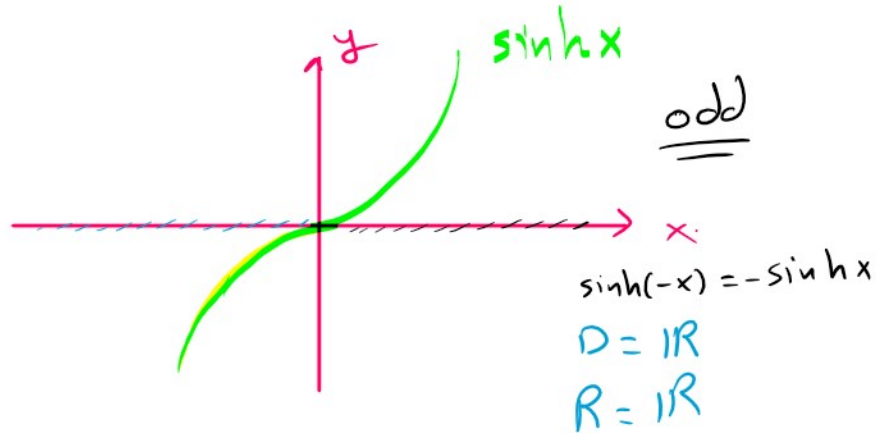
7.7 Hyperbolic functions

Monday, December 20, 2021 12:44 PM

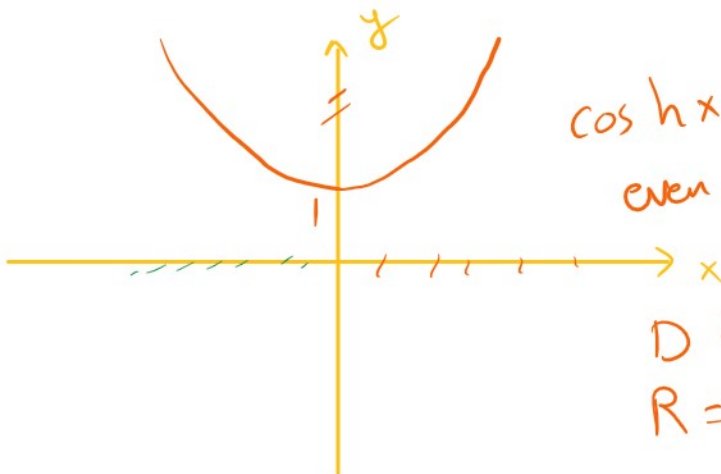
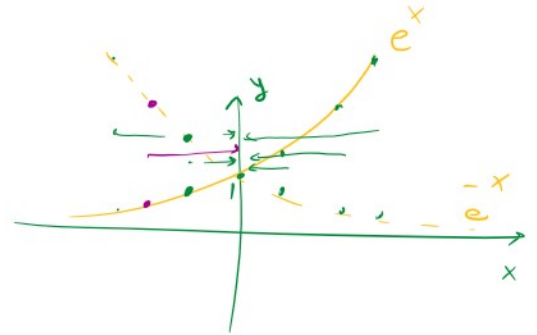


