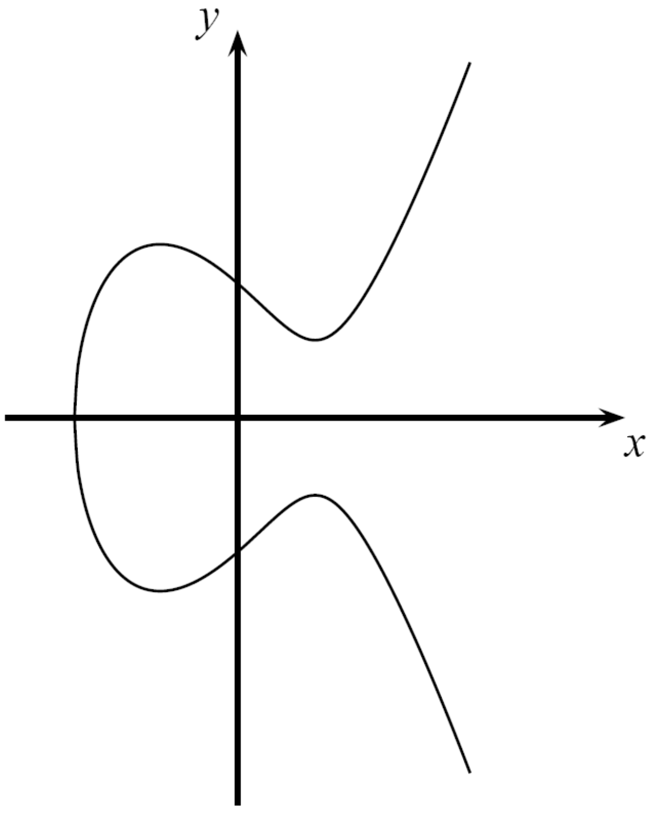
Elliptic Curve Cryptography (ECC)

Slides Original Source:

1. C. Paar and J. Pelzl, "Understanding Cryptography – A Textbook for Students and Practitioners," Springer (www.crypto-textbook.com)
2. M. Stamp, "Information Security: Principles and Practice," John Wiley

Outline

- Introduction
- Computations on Elliptic Curves
- The Elliptic Curve Diffie-Hellman Protocol
- Security Aspects
- Implementation in Software and Hardware



Motivation

- **Problem:**
  Asymmetric schemes like RSA and Elgamal require exponentiations in integer rings and fields with parameters of more than 1000 bits.
  - High computational effort on CPUs with 32-bit or 64-bit arithmetic
  - Large parameter sizes critical for storage on small and embedded
- **Motivation:**
  Smaller field sizes providing equivalent security are desirable
- **Solution:**
  Elliptic Curve Cryptography uses a group of points (instead of integers) for cryptographic schemes with coefficient sizes of 160-256 bits, reducing significantly the computational effort.

Elliptic Curve Cryptography (ECC)

- Elliptic curve is **not** a cryptosystem
- Elliptic curves are a different way to do the math in public key system
- Elliptic curve versions DH, RSA, etc.
- Elliptic curves may be more efficient
  - Fewer bits needed for same security
  - But the operations are more complex

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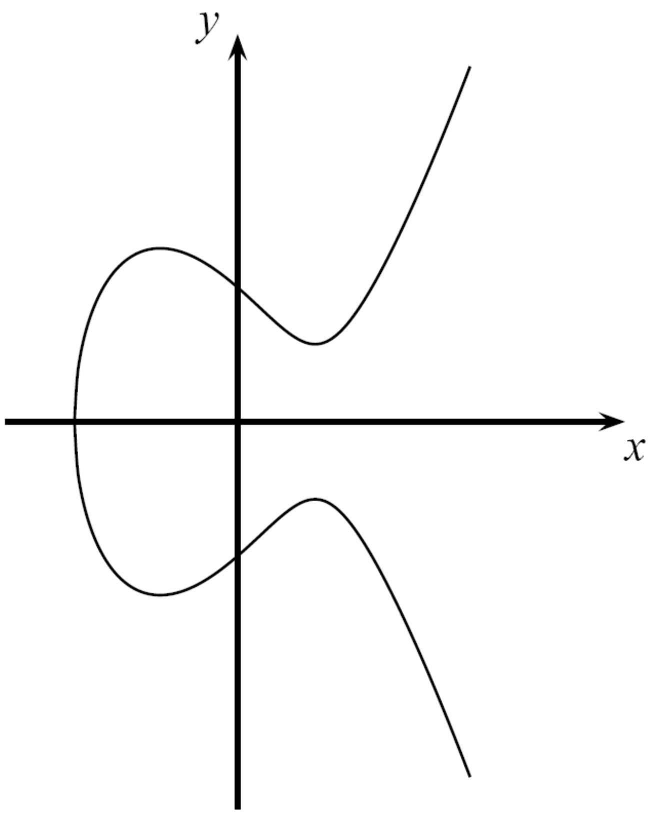


