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MATH1321

Calculus 2

Chapter 10.1

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10.1 Sequences → List of numbers seperated by comma (,)

div. conv.

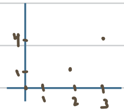
denoted by $a_n, b_n, c_n, \ldots$

Ex. $a_n = n^2, \quad n = 0, 1, 2, 3$

$a_0 = 0^2 = 0$ $\quad 1^{st}$ term

$a_1 = 1^2 = 1$ $\quad 2^{nd}$ term

$a_2 = 2^2 = 4$ $\quad 3^{rd}$ term



Div

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^2 = \infty$

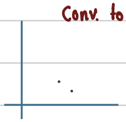
Ex. $b_n = \frac{1}{n}, \quad n = 1, 2, 3$

$b_1 = 1 \quad b_2 = \frac{1}{2}$

$b_3 = \frac{1}{3} \quad b_4 = \frac{1}{4}$

$b_5 = \frac{1}{5}$

Conv. to 0

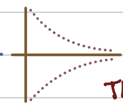


$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

Ex. $C_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, 3, \ldots$

$C_1 = 1 \quad C_2 = \frac{1}{4}$

$C_3 = -\frac{1}{9} \quad C_4 = \frac{1}{16}$



conv. to 0

$\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$

This sequence is called Alternating.

Assume $\lim\limits_{n\to\infty} a_n = A$ (رقم) and $\lim\limits_{n\to\infty} b_n = B$ (رقم)

Then 1) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B.$

2) $\lim\limits_{n\to\infty} (a_n b_n) = AB.$

3) $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}. \quad (B \neq 0)$

4) $\lim\limits_{n\to\infty} k a_n = kA.$

5) $\lim\limits_{n\to\infty} k = k.$

Ex. $\lim\limits_{n\to\infty} \frac{2n+1}{5+4n} = \frac{2}{4} = \frac{1}{2}.$

Ex. $\lim\limits_{n\to\infty} \frac{5n^2}{n+n^3} = 0$ since $n+n^3$ deg > $5n^2$ deg.

Ex. $\lim\limits_{n\to\infty} \frac{n^4}{n^3+1} = \infty$ since n^4 deg > n^3+1 deg.

Ex. Show that $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} = 0$?

$\frac{-1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$

$\lim\limits_{n\to\infty} \quad \lim\limits_{n\to\infty} \quad \lim\limits_{n\to\infty}$

$0 \leq \lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} \leq 0$

$= 0$ by sandwich th.

Sandwich Theorem

Assume $a_n \leq b_n \leq C_n$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} C_n = L$

Then $\lim\limits_{n\to\infty} b_n = L.$

✱ Reminder ✱

1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$

2) $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.$

3) $\lim\limits_{x\to\infty} \frac{\cos x}{x^3} = 0$

by sandwich th.

Ex. Find $\lim\limits_{n\to\infty} \frac{\sin^2 n}{2^n}$?

$0 \leq \sin^2 n \leq 1$

$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} \longrightarrow \lim\limits_{n\to\infty} \frac{1}{2^n} = \lim\limits_{n\to\infty} 0 = 0$

By S.T $\lim\limits_{n\to\infty} \frac{\sin^2 n}{2^n} = 0.$

Ex. Find $\lim\limits_{n\to\infty} (-1)^n \frac{1}{n}$?

$\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$

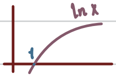
$\lim\limits_{n\to\infty} \frac{1}{n} = -\lim\limits_{n\to\infty} \frac{1}{n} = 0$

By S.T $\lim\limits_{n\to\infty} (-1)^n \frac{1}{n} = 0.$

Chapter 10.1 Theorem 1:

1) $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ (since $\frac{\infty}{\infty}$)

L'Hopital $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$



2) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ (since $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = \infty^0$)

$\lim\limits_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1.$

3) $\lim\limits_{n\to\infty} x^n = 0$ where $-1 < x < 1$.

