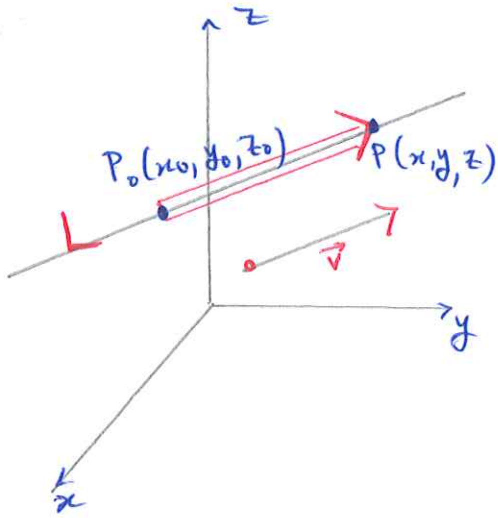
12.5 Lines and Planes in SPace.

Lines and Line Segments in SPace.

Suppose that $\overrightarrow{P_0P} \parallel \vec{v}$

where $\overrightarrow{P_0P} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$

and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$.



Then $L = \{ P(x,y,z) : \overrightarrow{P_0P} \parallel \vec{v} \}$

Thus $\overrightarrow{P_0P} = t\vec{v}$, where $-\infty < \underset{\text{parameter}}{t} < \infty$

$\Rightarrow \quad (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k} = t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$

$\Leftrightarrow \quad \underbrace{x\vec{i} + y\vec{j} + z\vec{k}}_{\Downarrow} = \underbrace{x_0\vec{i} + y_0\vec{j} + z_0\vec{k}}_{\Downarrow} + t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$

$$\boxed{\vec{r}(t) = \vec{r}_0(t) + t\vec{v}}, \quad -\infty < t < \infty$$

This equation is called <u>vector equation</u> for a line L through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$.

$\vec{r}(t)$: is the position vector of a point $P(x,y,z)$ on L

$\vec{r}_0(t)$: is the position vector of $P_0(x_0, y_0, z_0)$.

" parametrization "

The standard parametric equations of the line L through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ are

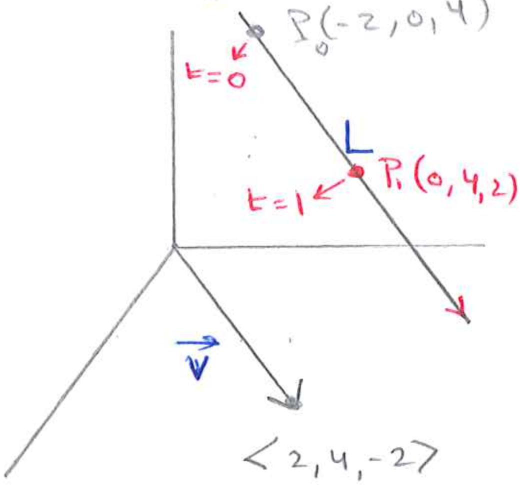
$$\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases}, \quad -\infty < t < \infty.$$

Example (1): Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\vec{v} = 2\vec{i} + 4\vec{j} - 2\vec{k}$

sol:

$$x = -2 + 2t$$
$$y = 0 + 4t$$
$$z = 4 - 2t$$

with $-\infty < t < \infty$.



Moreover:

The vector equation for the line L is given by:

$$\vec{r}(t) = \vec{r_0}(t) + t\vec{v} \quad \cdots (*)$$

$$\Leftrightarrow \langle x, y, z \rangle = \langle -2, 0, 4 \rangle + t\langle 2, 4, -2 \rangle$$

$$\Leftrightarrow x\vec{i} + y\vec{j} + z\vec{k} = -2\vec{i} + 4\vec{k} + (2\vec{i} + 4\vec{j} - 2\vec{k})t.$$

Example (2): Find parametric equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$

Sol: $\vec{V} = \vec{PQ} = (1-(-3))\vec{i} + (-1-2)\vec{j} + (4-{-3})\vec{k}$

$$= 4\vec{i} - 3\vec{j} + 7\vec{k}$$

$\vec{PQ}$ // Line with $P_0(x_0, y_0, z_0) = (-3, 2, -3)$ → or the other point.