Computations on Elliptic Curves

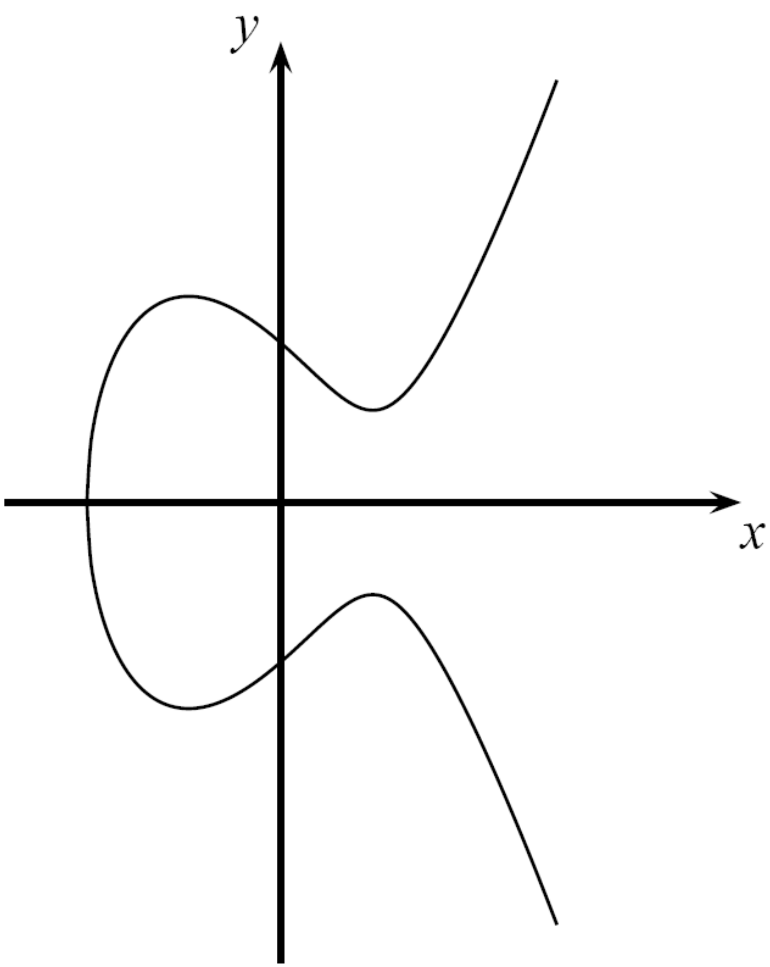
- Elliptic curves are polynomials that define points based on the equation:

$$y^2 = x^3 + ax + b$$

  for parameters ***a*** and ***b*** that specify the exact shape of the curve

- On the real numbers and with parameters $a, b \in R$, an elliptic curve looks like this →

- Elliptic curves are not just defined over the real numbers *R* but also over many other types of finite fields.



Example: $y^2 = x^3 - 3x + 3$ over R

Computations on Elliptic Curves (ctd.)

