Chapter

4

Fourier Transform:-

$$x(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

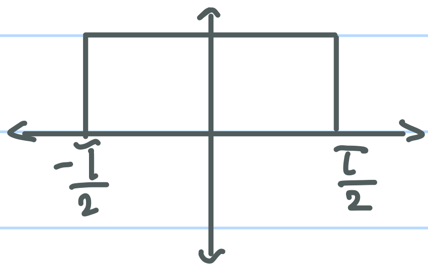
$$x(t) = \int_{-\infty}^{\infty} x(f)\, e^{j2\pi f t}\, dt$$

Ex. Evaluate the Fourier transform of the following signal

$x(t) = A\pi\left(\frac{t}{\tau}\right)$

Ans

$$x(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f}\, dt$$

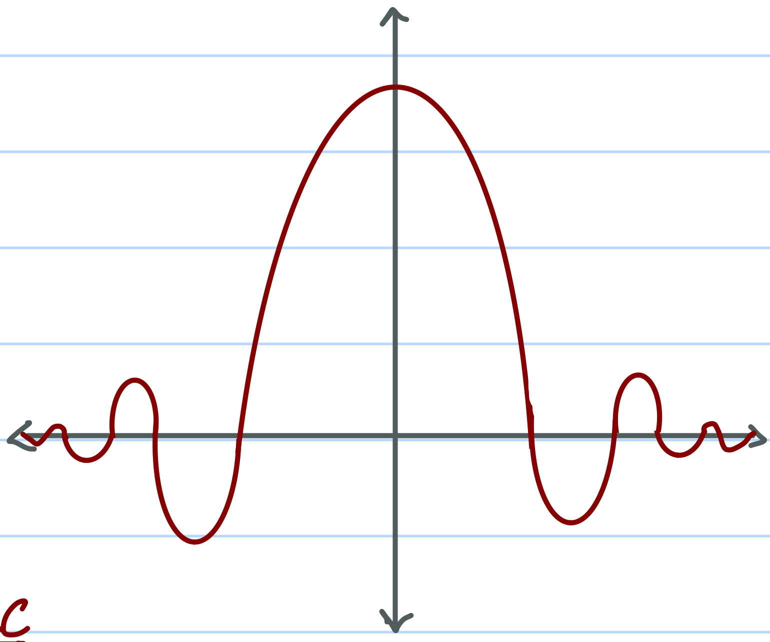


$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-j2\pi f t} = \frac{A}{-j2\pi f} e^{-j2\pi f t} \Big|_{-\tau/2}^{\tau/2}$$

$$x(f) = \frac{-A}{j2\pi f}\left(e^{-j\pi f \tau/2} - e^{j2\pi f \tau/2}\right)$$

$$= \frac{A}{\pi f}\left(\frac{e^{j\pi f\tau} - e^{-j\pi f \tau}}{j2}\right)$$

$$= \frac{A}{\pi f} \sin(\tau\pi f) \cdot \frac{\tau}{\tau}$$



$$= A\tau \frac{\sin(\tau\pi f)}{\tau\pi f} \rightarrow \underline{\underline{\text{Sinc}}}$$

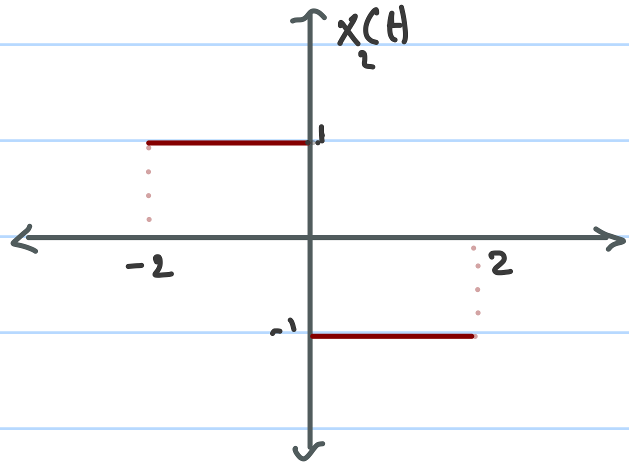
$$= A\tau\, \text{Sinc}(\tau f)$$

Note:

In general

- if $x(t)$ even $\Rightarrow x(t) = x(-t)$
- if $x(t)$ odd $\Rightarrow x(-t) = -x(t)$
- $|X(f)| = |X(-f)|$
- $\angle\theta_{X(f)} = -\angle\theta_{X(-f)}$
- $\text{Sinc}(\theta) = \frac{\sin(\theta\pi)}{\theta\pi}$

Ex. 2:



$$X(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-2}^{0} e^{-j2\pi f t}\, dt - \int_{0}^{2} e^{-j2\pi f t}\, dt$$

$$= \frac{-1}{j2\pi f} e^{-j2\pi f t}\Big|_{-2}^{0} + \frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{0}^{2}$$

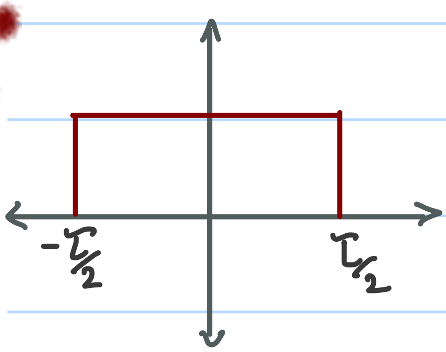
$$= \frac{-1}{j2\pi f}\left[1 - e^{j4\pi f}\right] + \frac{1}{j2\pi f}\cdot\left[e^{-j4\pi f} - 1\right]$$

$$= \frac{-1}{j2\pi f} + \frac{e^{j4\pi f}}{j2\pi f} + \frac{e^{-j4\pi f}}{j2\pi f} - \frac{1}{j2\pi f}$$

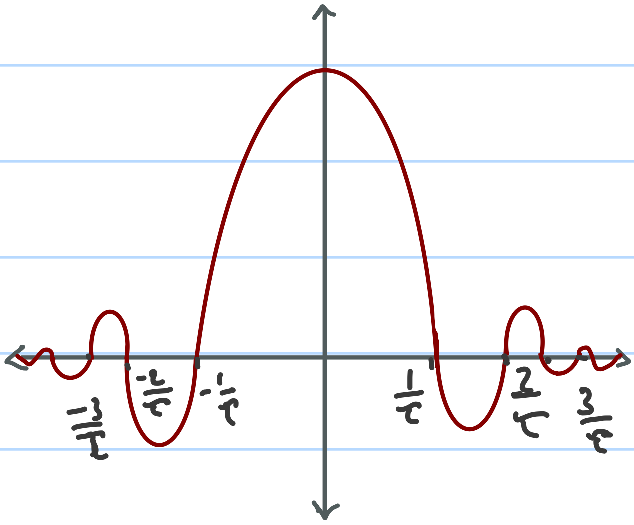
$=$

* موجود بحل الدكتور

Remember:-

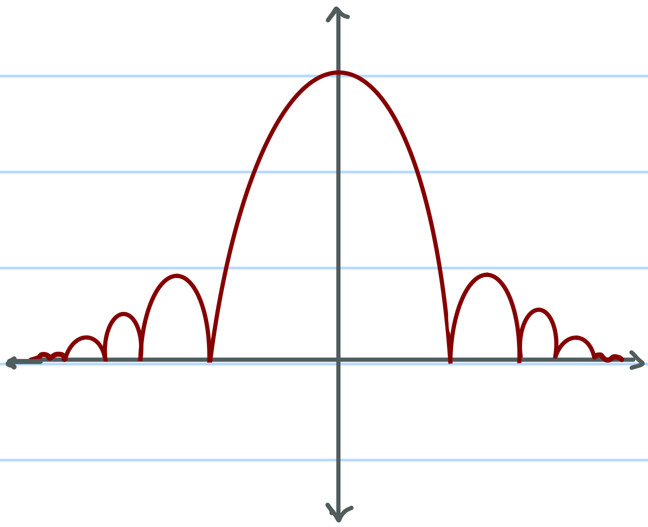


$\Rightarrow$



- $\text{Sinc}(\tau f) = \dfrac{\sin(\tau \pi f)}{\tau \pi f}$

- $\text{Sinc}^2(\tau f) = \left(\dfrac{\sin(\tau \pi f)}{\tau \pi f}\right)^2 \Rightarrow$