h: hyperbolic

① $\sinh x = \frac{e^x - e^{-x}}{2}$



② $\cosh x = \frac{e^x + e^{-x}}{2}$

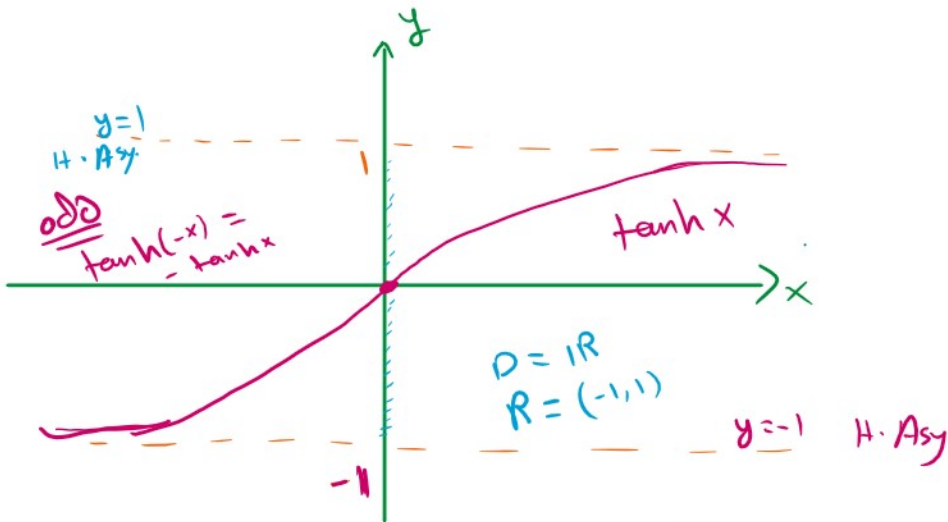


$\cosh x$
even $\Rightarrow \cosh(-x) = \cosh(x)$

$D = \mathbb{R}$
 $R = [1, \infty)$

③ $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$e + e$$



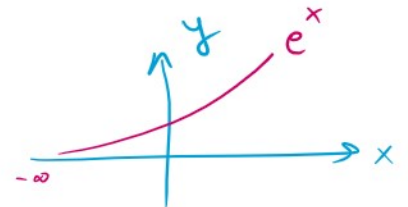
$$\lim_{x \rightarrow \infty} \tanh x =$$

$$\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{e^x} = \lim 1 = 1$$

$$\lim_{x \rightarrow -\infty} \tanh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$= \lim_{x \rightarrow -\infty} \frac{-e^{-x}}{e^{-x}} = - \lim_{x \rightarrow -\infty} 1 = -1$$



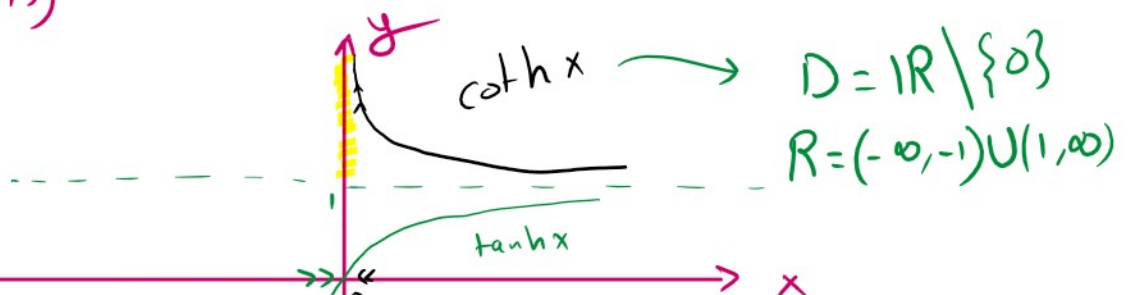
$$\tanh 0 = \frac{e^0 - e^0}{e^0 + e^0} = \frac{1 - 1}{1 + 1} = \frac{0}{2} = 0$$

$$\textcircled{4} \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$x = 0$ is V. Asy

$$\lim_{x \rightarrow 0^+} \coth x = \infty$$

$$\lim_{x \rightarrow 0^-} \coth x = -\infty$$



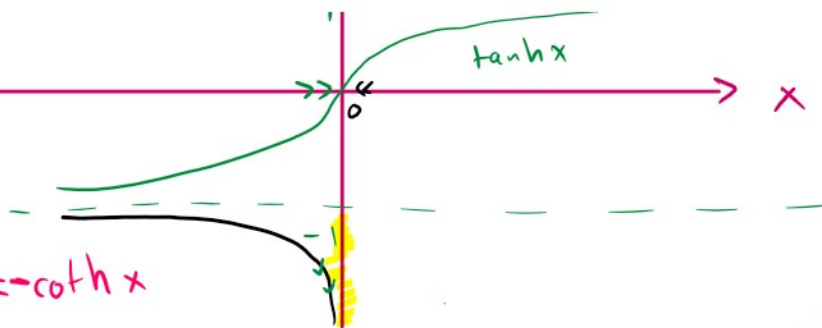
$$x \rightarrow 0^+$$

$$\lim_{x \rightarrow 0^+} \coth x = -\infty$$

$$x \rightarrow 0^-$$

odd

$$\coth(-x) = -\coth x$$



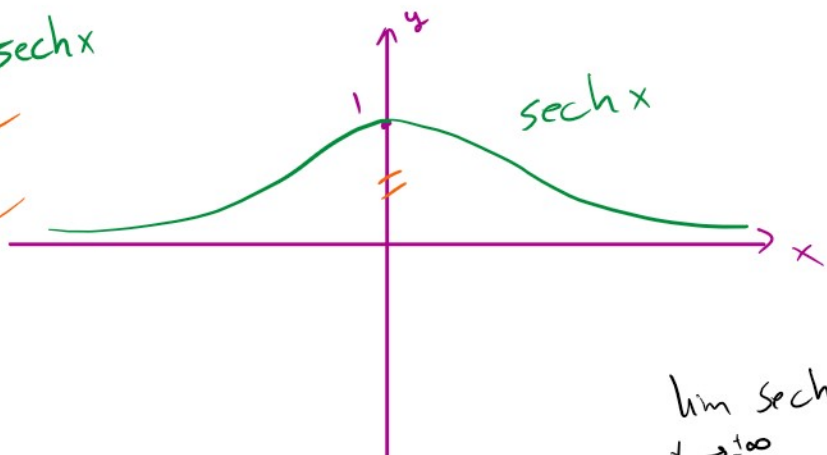
$$\textcircled{5} \operatorname{sech} x = \frac{1}{\cosh x}$$