- In cryptography, we are interested in elliptic curves modulo a prime p (i.e., EC over $GF(p)$):

Definition: Elliptic Curves over prime fields

The elliptic curve over Z_p, $p > 3$, is the set of all pairs $(x,y) \in Z_p$ which fulfill

$$y^2 = x^3 + ax + b \bmod p$$

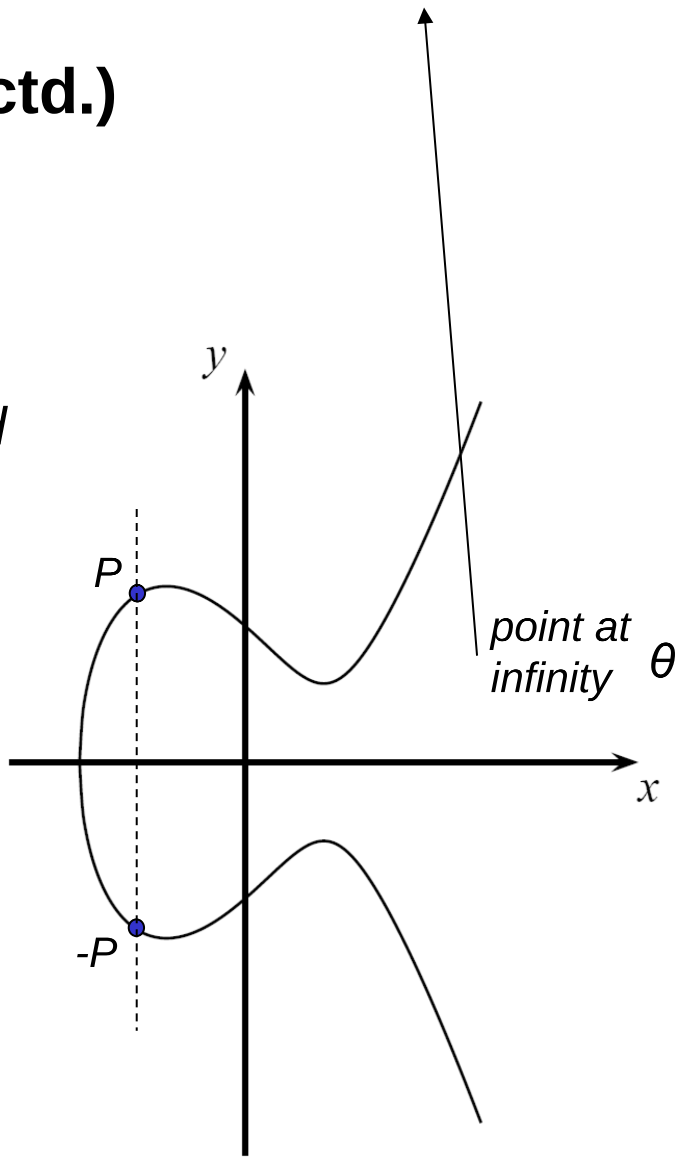
together with an imaginary point of infinity θ, where $a,b \in Z_p$ and the condition

$$4a^3+27b^2 \neq 0 \bmod p$$

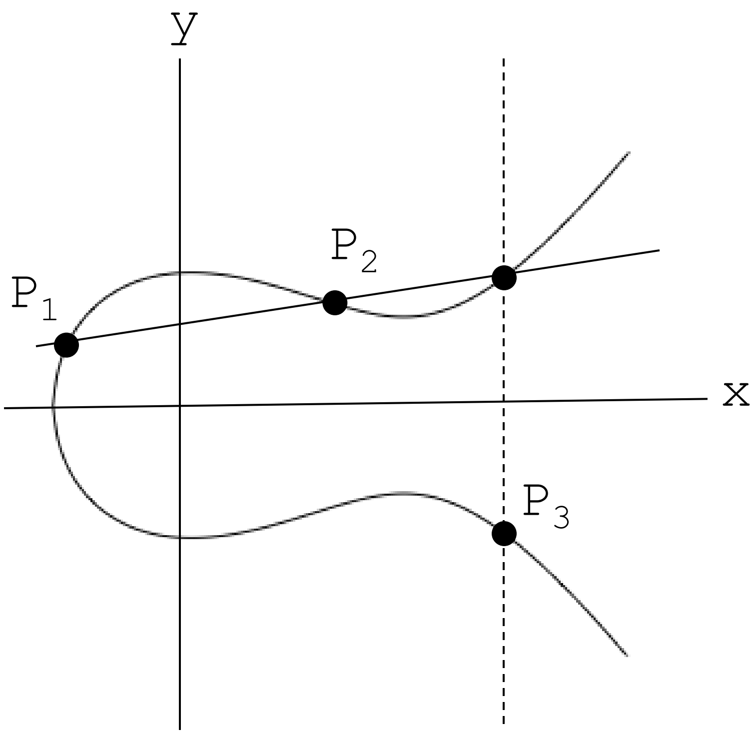
- Note that $Z_p = \{0,1,\ldots, p-1\}$ is a set of integers with modulo p arithmetic

Computations on Elliptic Curves (ctd.)