- $\text{Sinc}(0) = \text{Sinc}^2(0) = 1$

Theorems of F T

① Linearity:-

$$\mathcal{F}[x_1(t) + x_2(t)]$$

$$= \int_{-\infty}^{\infty} (x_1(t) + x_2(t)) e^{-j2\pi f t}$$

$$= \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} dt$$

$$= X_1(f) + X_2(f)$$

② scaling

$$\mathcal{F}[x(at)]$$

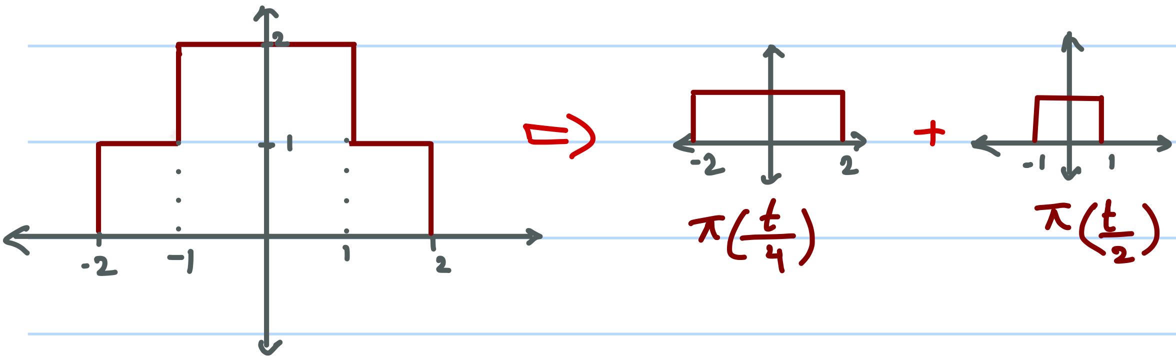
$$= \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt$$

let: $u = at \Rightarrow du = a\,dt \Rightarrow dt = \frac{du}{a}$

$$\int_{-\infty}^{\infty} \frac{1}{a} x(t) e^{-j2\pi f t} dt$$

$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

Ex. Evaluate the Fourier Transform of x(t) shown below



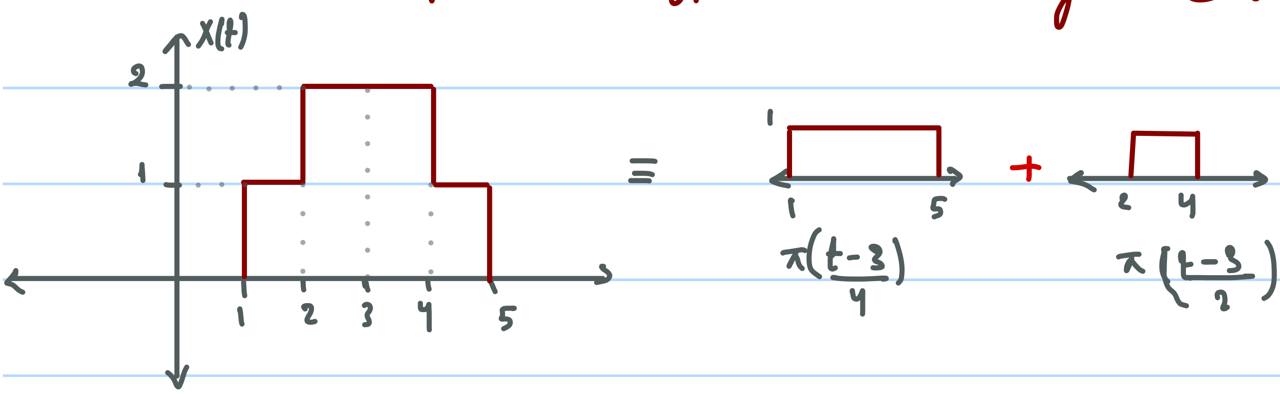
$$x(t) = \pi\left(\frac{t}{4}\right) + \pi\left(\frac{t}{2}\right)$$

$$\mathcal{F}[x(t)] = 4\,\text{Sinc}(4f) + 2\,\text{Sinc}(2f)$$

3 Time delay:

$$\mathcal{F}[x(t-t_0)] = X(f)\,e^{-j2\pi f t_0}$$

Ex. Evaluate the fourier transform for the signal shown below:-



$$x(t) = \pi\left(\frac{t-3}{4}\right) + \pi\left(\frac{t-3}{2}\right)$$

$$\mathcal{F}[x(t)] = 4\,\text{Sinc}(4f)\,e^{-j2\pi f(3)} + 2\,\text{Sinc}(2f)\,e^{-j2\pi f(3)}$$

Ex. Evaluate the fourier transform for the signal $x(t) = -3\pi(-2t-4)$

$x(t) = 3\pi(-2t-4) = -3\pi(-2(t+4))$ even func

$x(t) = \pi(2(t+2))$

$\mathcal{F}[x(t)] = \frac{-3}{2} \text{Sinc}\left(\frac{1}{2}f\right) e^{-j2\pi f(-2)}$

$$\boxed{X(f) = \frac{-3}{2} \text{Sinc}\left(\frac{f}{2}\right) e^{j2\pi f 2}}$$

④ Frequency transian theorem:-

$$\mathcal{F}[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t) e^{j2\pi f_0 t} \cdot e^{-j2\pi f t} \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi(f-f_0)} \, dt$$

$$= X(f-f_0)$$ → shift

⑤ Modulation theorem

$$\mathcal{F}[x(t)\cos(w_0 t)] = \mathcal{F}\left[\frac{1}{2} x(t) e^{j2\pi f_0 t} + \frac{1}{2} x(t) e^{-j2\pi f_0 t}\right]$$

$$= \mathcal{F}\left[\frac{1}{2} x(t) e^{j2\pi f_0 t}\right] + \mathcal{F}\left[\frac{1}{2} x(t) e^{-j2\pi f_0 t}\right]$$

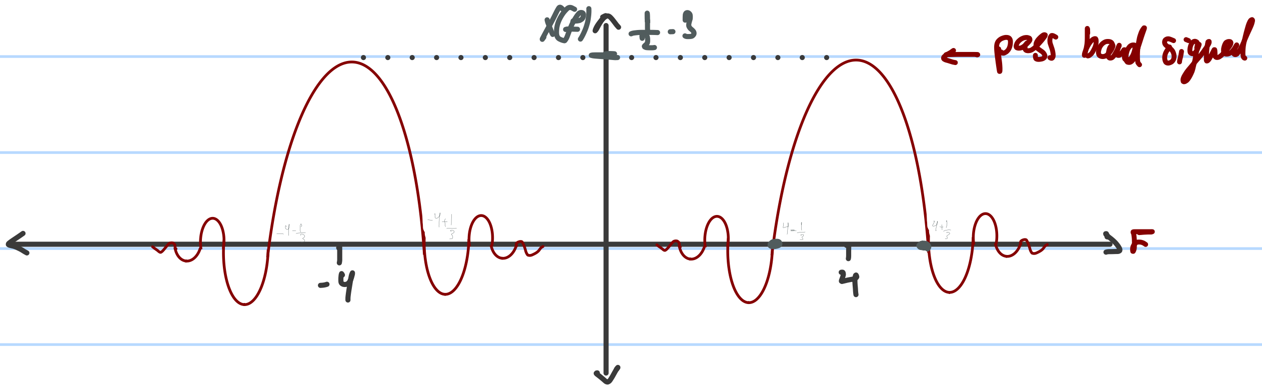
$$= \frac{1}{2} X(f-f_0) + \frac{1}{2} X(f+f_0)$$