$\Rightarrow$ Parametric equations are:

$$x = -3 + 4t$$
$$y = 2 - 3t \quad , \quad \text{with } -\infty < t < \infty$$
$$z = -3 + 7t$$

OR: We can choose $P_0(1,-1,4)$, then

$x = 1 + 4t$, $y = -1 - 3t$, $z = 4 + 7t$, $-\infty < t < \infty$

Note: The parametrizations are Not Unique.

Note: To parametrize a line Segment (قطعة مستقيمة) joining two points, we first parametrize the line through the points, then we find the t-values for the end points and restrict t to lie in the closed Interval.

Example (3): Parametrize the line segment joining the points $P(-3,2,-3)$ and $Q(1,-1,4)$

Sol: From example (2): $\vec{PQ} = 4\vec{i} - 3\vec{j} + 7\vec{k}$

and the parametrization is $\begin{cases} x = -3 + 4t \\ y = 2 - 3t \\ z = -3 + 7t. \end{cases}$

We observe that the point $(x, y, z) = (-3+4t, 2-3t, -3+7t)$ on the line passes through $P(-3,2,-3)$ at $t = 0$ and $Q(1,-1,4)$ at $t = 1$, therefore $0 \leq t \leq 1.$

In General: $P_0(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$

$$\Rightarrow \quad \begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \\ z &= z_0 + (z_1 - z_0)t \end{aligned} \quad , \text{ with } 0 \leq t \leq 1.$$

Note: The vector equation (*) for a line in space is equivalent to the path of a particle starting at $P_0(x_0, y_0, z_0)$ and moving in the direction of $\vec{v}$.

Rewriting Equation (*), we have:

$$\vec{r}(t) = \vec{r_0} + t\vec{v}$$

$$= \underbrace{r_0}_{\text{Initial position}} + \underbrace{t}_{\text{time}} \cdot \underbrace{|\vec{v}|}_{\text{speed}} \cdot \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{Direction}},$$

(i.e) The position of the particle at time t is its initial position plus its distance moved (speed x time) in the direction $\frac{\vec{v}}{|\vec{v}|}$ of its straight-line motion.

Example (4): A helicopter is to fly directly from a helipad (مهبط الهيلوكبتر) at the origin in the direction of the point $(1,1,1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec.?

Sol: Let the starting position $= (0,0,0)$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}. \quad \text{(flight direction)}$$

$$\Rightarrow \vec{r}(t) = \langle 0,0,0\rangle + t(60)\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$$

$$\vec{r}(t) = 20\sqrt{3}\, t\,(\vec{i} + \vec{j} + \vec{k})$$

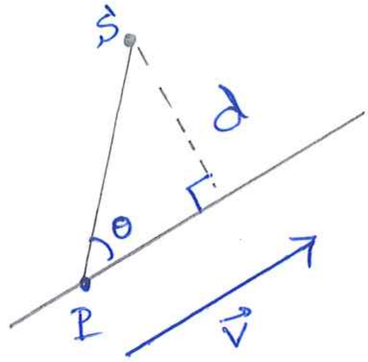
$$\Rightarrow \vec{r}(10) = \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\rangle$$

The Distance from a Point S to a line Through P parallel to $\vec{v}$ is

$$\boxed{d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}}$$

Recall: $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$.

$$\sin\theta = \frac{d}{|\vec{PS}|} \iff d = |\vec{PS}| \sin\theta = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$



Example (5): Find the Distance from the point $S(1,1,5)$ to the line

$$L: \quad x = 1+t, \quad y = 3-t, \quad z = 2t, \quad -\infty < t < \infty$$

Sol: Notice that $P_0(x_0, y_0, z_0) = (1,3,0)$

and $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$

$\vec{PS} = -2\vec{j} + 5\vec{k}$.

$$\Rightarrow \vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i} + 5\vec{j} + 2\vec{k}.$$