$\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$ $\quad$ $\lim\limits_{n\to\infty} \left(\frac{\pi}{e}\right)^n = \infty$

$\lim\limits_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$ $\quad$ $\lim\limits_{n\to\infty} \left(\frac{5}{3}\right)^n = \infty$

$\lim\limits_{n\to\infty} (0.9)^n = 0$ $\quad$ $\lim\limits_{n\to\infty} \left(\frac{-e}{\pi}\right)^n = 0$

4) $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$ where $x > 0$

$\lim\limits_{n\to\infty} 5^{\frac{1}{n}} = 5^0 = 1$

$\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^{\frac{1}{n}} = \left(\frac{1}{2}\right)^0 = 1$

5) $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

$\lim\limits_{n\to\infty} \left(1 - \frac{3}{n}\right)^n = e^{-3}$

Proof: $\lim\limits_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = \lim e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}}$

$= \lim\limits_{n\to\infty} e^{\frac{\frac{-x}{n^2}}{1+\frac{x}{n}} \Big/ \frac{-1}{n^2}} = \lim\limits_{n\to\infty} e^{\frac{x}{1+\frac{x}{n}}} = e^x.$

6) $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0.$

for any value of x.

(Taylor Series)

Ex. Find the following limits?

1) $\lim\limits_{n\to\infty} \pi^{\frac{1}{n}} = 1.$

2) $\lim\limits_{n\to\infty} \frac{e^n}{n!} = 0.$

3) $\lim\limits_{n\to\infty} (-0.35)^n = 0.$

4) $\lim\limits_{n\to\infty} \left(1 - \frac{\sqrt{2}}{n}\right)^n = e^{-\sqrt{2}}$

5) $\lim\limits_{n\to\infty} 7^n = \infty.$

6) $\lim\limits_{n\to\infty} \frac{\ln n^3}{n} = 3(0) = 0.$

7) $\lim\limits_{n\to\infty} \sqrt[n]{n^3} = \lim\limits_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3 = 1^3 = 1.$

8) $\lim\limits_{n\to\infty} \sqrt[n]{\pi n} = \lim\limits_{n\to\infty} \pi^{\frac{1}{n}} n^{\frac{1}{n}} = (1)(1) = 1.$

9) $\lim\limits_{n\to\infty} \frac{\pi^{-n}}{e^{-n}} = \lim\limits_{n\to\infty} \left(\frac{e}{\pi}\right)^n = 0.$

10) $\lim\limits_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \lim\limits_{n\to\infty} \left(\frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(1-\frac{1}{n}\right)}\right)^n = \frac{e^1}{e^{-1}} = e^2.$

or $\lim\limits_{n\to\infty} \left(\frac{n-1+2}{n-1}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^n$ $\quad$ $u = n-1$

$= \lim\limits_{u\to\infty} \left(1 + \frac{2}{u}\right)^{u+1}$ $\quad$ $n\to\infty \quad u\to\infty$

$= \lim\limits_{u\to\infty} \left(1 + \frac{2}{u}\right)^u \left(1 + \frac{2}{u}\right)^1$

$= (e^2)(1) = e^2.$

11) $\lim\limits_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} = \frac{\lim\limits_{n\to\infty} \ln n}{\lim\limits_{n\to\infty} \sqrt[n]{n}} = \frac{\infty}{1} = \infty$

12) $\lim\limits_{n\to\infty} \sqrt[n]{n^3} = \lim\limits_{n\to\infty} \left(\sqrt[n]{n}\right)^3 = 1^3 = 1.$

13) $\lim\limits_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \ln e = 1.$

14) $\lim\limits_{n\to\infty} \sqrt[n]{n^2+n} = \lim\limits_{n\to\infty} e^{\ln \sqrt[n]{n^2+n}} = e^{\lim\limits_{n\to\infty} \frac{\ln(n^2+n)}{n}}$

$= e^{\lim\limits_{n\to\infty} \frac{2n+1}{n^2+n}} = e^0 = 1.$

Ex. Find the n^{th} term of the following sequences?