Ex. Evaluate the Fourier Transform for the following signal

① $x_1(t) = \Pi\left(\frac{t}{3}\right)\cos(8\pi t)$

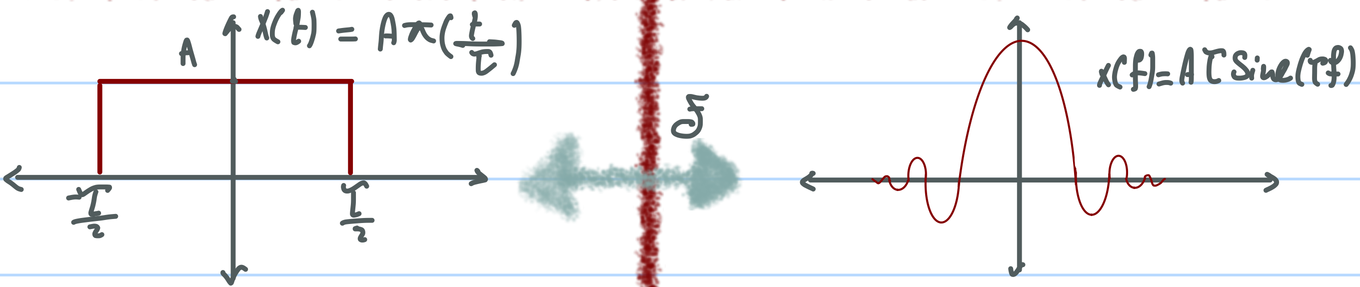
$$= \left[\frac{1}{2}\Pi\left(\frac{t}{3}\right)e^{j8\pi t} + \frac{1}{2}\Pi\left(\frac{t}{3}\right)e^{-j8\pi t}\right]$$

$$= \frac{1}{2}\,3\,\mathrm{Sinc}(3(f-4)) + \frac{1}{2}\cdot 3\,\mathrm{Sinc}(3(f+4))$$



Time-domain	Freq-domain

1



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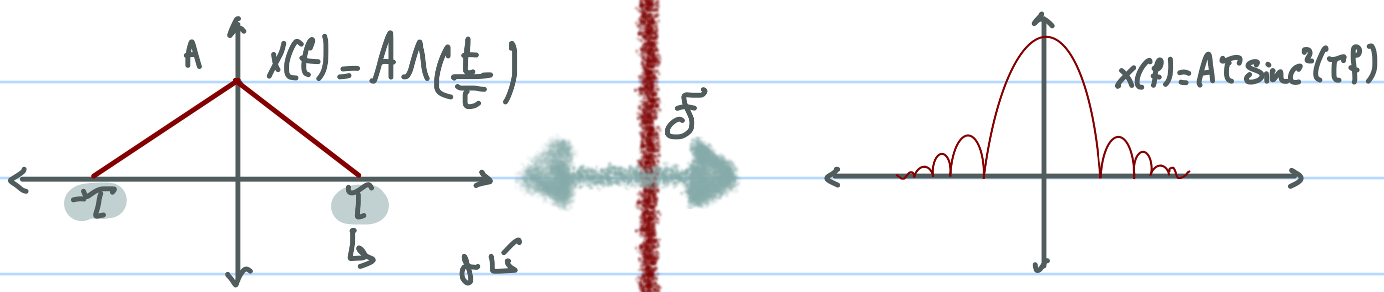

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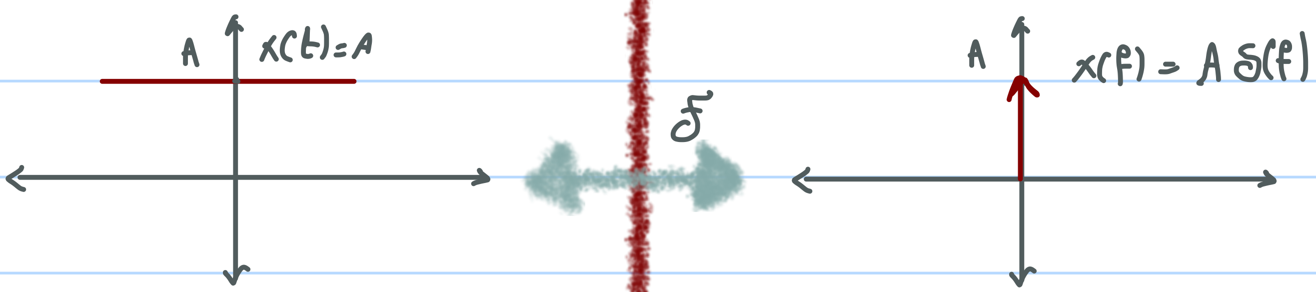


Ex.

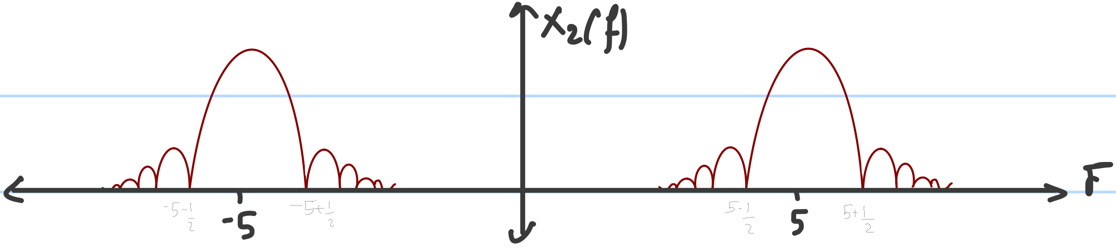
② $x_2(t) = \Lambda\left(\frac{t}{2}\right)\cos(10\pi t)$

$$= \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{j10\pi t} + \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{-j10\pi t}$$

$$\mathcal{F}\left[\frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{j10\pi t} + \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{-j10\pi t}\right]$$

$$= \frac{1}{2}\cdot 2\,\mathrm{Sinc}^2(2(f-5)) + \frac{1}{2}\cdot 2\,\mathrm{Sinc}^2(2(f+5))$$

$$= \mathrm{Sinc}^2(2(f-5)) + \mathrm{Sinc}^2(2(f+5))$$



⑥ Dualing theorem

$x(t) \longrightarrow X(f)$

$X(t) \longrightarrow x(-f) \rightarrow$ check if its even or odd

- Even $\rightarrow$ $x(f) = x(-f)$
- odd $\rightarrow$ $x(-f) = -x(f)$

Ex.