- Some special considerations are required to convert elliptic curves into a group of points
  - *In any group, a special element is required to allow for the identity operation, i.e., given $P \in E$: $P + \theta = P = \theta + P$*
  - *This identity point (which is not on the curve) is additionally added to the group definition*
  - *This (infinite) identity point is denoted by θ*
- Elliptic Curve are symmetric along the *x*-axis
  - Up to two solutions *y and -y* exist for each quadratic residue *x* of the elliptic curve
  - For each point $P = (x,y)$, the inverse or negative point is defined as $-P = (x,-y)$



Elliptic Curve Picture



- Consider elliptic curve

$$E: y^2 = x^3 - x + 1$$

- If P_1 and P_2 are on E, we can define

$$P_3 = P_1 + P_2$$

as shown in picture

- Addition is all we need

Points on Elliptic Curve

- Find the points on the elliptic curve

  $y^2 = x^3 + 2x + 3 \pmod 5$

  $x = 0 \Rightarrow y^2 = 3 \Rightarrow \text{no solution} \pmod 5$

  $x = 1 \Rightarrow y^2 = 6 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 2 \Rightarrow y^2 = 15 = 0 \Rightarrow y = 0 \pmod 5$

  $x = 3 \Rightarrow y^2 = 36 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 4 \Rightarrow y^2 = 75 = 0 \Rightarrow y = 0 \pmod 5$

  - Then points on the elliptic curve are

    (1,1) (1,4) (2,0) (3,1) (3,4)
    (4,0) and the point at infinity: θ

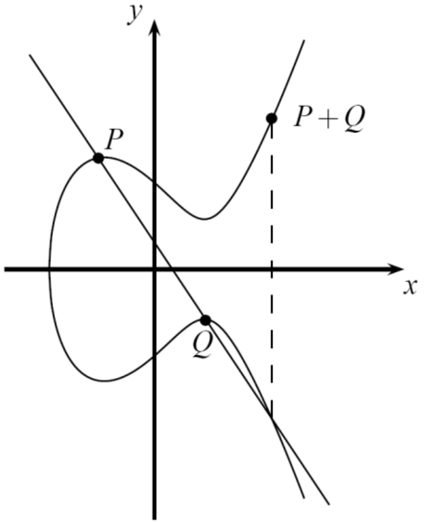
Computations on Elliptic Curves (ctd.)

- Generating a *group of points* on elliptic curves based on **point addition operation** $P+Q = R$, *i.e.*, $(x_P,y_P)+(x_Q,y_Q) = (x_R,y_R)$
- Geometric interpretation of point addition operation
  - *Draw straight line through P and Q; if P=Q use tangent line instead*
  - *Mirror third intersection point of drawn line with the elliptic curve along the x-axis*
- Elliptic Curve Point Addition and Doubling Formulas

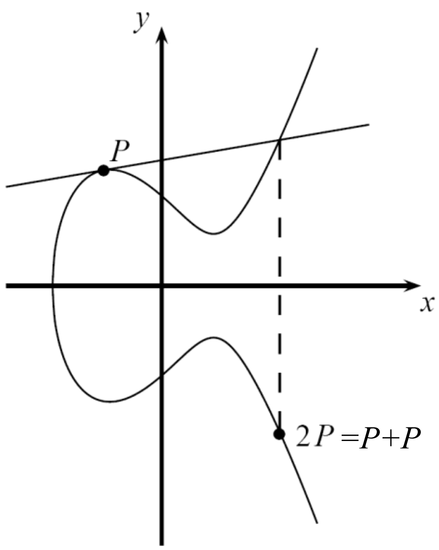
$x_3 = s^2 - x_1 - x_2 \pmod p$ and $y_3 = s(x_1 - x_3) - y_1 \pmod p$

where

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} \bmod p & ; \text{if } P \neq Q \text{ (point addition)} \\[2ex] \dfrac{3x_1^2 + a}{2y_1} \bmod p & ; \text{if } P = Q \text{ (point doubling)} \end{cases}$$