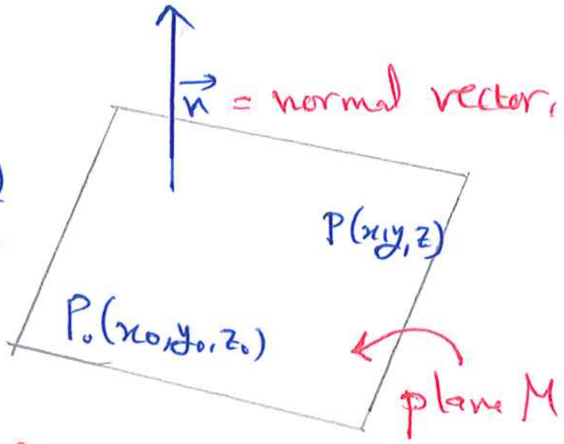
$$\Rightarrow |\vec{PS} \times \vec{v}| = \sqrt{1^2 + 5^2 + 2^2} = \sqrt{30} \quad \& \quad |\vec{v}| = \sqrt{6}$$

$$\Rightarrow d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation For a Plane In Space:

Let $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$

is a normal vector to the plane M, through P_0



$\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$.

(i.e) $\overrightarrow{P_0P} \cdot \vec{n} = 0.$ vector equation.

that is $\left[(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}\right] \cdot \left[A\vec{i} + B\vec{j} + C\vec{k}\right] = 0$

$\Longleftrightarrow$ $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$ Component Equation

$\Longleftrightarrow$ $Ax + By + Cz = \underbrace{Ax_0 + By_0 + Cz_0}_{=D}$

$\Longleftrightarrow$ $Ax + By + Cz = D.$ component equation Simplified

Example (6): Find an equation for the plane through

$P_0(-3,0,7)$ perpendicular to $\vec{n} = 5\vec{i} + 2\vec{j} - \vec{k}$.

Sol: $5(x+3) + 2(y) - 1(z-7) = 0$

$5x + 15 + 2y - z + 7 = 0$

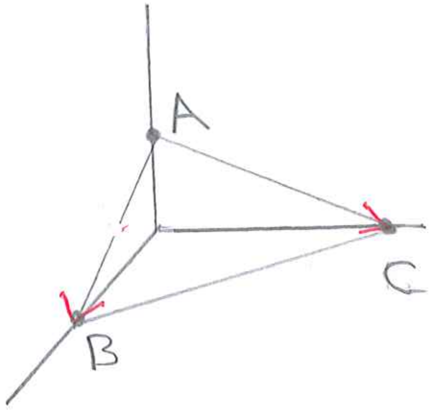
$\Longrightarrow$ $5x + 2y - z = -22.$

$\downarrow$ A $\quad \downarrow$ B $\quad \downarrow$ C

Example (7): Find an equation for the plane through

$A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$

Sol: First Find $\vec{n} \perp$ plane



Note that $\underbrace{(\vec{AB} \times \vec{AC})}_{= \vec{n}} \perp$ plane

$$\Rightarrow \vec{n} = (\vec{AB} \times \vec{AC}) = (2\vec{i} - \vec{k}) \times (3\vec{j} - \vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

Now, You Can take any point (say $A(0,0,1)$), then

The equation for the plane is

$$3(x-0) + 2(y-0) + 6(z-1) = 0 \iff 3x + 2y + 6z = 6$$

Lines of Intersection:

Remark: ① Two planes are parallel iff their normals are parallel. (i.e) $M_1 \parallel M_2$ iff $\vec{n}_1 = k\vec{n}_2$

② If the two planes are not parallel, then they intersect in a line.

Example (8): Find a vector parallel to the line of intersection of the planes :

$M_1 : 3x - 6y - 2z = 15$ and $M_2 : 2x + y - 2z = 5$

Sol: $\vec{n}_1 = 3\vec{i} - 6\vec{j} - 2\vec{k}$ & $\vec{n}_2 = 2\vec{i} + \vec{j} - 2\vec{k}$

$\vec{n}_1 \perp M_1$ and $\vec{n}_2 \perp M_2$

$\Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2$ // to the line of Intersections.

$$\Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\vec{i} + 2\vec{j} + 15\vec{k}$$

Example (9): Find parametric equations for the line in which the planes :

$3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ Intersect.