$x(t) = 3\,\text{Sinc}(2t)$ $\quad X(f) = ??$

$x(t) = 3\,\text{Sinc}(2t) \xrightarrow{\mathcal{F}} 3 \cdot \frac{1}{2} \pi\left(\frac{-f}{2}\right)$

Since $\pi(t)$ even fun $\Rightarrow$

$X(f) = \frac{3}{2} \pi\left(\frac{f}{2}\right)$

Ex. Evaluate the fourier transform of the following signals

$x(t) = 4\,\text{Sinc}(3(t-2))$

$\mathcal{F}[x(t)] = \mathcal{F}\left[4\,\text{Sinc}(3(t-2))\right]$

$= 4 \cdot \frac{1}{3} \pi\left(-\frac{f}{3}\right) \cdot e^{-j2\pi f(2)}$

Even function $\Rightarrow \frac{4}{3} \pi\left(\frac{f}{3}\right) \cdot e^{-j2\pi f(2)}$

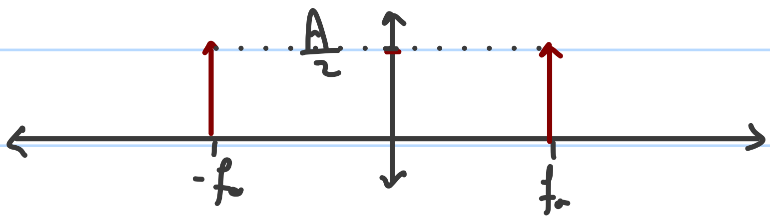
Ex. Evaluate the fourier transform of the following signals

$$A\,\delta(t) \xleftrightarrow{\mathcal{F}} A$$

$$A \xleftrightarrow{\mathcal{F}} A\,\delta(f)$$

$$A\,\delta(t-t_0) \xleftrightarrow{\mathcal{F}} A\,e^{-j2\pi f t_0}$$

$$A\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{A}{2}\,\delta(f-f_0) + \frac{A}{2}\,\delta(f+f_0)$$



(7) Differentiation and Integration theorem

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt}\, e^{-j2\pi f t}\, dt$$

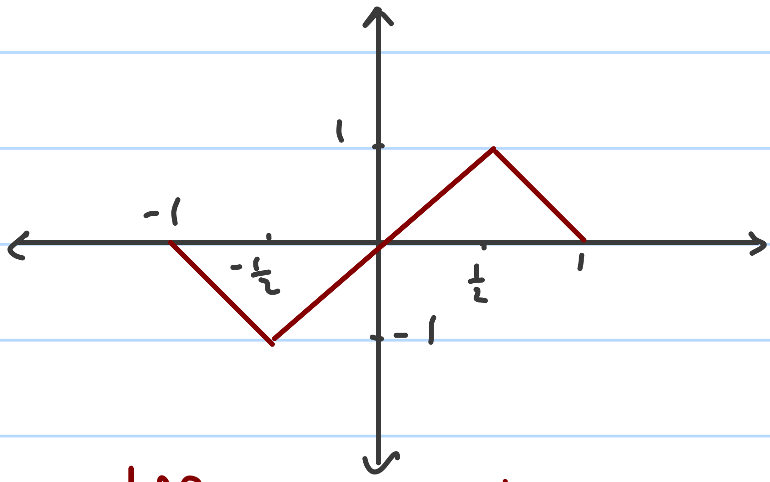
$$= j2\pi f\, X(f)$$

In general

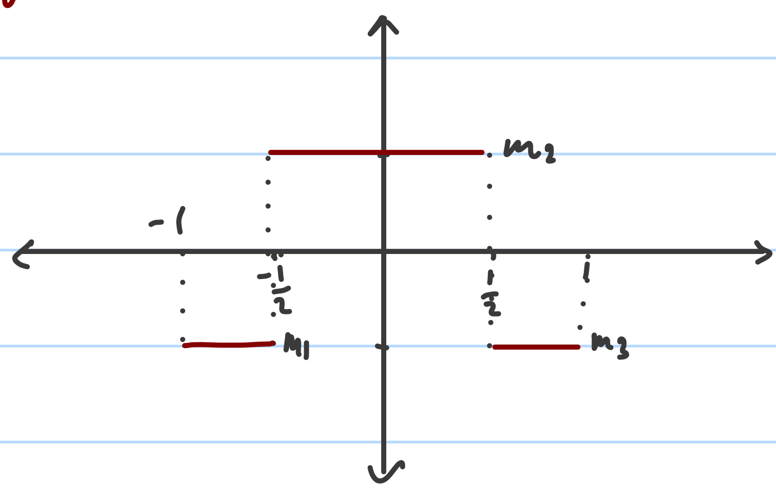
$$\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right] = (j2\pi f)^n X(f)$$

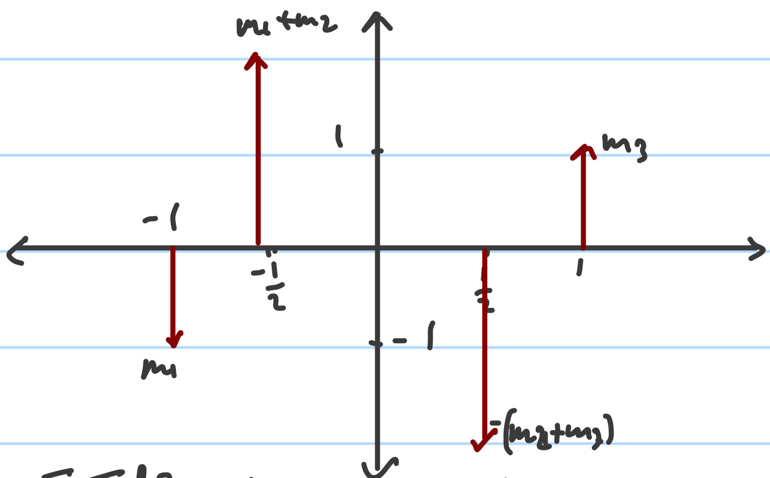
$$\mathfrak{F}\left[\int_{-\infty}^{\infty} x(\lambda)\,d\lambda\right] = \frac{1}{j2\pi f}\,X(f)$$

Ex. Evaluate the fourier transform of the following signals



using differentiation theorem





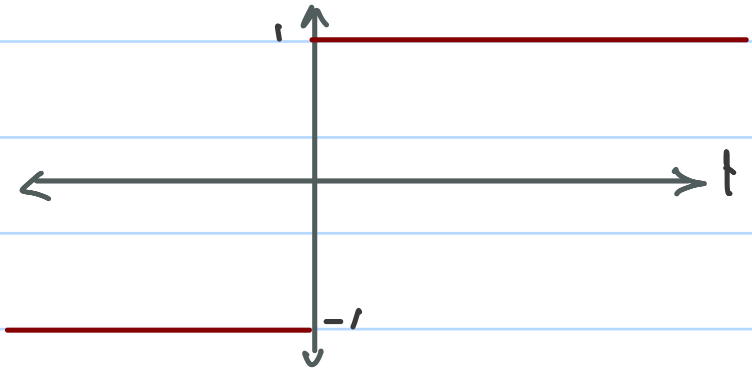
$$\mathfrak{F}\left[\frac{d^2 x(t)}{dt}\right] = (j2\pi f)^2 X(f)$$

$$\mathfrak{F}\left[m_1\delta(f+1) + (m_1+m_2)\delta(f+\tfrac{1}{2}) - (m_2+m_3)\delta(f-\tfrac{1}{2}) + m_3\delta(f-1)\right] =$$

$$= \frac{-m e^{j2\pi f} + (m_1+m_2)\, e^{j\cancel{2}\pi f \cdot \frac{1}{\cancel{2}}} - (m_1+m_2)\, e^{-j\cancel{2}\pi \cdot \frac{1}{\cancel{2}}} + m_3 e^{-j2\pi f}}{(j2\pi f)^2}$$

Ex. Evaluate the fourier transform for the signum function, which is defined as

$$\mathrm{sgn}(t) = \begin{cases} +1 & t>0 \\ -1 & t<0 \\ 0 & t=0 \end{cases}$$



$$\mathrm{sgn}(t) = 2u(t) - 1$$

$$\mathfrak{F}\left[\frac{d}{dt}\,\mathrm{sgn}(t)\right] = \mathfrak{F}[2\delta(t)]$$

$$j2\pi f\, \mathfrak{F}[\mathrm{sgn}(t)] = 2 \Rightarrow \mathfrak{F}[\mathrm{sgn}(t)] = \frac{2}{j2\pi f} = \boxed{\frac{1}{j\pi f}} \rightarrow \text{odd fun}$$

$$x(t) = \frac{1}{\pi t}$$

$$x(t) = \frac{1}{\pi t} \cdot \frac{j}{j} \Rightarrow j \cdot \frac{1}{j\pi t} =$$

$$\mathfrak{F}[x(t)] = \mathfrak{F}\left[j \cdot \frac{1}{j\pi t}\right] = -j\,\mathrm{sgn}(f)$$