1) $1, 4, 9, 16, 25, \ldots$

$a_n = n^2, \quad n = 1, 2, 3, \ldots$

2) $1, 3, 5, 7, \ldots$

$a_n = 2n - 1, \quad n = 1, 2, 3, \ldots$

$b_n = 2n + 1, \quad n = 0, 1, 2, \ldots$

3) $-1, 4, -9, 16, -25, \ldots$

$a_n = (-1)^n n^2, \quad n = 1, 2, 3, \ldots$

4) $1, -4, 9, -16, 25, \ldots$

$a_n = (-1)^{n+1} n^2, \quad n = 1, 2, 3, \ldots$

Note

If the sequence was Alternating always think of $(-1)^n$.

Recursive Sequence:

Assume $a_0 = 2$. $a_{n+1} = \left(\frac{a_n}{2}\right)^2$

Find $\lim_{n \to \infty} a_n = \dots$

Assuming it converges $\to L = \left(\frac{L}{2}\right)^2$

$L = \frac{L^2}{4} \to \frac{L^2}{4} - L = 0 \to L\left(\frac{L}{4} - 1\right) = 0$

$L = 0$ or $L = 4$.

$a_1 = a_{0+1} = \left(\frac{a_0}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1.$

$a_2 = a_{1+1} = \left(\frac{a_1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$

$a_3 = a_{2+1} = \left(\frac{a_2}{2}\right)^2 = \frac{1}{64}.$

$a_4 = a_{3+1} = \left(\frac{a_3}{2}\right)^2 = \frac{1}{17424}.$

$a_n \to 0$ so $\lim_{n \to \infty} a_n = 0$.

Remark:

1) The sequence a_n is bounded from above if $\exists M$ (عدد) such that $M \geq a_n \ \forall n$ and bounded from below if $\exists m$ such that $m \leq a_n \ \forall n$.

2) Bounded sequence is sequence bounded from above and below.

Ex. 1) $1, 2, 3, 4, \dots, n, \dots$

$a_n = n \quad (n = 1, 2, 3, \dots)$

so $\lim_{n \to \infty} a_n = \lim_{n \to \infty} n = \infty$

This sequence diverges.

Not bounded from above.

$m = 1, \frac{1}{2}, 0, -5, \dots$ (lower bounds)

greatest lower bounded.

This sequence is only bounded from below so its not bounded.

(nondecreasing)

Monotonic Sequence

2) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

$b_n = \left(\frac{1}{2}\right)^n \quad (n = 0, 1, 2, \dots)$

so $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0$

This sequence converges to 0.

$M = 1, 2, 3, 3.5, e, \dots$ (upper bounds)

least upper bound.

$m = 0, -1, -2, \dots$ (lower bounds)

greatest lower bound.

b_n is bounded.

(nonincreasing)

Monotonic Sequence

3) $3, 3, 3, 3, 3, \dots$

$C_n = 3 \quad (\forall n)$

so $\lim_{n \to \infty} C_n = \lim_{n \to \infty} 3 = 3$

This sequence converges to 3.

$M = 3, 3.1, \pi, \dots$ (upper bounds)

least upper bound.

$m = 3, 2, 1, 0, \dots$

greatest lower bound.

C_n is bounded.

(nonincreasing and nondecreasing)

Monotonic Sequence

The sequence a_n is nondecreasing if

$a_{n+1} \geq a_n \; \forall n.$

The sequence a_n is nonincreasing if

$a_{n+1} \leq a_n \; \forall n.$

Monotonic Sequence either → nondecreasing or → nonincreasing

Theorem: The sequence a_n conv. iff it is bounded and monotonic.

Ex. 1) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ is nonincreasing (monotonic)

bounded above $M = 1, 2, 3, 4, \dots$ → Bounded

bounded below $m = 0, -1, -2, \dots$

$a_n = (\frac{1}{2})^n ; \; n = 0, 1, 2, \dots$

$\lim_{n \to \infty} a_n = 0$ so the sequence conv. to 0.

2) $2, 4, 6, 8, 10, \dots$ is nondecreasing (monotonic)

not bounded (not bounded from above).

3) $-3, -3, -3, -3, \dots$ is nondecreasing and nonincreasing

(monotonic) it is bounded.

so the sequence conv. to -3.

4) $-1, 0, 1, 3, 5, 8, -10, \dots$

it is not monotonic.

not alternating.