Point Addition



Point Doubling

Elliptic Curve Addition

- Consider $y^2 = x^3 + 2x + 3 \pmod 5$.
  Points on the curve are $(1,1)$ $(1,4)$ $(2,0)$ $(3,1)$ $(3,4)$ $(4,0)$ and θ
- What is $(1,4) + (3,1) = P_3 = (x_3, y_3)$?

$$s = (y_2 - y_1) * (x_2 - x_1)^{-1} \pmod p$$
$$= (1-4) * (3-1)^{-1} = -3 * 2^{-1}$$
$$= 2(3) = 6 = 1 \pmod 5$$
$$x_3 = s^2 - x_1 - x_2 \pmod p$$
$$= 1 - 1 - 3 = 2 \pmod 5$$
$$y_3 = s(x_1 - x_3) - y_1 \pmod p$$
$$= 1(1-2) - 4 = 0 \pmod 5$$

- On this curve, $(1,4) + (3,1) = (2,0)$

Computations on Elliptic Curves (ctd.)

- **Example**: Given $E: y^2 = x^3+2x+2 \bmod 17$ and point $P=(5,1)$

  Goal: Compute $2P = P+P = (5,1)+(5,1)= (x_3,y_3)$

$$s = \frac{3x_1^2 + a}{2y_1} = (2 \cdot 1)^{-1}(3 \cdot 5^2 + 2) = 2^{-1} \cdot 9 \equiv 9 \cdot 9 \equiv 13 \bmod 17$$

$$x_3 = s^2 - x_1 - x_2 = 13^2 - 5 - 5 = 159 \equiv 6 \bmod 17$$

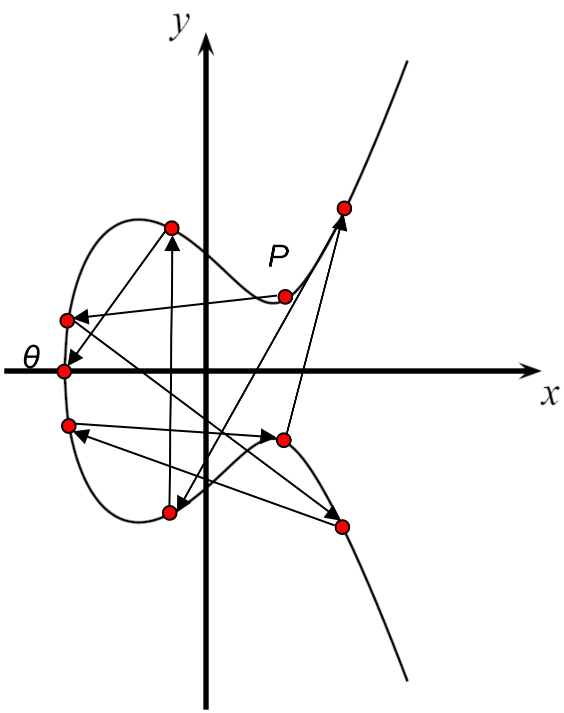
$$y_3 = s(x_1 - x_3) - y_1 = 13(5 - 6) - 1 = -14 \equiv 3 \bmod 17$$

Finally, $2P = (5,1) + (5,1) = (6,3)$

Computations on Elliptic Curves (ctd.)

- The points on an elliptic curve and the point at infinity θ form cyclic subgroups

$2P = (5,1)+(5,1) = (6,3)$	$11P = (13,10)$
$3P = 2P+P = (10,6)$	$12P = (0,11)$
$4P = (3,1)$	$13P = (16,4)$
$5P = (9,16)$	$14P = (9,1)$
$6P = (16,13)$	$15P = (3,16)$
$7P = (0,6)$	$16P = (10,11)$
$8P = (13,7)$	$17P = (6,14)$
$9P = (7,6)$	$18P = (5,16)$
$10P = (7,11)$	$19P = \theta$



- This elliptic curve has order #E = |E| = 19 since it contains 19 points in its cyclic group

Number of Points on an Elliptic Curve

- How many points can be on an arbitrary elliptic curve?
  - Consider previous example: $E: y^2 = x^3+2x+2 \bmod 17$ has 19 points
  - However, determining the point count on elliptic curves in general is hard
- But Hasse's theorem bounds the number of points to a restricted interval

Definition: Hasse's Theorem:

Given an elliptic curve modulo p, the number of points on the curve is denoted by #E and is bounded by

$$p+1-2\sqrt{p} \leq \#E \leq p+1+2\sqrt{p}$$

- **Interpretation:** The number of points is "close to" the prime *p*
- **Example:** To generate a curve with about 2^{160} points, a prime with a length of about 160 bits is required

Elliptic Curve Discrete Logarithm Problem

- Cryptosystems rely on the hardness of the Elliptic Curve Discrete Logarithm Problem (ECDLP)

Definition: Elliptic Curve Discrete Logarithm Problem (ECDLP)

Given a primitive element P and another element T on an elliptic curve E. The ECDL problem is finding the integer d, where $1 \leq d \leq \#E$ *such that*

$$\underbrace{P + P + \ldots + P}_{d \text{ times}} = dP = T$$

- Cryptosystems are based on the idea that *d* is large and kept secret and attackers cannot compute it easily
- If *d* is known, an efficient method to compute the point multiplication *dP* is required to create a reasonable cryptosystem
  - Known Square-and-Multiply Method can be adapted to Elliptic Curves
  - The method for efficient point multiplication on elliptic curves: **Double-and-Add Algorithm**

Double-and-Add Algorithm for Point Multiplication

Double-and-Add Algorithm

Input: Elliptic curve *E*, an elliptic curve point *P* and *a* scalar *d* with bits d_i

Output: $T = dP$

Initialization:

$T = P$

Algorithm:

FOR $i = t - 1$ DOWNTO 0

$T = T + T \bmod n$

IF $d_i = 1$

$T = T + P \bmod n$

RETURN (T)

Example: $26P = (11010)_2\ P = (d_4 d_3 d_2 d_1 d_0)_2\ P$

Step		
#0	$P = \mathbf{1}_2 P$	inital setting
#1*a*	$P+P = 2P = \mathbf{10}_2 P$	DOUBLE (bit d_3)
#1*b*	$2P+P = 3P = 10_2\ P + 1_2 P = \mathbf{11}_2 P$	ADD (bit d_3=1)
#2*a*	$3P+3P = 6P = 2(11_2 P) = \mathbf{110_2} P$	DOUBLE (bit d_2)
#2*b*		no ADD ($d_2 = 0$)
#3*a*	$6P = 12P = 2(110_2 P) = \mathbf{1100_2} P$	DOUBLE (bit d_1)
#3*b*	$12P+P = 13P = 1100_2 P + 1_2\ P = \mathbf{1101}_2 P$	ADD (bit d_1=1)
#4*a*	$13P+13P = 26P = 2(1101_2 P) = \mathbf{11010_2} P$	DOUBLE (bit d_0)
#4*b*		no ADD ($d_0 = 0$)

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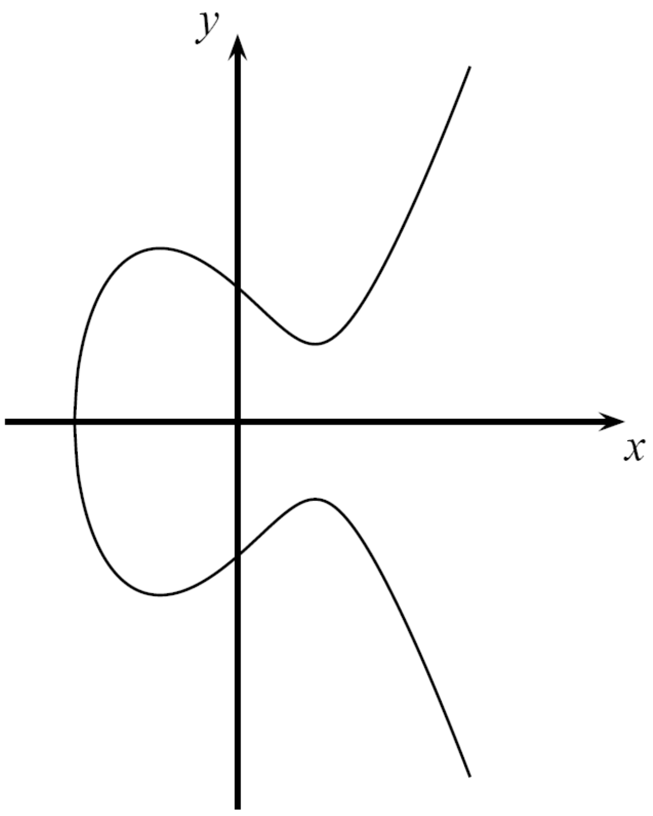