8) Convolution theorem

$$\left.\begin{array}{l} x(t) \longrightarrow \boxed{\;h(t)\;} \longrightarrow y(t) \\ y(t) = x(t) * h(t) \end{array}\right] \longrightarrow \text{Time domain}$$

$$\mathfrak{F}[y(t)] = \mathfrak{F}[x(t) * h(t)] = \int_{-\infty}^{\infty} [x(t) * h(t)]\, e^{-j2\pi f t}\, dt$$

Since:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$\mathfrak{F}[y(t)] = \mathfrak{F}\left[\int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda\, e^{-j2\pi f t}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x(\lambda)}_{②} \Big(\underbrace{\int_{-\infty}^{\infty} h(t-\lambda)\, e^{-j2\pi f t}\, dt}_{① \text{ change of var.}} \Big)\, d\lambda$$

let $u = t - \lambda$

$du = dt$

when $t = -\infty \Rightarrow u = -\infty$

when $t = \infty \Rightarrow u = \infty$

$$\mathfrak{F}[y(t)] = \int_{-\infty}^{\infty} x(\lambda) \Big(\int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f (u+\lambda)}\, du \Big)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda) \Big(\underbrace{\int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f (u)}\, du}_{H(f)} \Big)\, e^{-j2\pi f \lambda}\, d\lambda$$

$$= H(f) \underbrace{\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda} d\lambda}_{X(f)}$$

$$= H(f) \cdot X(f)$$

In general

$$* \Rightarrow \mathcal{F}[x_1(t) * x_2(t)] = \mathcal{F}[x_1(t)] \cdot \mathcal{F}[x_2(t)] = X_1(f) \cdot X_2(f)$$

Ex. Evaluate the fourier transform of $y(t) = x_1(t) * x_2(t)$ where $x_1(t) = \Pi\left(\frac{t}{3}\right)$ and $x_2(t) = \Pi\left(\frac{t}{3}\right)$

$$y(t) = x_1(t) * x_2(t)$$

$$\mathcal{F}[y(t)] = \mathcal{F}[x_1(t) * x_2(t)]$$

$$= \mathcal{F}\left[\Pi\left(\tfrac{t}{3}\right)\right] \cdot \mathcal{F}\left[\Pi\left(\tfrac{t}{3}\right)\right]$$

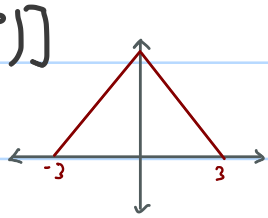
$$= 3\,\text{sinc}(3f) \cdot 3\,\text{sinc}(3f)$$

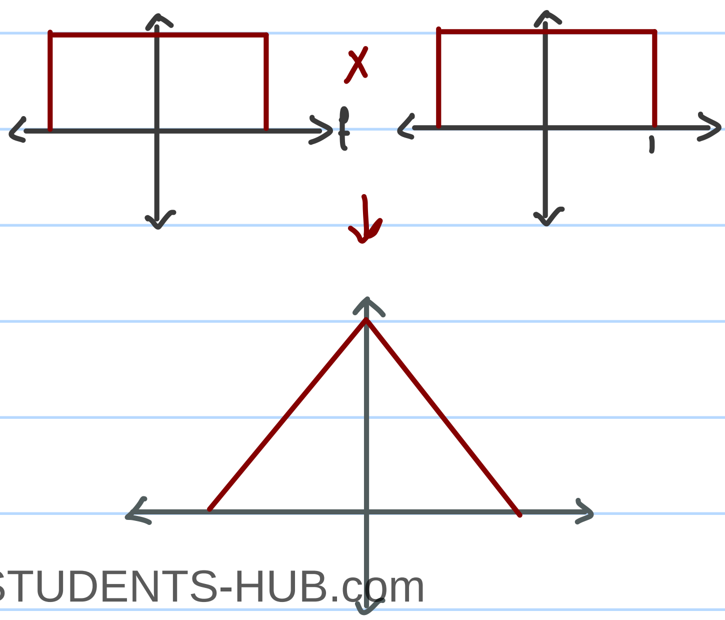
$$= 9\,\text{sinc}^2(3f)$$

$$y(t) = \mathcal{F}^{-1}[9\,\text{sinc}^2(3f)]$$

$$= \mathcal{F}^{-1}[3 \cdot 3\,\text{sinc}^2(3f)]$$

$$y(t) = 3\,\Lambda\left(\frac{t}{3}\right) \Rightarrow$$





Remember: $\mathfrak{F}\left[\pi\left(\frac{t}{\tau}\right) * \pi\left(\frac{t}{\tau}\right)\right] = \mathfrak{F}\left[\tau\Lambda\left(\frac{t}{\tau}\right)\right]$

Ex. Find the fourier transform of the Hilber transform function which donate as

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda}\, d\lambda$$

$$= \frac{1}{\pi t} * x(t)$$

$$\mathfrak{F}[\hat{x}(t)] = \mathfrak{F}\left[\frac{1}{\pi t}\right] \mathfrak{F}[x(t)]$$

Remember $\mathfrak{F}[\operatorname{sgn}(t)] = \frac{1}{j\pi f}$

$$\mathfrak{F}\left[\frac{1}{\pi t} \cdot \frac{j}{j}\right] = \mathfrak{F}\left[\frac{j}{j\pi t}\right] = j\operatorname{sign}(-f)$$ → odd function

$$\boxed{= -j\operatorname{sgn}(f)}$$

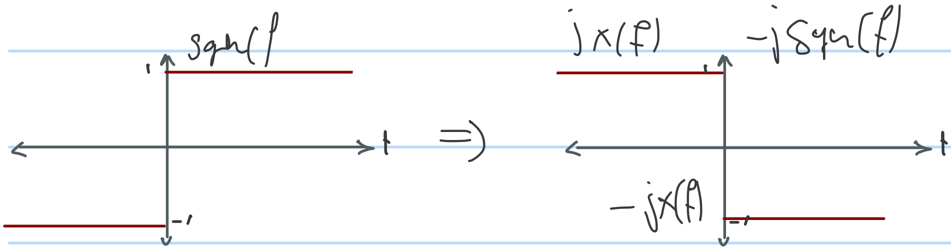
$$\mathfrak{F}[\hat{x}(t)] = -j\operatorname{sgn}(f) \cdot X(f)$$

Ex. Evaluate the hilbert transform of $x(t) = \cos(\omega_0 t)$

b) Evaluate $X^H(f)$

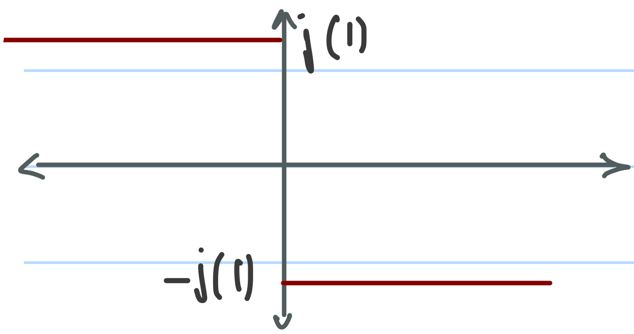
$$x(t) \longrightarrow \boxed{X^H(f)} \longrightarrow \sin(\omega_0 t)$$

* Remember

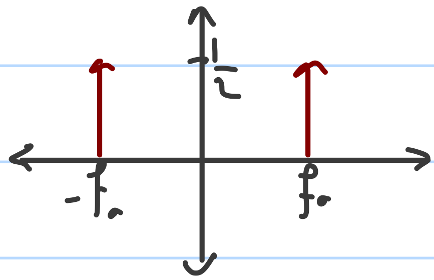


$$\mathcal{F}[x^H(t)] = -j\,\text{sgn}(f)\,X(f)$$

$$\mathcal{F}[x(t)] = \mathcal{F}[\cos(\omega_0 t)] = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