even

$$\operatorname{sech}(-x) = \operatorname{sech} x$$

$$D = \mathbb{R}$$

$$R = (0, 1]$$



$$y = 0 \text{ is}$$

H. Asy

$$\lim_{x \rightarrow \infty} \operatorname{sech} x = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{\cosh x} = \lim_{x \rightarrow \infty} \frac{2}{e^x + e^{-x}}$$

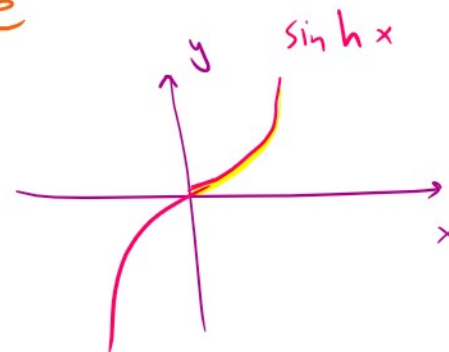
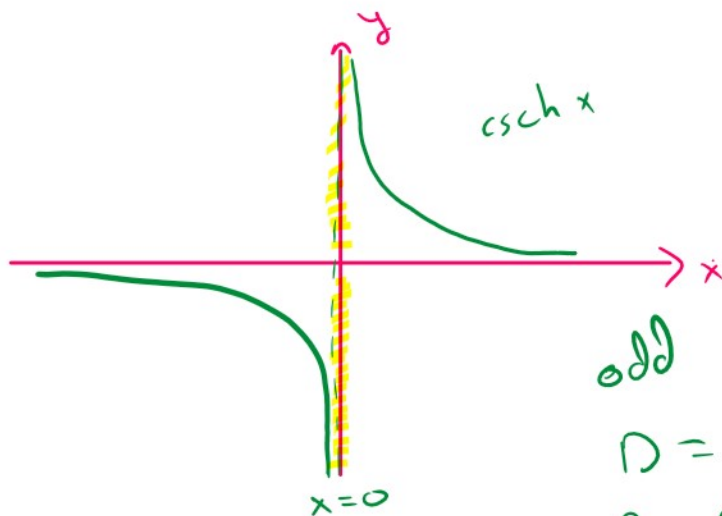
$$= \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{2}{e^x + e^{-x}} = \frac{2}{\infty} = 0$$



$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\textcircled{6} \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



odd $\Rightarrow \operatorname{csch}(-x) = -\operatorname{csch} x$

$$D = \mathbb{R} \setminus \{0\}$$

$$R = (-\infty, 0) \cup (0, \infty)$$

$$\bullet \cos^2 x + \sin^2 x = 1$$

$$\bullet \sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \bullet \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \end{aligned}$$

$$\bullet 1 + \tan^2 x = \sec^2 x$$

$$\bullet \cot^2 x + 1 = \csc^2 x$$

$$\textcircled{1} \Rightarrow \cosh^2 x - \sinh^2 x = 1$$

$$\textcircled{2} \sinh 2x = 2 \sinh x \cosh x$$

$$\begin{aligned} \textcircled{3} \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2 \cosh^2 x - 1 \\ &= 1 + 2 \sinh^2 x \end{aligned}$$

$$\textcircled{4} 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\textcircled{5} -\coth^2 x + 1 = -\operatorname{csch}^2 x$$

$$1 = \coth^2 x - \operatorname{csch}^2 x$$

$$1 = \cosh x - \operatorname{csch} x$$

Proof

$$(1) \cosh^2 x - \sinh^2 x = 1$$

$$\left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \stackrel{?}{=} 1$$

$$\frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \stackrel{?}{=} 1$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \stackrel{?}{=} 1$$

$$\frac{e^{2x} + \cancel{2} + e^{-2x}}{4} - \frac{e^{2x} - \cancel{2} + e^{-2x}}{4} \stackrel{?}{=} 1$$

$$\frac{4}{4} = 1$$

$$(1) (\sinh x)' = \cosh x \Rightarrow \int \cosh x \, dx = \sinh x + c$$

$$\textcircled{2} (\cosh x)' = \sinh x \Rightarrow \int \sinh x \, dx = \cosh x + C$$

$$\textcircled{3} (\tanh x)' = \operatorname{sech}^2 x \Rightarrow \int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\textcircled{4} (\coth x)' = -\operatorname{csch}^2 x \Rightarrow \int \operatorname{csch}^2 x \, dx = -\coth x + C$$

$$\textcircled{5} (\operatorname{sech} x)' = -\operatorname{sech} x \tanh x \Rightarrow \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\textcircled{6} (\operatorname{csch} x)' = -\operatorname{csch} x \coth x \Rightarrow \int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