Sol: From example (8) : $\vec{v} = 14\vec{i} + 2\vec{j} + 15\vec{k}$. // L

To find a point : Let $\boxed{z = 0}$ (or x or y)

$$\Rightarrow \left.\begin{aligned} 3x - 6y &= 15 \\ 2x + y &= 5 \end{aligned}\right\} \Rightarrow \boxed{x = 3} \ \& \ \boxed{y = -1}$$

$P(3, -1, 0)$

$\Rightarrow$ The parametric equations are :

$x = 3 + 14t$, $y = -1 + 2t$, $z = 15t$, $-\infty < t < \infty$

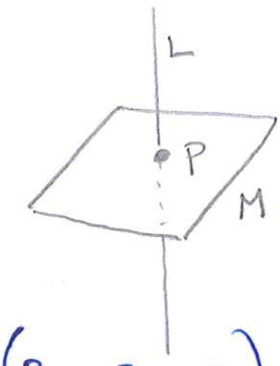
Example (10): Find the point where the line

$$x = \frac{8}{3} + 2t \quad , y = -2t \quad , z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Sol: The point $\left(\frac{8}{3} + 2t, -2t, 1+t\right)$ lies in the plane if its coordinates satisfy the equation of the plane:

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$



$$\Rightarrow 8 + 6t - 4t + 6 + 6t = 6$$

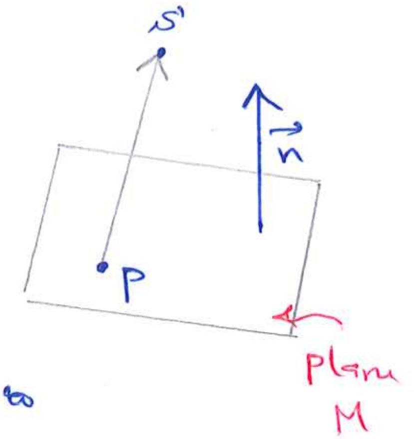
$$\Rightarrow \quad \boxed{t = -1}$$

$\therefore$ The point of Intersection is $(x, y, z)\Big|_{t=-1} = \left(\frac{2}{3}, 2, 0\right)$.

The distance from a Point to a Plane:

The distance from S to the plane M is:

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$



where $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ is normal to the plane M.

Example (11): Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z=6$

sol: $\vec{n} = 3\vec{i}+2\vec{j}+6\vec{k}$, $|\vec{n}| = \sqrt{9+4+36} = 7$

Let $x=y=0 \Rightarrow \boxed{z=1} \Rightarrow P(0,0,1)$

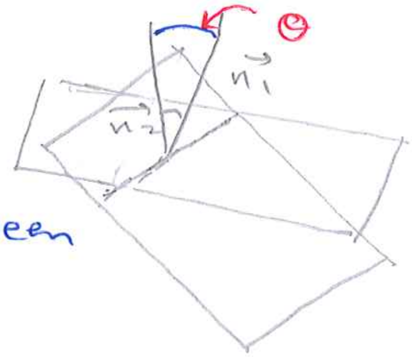
$\overrightarrow{PS} = \vec{i}+\vec{j}+2\vec{k}$

$\Rightarrow \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} = (\vec{i}+\vec{j}+2\vec{k}) \cdot \left(\frac{3}{7}\vec{i}+\frac{2}{7}\vec{j}+\frac{6}{7}\vec{k}\right) = \frac{17}{7}$

$\therefore$ Distance $= \left|\overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right| = \left|\frac{17}{7}\right| = \frac{17}{7}$.

Angles Between Planes:

The angle between two intersecting planes is defined to be the acute angle between their Normal Vecters.



Example (12): Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

sol: $\vec{n}_1 = 3\vec{i}-6\vec{j}-2\vec{k}$, $\vec{n}_2 = 2\vec{i}+\vec{j}-2\vec{k}$.

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|n_1||n_2|}\right) = \cos^{-1}\left(\frac{3(2)+(-6)(1)+(-2)(-2)}{\sqrt{49}\ \sqrt{9}}\right) \approx 1.38$$

radians