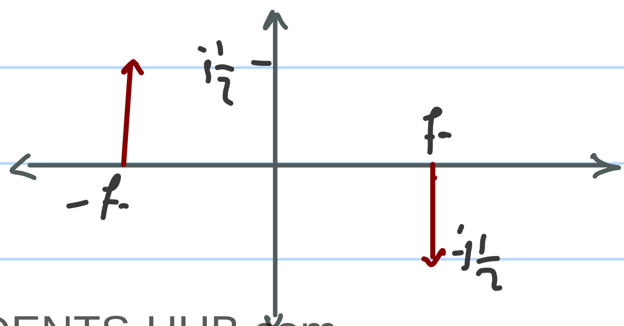


x



$$X^H(f) = \frac{1}{2} j\,\delta(f+f_0) - \frac{1}{2}\delta(f-f_0)$$

=)



$$x^H(t) = \mathcal{F}^{-1}[X^H(f)]$$

$$= \frac{1}{2} j e^{-j2\pi f_0 t} - \frac{1}{2} j e^{j2\pi f_0 t}$$

$$= \frac{1}{\sqrt{2}}\left(e^{j2\pi f t} - e^{-j2\pi f_0 t}\right)$$

$$= \sin(2\pi f_0 t)$$

Energy spectral density and power spectral density
(ESD) (PSD)

To evaluate the total spectral density ⇒ periodic

To evaluate the total power "Periodic"

Ex. $x(t) = 3\sin(20\pi t)$ evaluate the total power:-

$\boxed{1}$

$$P_{avg} = \frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_0^T (3)^2 \sin^2(20\pi t)\, dt$$

$$= \frac{1}{1/10}\int_0^{1/10} 9 \sin^2(20\pi t)\, dt \qquad \frac{1}{2} - \frac{1}{2}\cos(40\pi t)$$

$$= 10\int_0^{1/10} \frac{9}{2}\, dt \quad \cancel{+\frac{10}{2}\int_0^{1/10} \cos(40\pi t)\, dt} \to 0$$

$$= \cancel{10} \cdot \frac{9}{2} \cdot \frac{1}{\cancel{10}}$$

In general

$x(t) = A\cos(20\pi t) + B\sin(30\pi t) + C\cos(60\pi t)$

$$P_{avg} = \frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2}$$

$\boxed{2}$ by using parseval's theorem

$$P_{avg} = \sum_{-\infty}^{\infty} |X_n|^2$$

$$X(t) = 3\sin(20\pi t)$$

$$X_n \begin{cases} \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_n + b_n) & n<0 \\ a_0 & n=0 \end{cases} \Rightarrow X_n = \begin{cases} \frac{1}{2}(0-3j) & n=1 \\ \frac{1}{2}(0+3j) & n=-1 \\ 0 & o.w \end{cases}$$

$$P_{avg} = 2|X_1|^2 = 2 \cdot \left(\frac{3}{2}\right)^2 = 2 \cdot \frac{9}{4} = \frac{9}{3}$$

OR

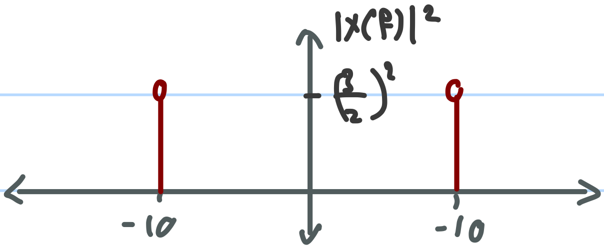
$$X(t) = 3\sin(20\pi t) = 3\left(\frac{e^{j20\pi t} - e^{-j20\pi t}}{j2}\right)$$

$$= \frac{3}{j2} e^{j20\pi t} - \frac{3}{j2} e^{-j20\pi t}$$

$$X_1 = \frac{3}{j2} \quad , X_{-1} = -\frac{3}{j2}$$

3 By using Fourier transform

$x(t) = 3\sin(20\pi t) \longrightarrow$



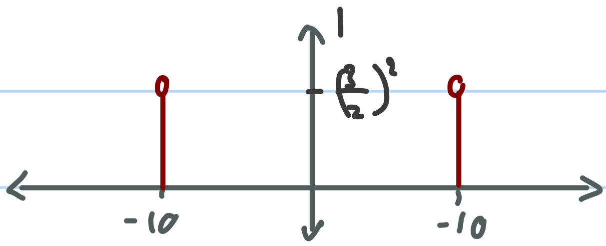
$$= 3\left(\frac{e^{j20\pi t} - e^{-j20\pi t}}{j2}\right)$$

$$X(f) = \frac{3}{j2}\,\delta(f-10) - \frac{3}{j2}\,\delta(f+10) \longrightarrow (1)$$

$$P_{avg} = \frac{1}{T}\int_T |x(t)|^2 = \frac{1}{T}\int_T x(t)\,x^*(t)\,dt$$

$$P_{avg} = \left(\frac{3}{2}\right)^2 \cdot 2$$

$$= \frac{9}{2}$$



Ex. Consider the following signal

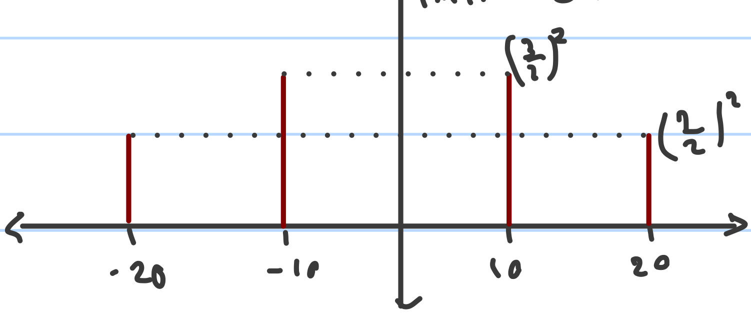
$$x(t) = 3\cos(20\pi t) + 2\sin(40\pi t)$$

$$\omega_1 = 20\pi = 2\pi f_1 \qquad \omega_2 = 40\pi = 2\pi f_2$$

$$f_1 = 10 \qquad f_2 = 20$$

a) Evaluate the power spectral density

$|X(f)|^2 = S(f) =$



$$= \left(\frac{3}{2}\right)^2 \delta(f-10) + \left(\frac{3}{2}\right)^2 \delta(f+10) + \left(\frac{2}{2}\right)^2 \delta(f-20) + \left(\frac{2}{2}\right)^2 \delta(f+20)$$

b) Evaluate the total power

$$P_{tot} = 2 \cdot \left(\frac{3}{2}\right)^2 + 2 \cdot \left(\frac{2}{2}\right)^2$$

$$= \frac{9}{2} + 2$$

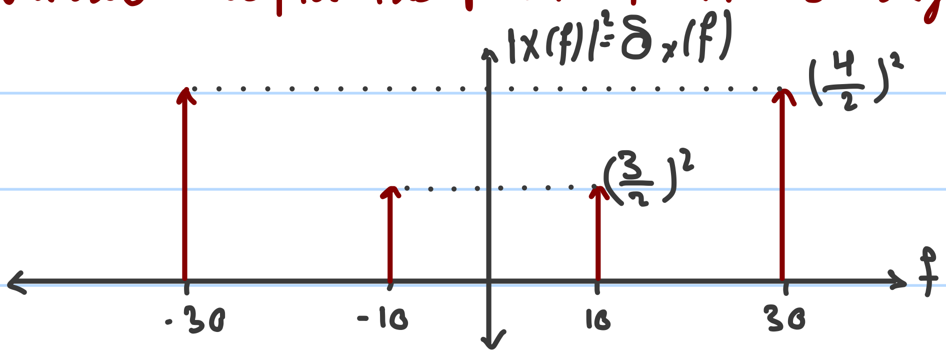
$$= 6.5$$

"Energy spectral density and power spectral density"