Proof $(\sinh x)' \stackrel{?}{=} \cosh x$

$$\left(\frac{e^x - e^{-x}}{2} \right)' = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Exp Find y' if ① $y = \ln(\sinh \sqrt{x})$

$$y' = \frac{(\cosh \sqrt{x})^{\frac{1}{2\sqrt{x}}}}{\sinh \sqrt{x}}$$

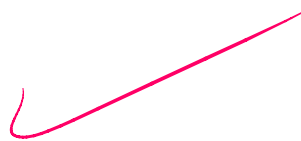
$$= \frac{\cosh \sqrt{x}}{\sinh \sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\coth \sqrt{x}}{2\sqrt{x}}$$

② $y = 4 \cosh \frac{x^3}{3}$

$$y' = 4 \sinh \frac{x^3}{3} \left(\frac{x^2}{3} \right)$$

$$= 4x^2 \sinh \frac{x^3}{3}$$



Exp ① $\int \sinh \underline{2x} dx = \int \sinh u \frac{du}{2}$

$\int \sinh u du$

$$\begin{aligned} u &= 2x \\ du &= 2 dx \\ \frac{du}{2} &= dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int \sinh u \, du \\
 &= \frac{1}{2} \cosh u + C \\
 &= \frac{1}{2} \cosh(2x) + C \quad \checkmark
 \end{aligned}$$

$$\textcircled{2} \int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx$$

or
 $u = \sinh x$
 $du = \cosh x \, dx$

$$= \ln |\sinh x| + C \quad \checkmark$$

$$\int \frac{du}{u} = \ln |u| + C \quad \nearrow$$

Ex 15 Find y' if $y = 2\sqrt{t} \tanh \sqrt{t}$

$$\begin{aligned}
 y' &= \cancel{2\sqrt{t}} \text{ sech}^2 \sqrt{t} \left(\frac{1}{\cancel{2\sqrt{t}}} \right) + \underbrace{\tanh \sqrt{t}}_{\text{is}} \cancel{2} \frac{1}{\cancel{2\sqrt{t}}} \\
 &= \text{sech}^2 \sqrt{t} + \frac{1}{\sqrt{t}} \tanh \sqrt{t}
 \end{aligned}$$

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$$\int_0^{\ln 2} 4 e^{-\theta} \sinh \theta \, d\theta$$

$$\int_0^{\ln 2} \cancel{4}^2 e^{-\theta} \left(\frac{e^{\theta} - e^{-\theta}}{\cancel{2}_1} \right) d\theta$$

$$2 \int_0^{\ln 2} (1 - e^{-2\theta}) \, d\theta$$

$$2 \left(\theta - \frac{e^{-2\theta}}{-2} \right) \Big|_0^{\ln 2}$$

$$2 \left(\theta + \frac{1}{2} e^{-2\theta} \right) \Big|_0^{\ln 2}$$

$$2 \left[\left(\ln 2 + \frac{1}{2} e^{-2 \ln 2} \right) - \left(0 + \frac{1}{2} e^0 \right) \right]$$

$$2 \left[\ln 2 + \frac{1}{2} \left(\frac{\ln 2^{-2}}{e} \right) - \frac{1}{2} \right]$$

$$2 \left(\ln 2 + \frac{1}{2} (e) - \frac{1}{2} \right)$$

$$2 \left[\ln 2 + \frac{1}{2} \cdot 2^{-2} - \frac{1}{2} \right]$$

$$2 \ln 2 + 2^{-2} - 1$$

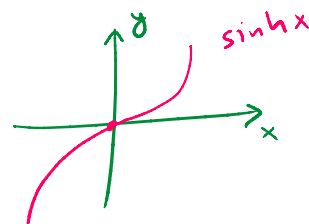
$$\ln 4 + \frac{1}{4} - 1$$

$$\ln 4 - \frac{3}{4} \checkmark$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\frac{0}{0} \checkmark$$

$$\lim_{x \rightarrow 0} \frac{\sinh x}{x} = 1 \quad \frac{0}{0}$$



$$\lim_{x \rightarrow 0} \frac{\cosh x}{1} = \cosh 0 = 1$$