EC Diffie-Hellman

- **Public:** Elliptic curve and point (x, y) on curve
- **Private:** Alice's A and Bob's B

$\xrightarrow{A(x, y)}$

$\xleftarrow{B(x, y)}$

Alice, A Bob, B

- Alice computes $T_{AB} = A(B(x, y))$
- Bob computes $T_{BA} = B(A(x, y))$
- T_{AB} and T_{BA} are the same since $AB = BA$
- Session key for symmetric encryption can be derived by taking one of the coordinates of the point T_{AB} (usually the x-coordinate)

EC Diffie-Hellman

- **Public:** Curve $y^2 = x^3 + 7x + b \pmod{37}$ and point **(2,5)** $\Rightarrow b = 3$
- ***Alice's private value:*** $A = 4 \in \{2,3,\ldots,\#E-1\}$
- **Bob's private value:** $B = 7 \in \{2,3,\ldots,\#E-1\}$

- *Alice sends Bob:* 4**(2,5)** $= (7,32)$
- Bob sends *Alice:* 7**(2,5)** $= (18,35)$

- *Alice computes:* $4(18,35) = (22,1)$
- Bob computes: $7(7,32) = (22,1)$

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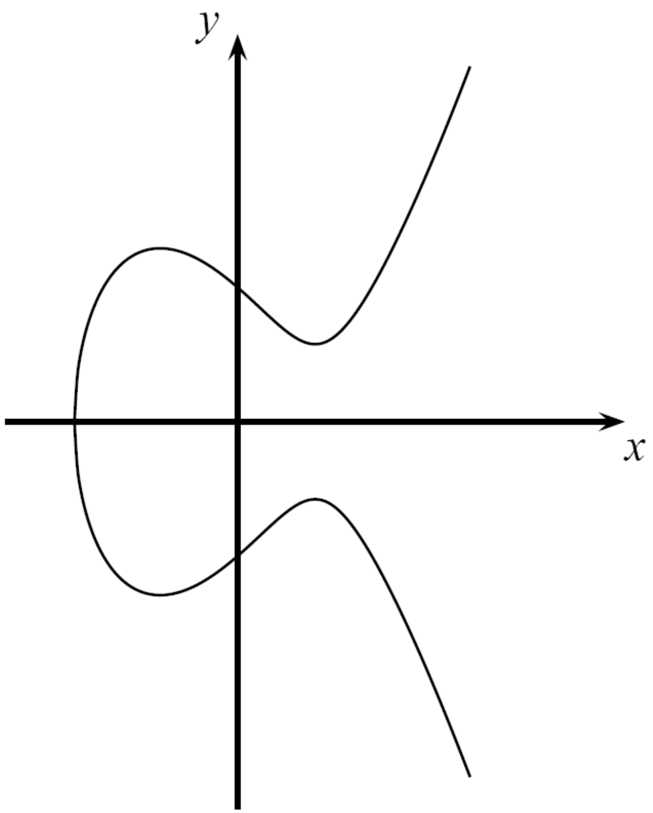

Security Aspects

- Why are parameters signficantly smaller for elliptic curves (160 - 256 bit) than for RSA (1024 - 3076 bit)?
  - Attacks on groups of elliptic curves are weaker than available factoring algorithms or integer DL attacks
  - Best known attacks on elliptic curves (chosen according to cryptographic criterions) are the Baby-Step Giant-Step and Pollard-Rho method
  - Complexity of these methods: on average, roughly $\sqrt{p}$ steps are required before the ECDLP can be successfully solved

- Implications to practical parameter sizes for elliptic curves:
  - An elliptic curve using a prime *p* with 160 bit (and roughly 2^{160} points) provides a security of 2^{80} steps that required by an attacker (on average)
  - An elliptic curve using a prime *p* with 256 bit (roughly 2^{256} points) provides a security of 2^{128} steps on average