- power spectral density (PSD) ⇒ periodic

Ex. Consider the following signal $x(t) = 3\cos(20\pi t) + 4\sin(60\pi t)$

a) Evaluate and plot the power spectral density.



$$S_x(f) = \left(\frac{3}{2}\right)^2 \delta(f-10) + \left(\frac{3}{2}\right) \delta(f+10) + \left(\frac{4}{2}\right)^2 \delta(f+30) + \left(\frac{4}{2}\right)^2 \delta(f-30)$$

b) Evaluate the average power

In time-domain

$$P_{avg} = \frac{(3)^2}{2} + \frac{(4)^2}{2}$$

by using parseval's theorem

Since $x(t)$ expressed as Trigonometric fourier series

$$x(t) = 3\cos(20\pi t) + 4\sin(60\pi t)$$

$w_1 = 2\pi f_1 = 20\pi t$ $\qquad$ $w_2 = 2\pi f_2 = 60\pi$

$f_1 = 10$ $\qquad$ $f_2 = 30$

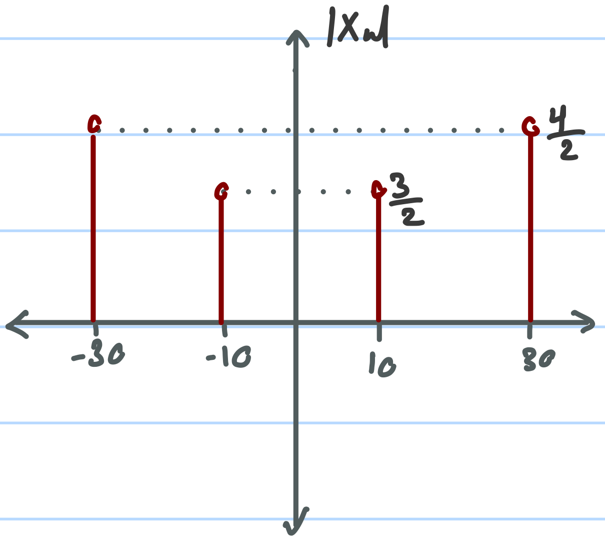
$GCD(10, 30) \Rightarrow f_0 = 10 \quad n_1 = 1, \; n_2 = 3$

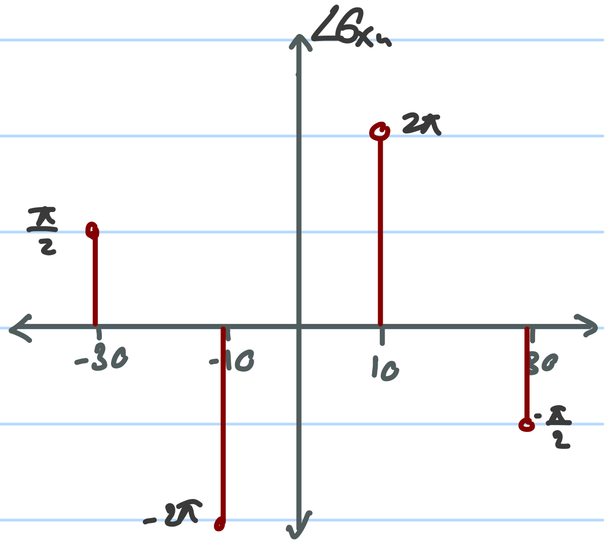
$a_n = \begin{cases} 3 & n=0 \\ 0 & o.w \end{cases}$ $\qquad$ $b_n \begin{cases} 4 & n=3 \\ 0 & o.w \end{cases}$

$$X_n = \begin{cases} \frac{1}{2}(a_n - j b_n) & n>0 \\ \frac{1}{2}(a_n + j b_n) & n<0 \\ a_0 & n=0 \end{cases} \Rightarrow X_n = \begin{cases} \frac{3}{2} & n=1 \\ -j\frac{4}{2} & n=3 \\ \frac{-3}{2} & n=-1 \\ j\frac{4}{2} & n=-3 \end{cases}$$

$$P_{avg} = \sum_{-\infty}^{\infty} |X_n|^2 = 2|X_1|^2 + 2|X_3|^2$$

$$= 2\cdot\left(\frac{3}{2}\right)^2 + 2\cdot\left(\frac{4}{2}\right)^2$$





Energy spectral density:

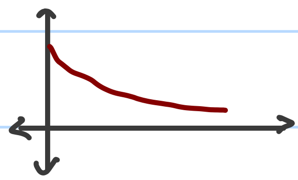
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \Rightarrow E_{tot} = \lim_{B\to\infty} \int_{-B}^{B} |X(f)|^2 df =$$

where $|X(f)| = G(f) \Rightarrow$ energy spectral density

Ex. For the following signal $x(t) = e^{-\alpha t} u(t)$, $\alpha > 0$

a) Find the fourier transform of $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-\alpha t} e^{-j2\pi f t} dt$$



$$= \frac{1}{\alpha + j2\pi f}$$

In general

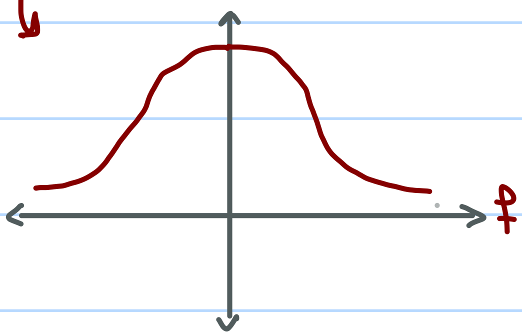
$$\mathcal{F}[e^{-\alpha t} u(t)] = \frac{1}{\alpha + j2\pi f}$$

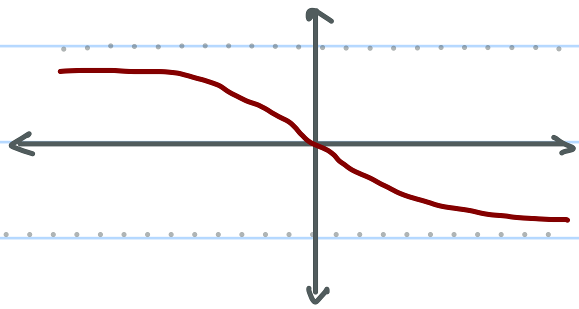
b) Evaluate and plot the spectrum amplitude and phase of $x(t)$

$$X(f) = \frac{1}{\alpha + j2\pi f} = \frac{1 \angle 0}{\sqrt{(\alpha)^2 + (2\pi f)^2} \tan^{-1}\left(\frac{2\pi f}{\alpha}\right)}$$

$$|X(f)| = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}}$$

$$\angle \theta_{X(f)} = -\tan^{-1}\left(\frac{2\pi f}{\alpha}\right)$$





c) Evaluate the cutoff freq (f_{3dB})

$$|X(f)|_{max} = |X(0)| = \frac{1}{\alpha}$$

$$|X(f_{3dB})| = \frac{1}{\sqrt{2}}|X(0)|$$

$$\frac{1}{\sqrt{\alpha^2+(2\pi f_{3dB})^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\alpha}$$