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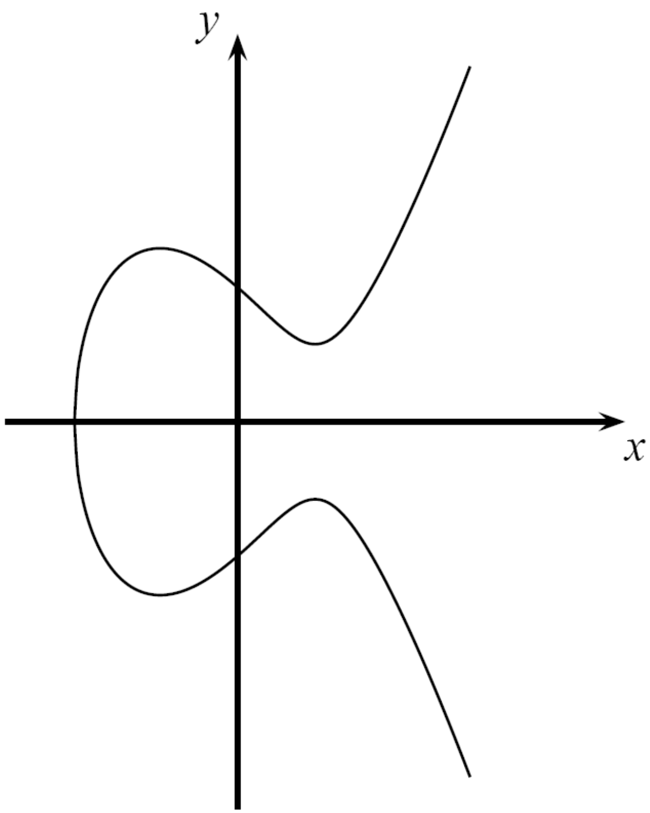
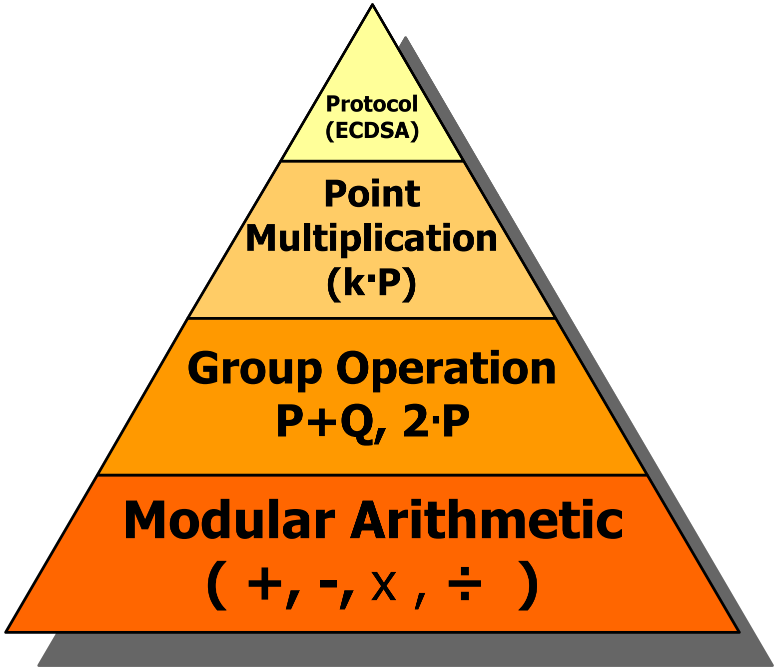


Implementations in Hardware and Software

- Elliptic curve computations usually regarded as consisting of four layers:
  - Basic modular arithmetic operations, computationally most expensive
  - Group operation implements point doubling and point addition
  - Point multiplication can be implemented using the Double-and-Add method
  - Upper layer protocols like ECDH and ECDSA
- Most efforts should go in optimizations of the modular arithmetic operations, such as
  - Modular addition and subtraction
  - Modular multiplication
  - Modular inversion



Implementations in Hardware and Software

- Software implementations
  - Optimized 256-bit ECC implementation on 3GHz 64-bit CPU requires about 2 *ms* per point multiplication
  - Less powerful microprocessors (e.g, on SmartCards or cell phones) take significantly longer time (>10 *ms*)
- Hardware implementations
  - High-performance implementations with 256-bit special primes can compute a point multiplication in a few hundred μs on reconfigurable hardware
  - Dedicated chips for ECC can compute a point multiplication even in a few ten μs