$$\Rightarrow 2\alpha^2 = \alpha^2 + (2\pi f_{3dB})^2$$

$$\boxed{f_{3dB} = \frac{\alpha}{2\pi}}$$

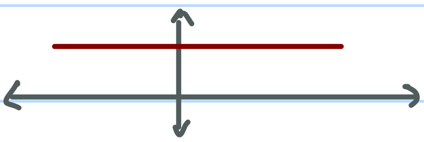
Evaluate α if $f_{3dB} = 1KHz$

$$1KHz = \frac{\alpha}{2\pi} \Rightarrow \boxed{\alpha = 2\pi \times 10^3}$$

d) Determine the type of the distortion

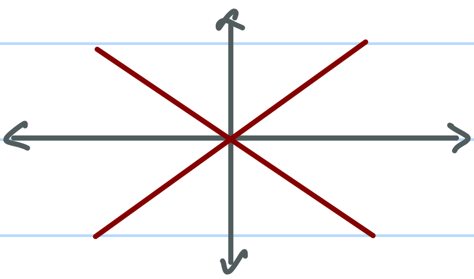
Notes:

the signal has distortionless of amp if $|X(f)| \to const$



the signal has distortionless of phase if $\angle\theta = K\omega$

$$= K(2\pi f)$$



In our example:

we have distortion in Amp

since $|X(f)|$ not constant and distortion in phase

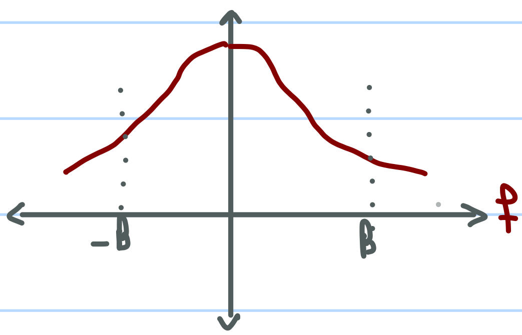
since $\angle G_{x(f)}$ non linear

$\Rightarrow$ we have distortion

ⓒ Evaluate the energy spectral density

$$G_x(f) = |X(f)|^2$$

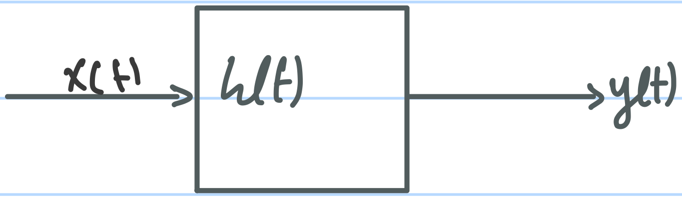
$$= \frac{1}{\alpha^2 + (2\pi f)^2}$$



ⓕ Evaluate the total energy

$$E = \lim_{B \to \infty} \int_{-B}^{B} \frac{1}{\alpha^2 + (2\pi f)^2} \, df$$

System Analysis with Fourier Transform:



LTI

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)\,H(f)$$

$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} \longrightarrow \text{transfer function}$$

Ex. Consider the following LTI system, which has the following DFE

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

a) Evaluate the spectrum of the response of the system

Ans.

$$j2\pi f RC\, Y(f) + Y(f) = X(f)$$

$$Y(f)\cdot[1 + j2\pi f RC] = X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1+j2\pi f RC}$$

b) Evaluate the impulse response, $h(t)$

Ans.

Since: $\mathfrak{F}[e^{-\alpha t}u(t)] = \frac{1}{\alpha + j2\pi f}$

$$H(f) = \frac{1}{RC} \cdot \frac{1}{1/RC + j2\pi f}$$

$$= \frac{1}{RC} e^{-t/RC}$$

c) Evaluate the output of the system if $x(t) = e^{-\alpha t}u(t)$

$$Y(f) = X(f)\,H(f)$$

$$= \frac{1}{\alpha + j2\pi f} \cdot \frac{1}{1 + 2\pi f RC}$$

$$= \frac{A}{\alpha + j2\pi f} + \frac{B}{1 + j2\pi f RC}$$

when $j2\pi f = \frac{-1}{RC}$

$\Rightarrow \quad 1 = B\left(\alpha - \frac{1}{RC}\right) \Rightarrow B = \frac{1}{\alpha - 1/RC}$

when $j2\pi f = -\alpha$

$1 = A(1 - \alpha RC) \Rightarrow A = \frac{1}{1 - \alpha RC}$

$$Y(f) = \frac{1}{1 - \alpha RC} \cdot \frac{1}{\alpha + j2\pi f} + \frac{1}{\left(\alpha - \frac{1}{RC}\right)} \cdot \frac{1}{RC} \cdot \frac{1}{1/RC + j2\pi f}$$

$$y(t) = \frac{1}{1-\alpha RC} \cdot e^{-\alpha t} u(t) + \frac{1}{RC\left(\alpha - \frac{1}{RC}\right)} e^{-t/RC} u(t)$$

Ex. Consider LTI system with response $h(t) = Sinc(2t)$ and it's input $x(t) = Sinc(t)$ Evaluate the spectrum output of the system, $Y(f)$

Ans.

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f)$$

where $X(f) = \mathcal{F}[Sin(t)] = \pi(f)$

and $H(f) = \mathcal{F}[Sinc(2t)] = \frac{1}{2}\pi(\frac{f}{2})$

$$Y(f) = \pi(f) \cdot \frac{1}{2}\pi(\frac{f}{2})$$

Ex. Consider LTI system with response $h(t) = Sin(2t)$ and input $x(t) = Sinc(t)$. Evaluate the out put $y(t) = x(t) \cdot h(t)$

Ans.

$$y(t) = x(t) \cdot h(t)$$

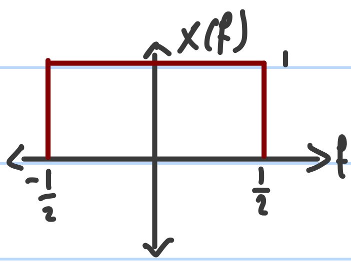
$$Y(f) = X(f) * H(f)$$

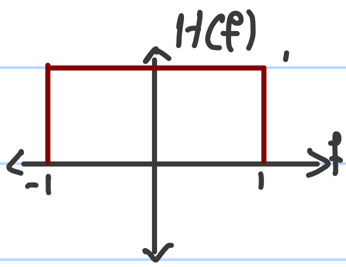
where $X(f) = \mathcal{F}[Sinc(t)] = \pi(f)$

and $H(f) = \mathcal{F}[Sinc(\frac{t}{2})] = \frac{1}{2}\pi(\frac{f}{2})$

$$Y(f) = X(f) * H(f)$$

=>

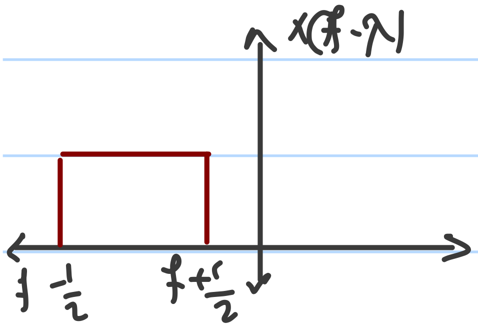




$[-\frac{1}{2}, \frac{1}{2}]$ $[-1, 1]$

$$= \left[-\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{3}{2}\right]$$

$$= \left[-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right]$$



for $f < -\frac{3}{2}$

$y(f) = 0$

for $-\frac{3}{2} < f < -\frac{1}{2}$

$$y(f) = \int_{-1}^{f+\frac{1}{2}} (1)\left(\tfrac{1}{2}\right) d\lambda$$

$$= \frac{1}{2}\left(f + \frac{3}{2}\right)$$

for $-\frac{1}{2} < f < \frac{1}{2}$

$$Y(f) = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} (1)\left(\tfrac{1}{2}\right) d\lambda$$

$$= \frac{1}{2}\left[\left(f + \frac{1}{2}\right) - \left(f - \frac{1}{2}\right)\right]$$

for $\frac{1}{2} < f < \frac{3}{2}$

$$Y(f) = \int_{f-\frac{1}{2}}^{1} (1)\left(\tfrac{1}{2}\right) d\lambda$$

$$Y(f) = \frac{1}{2}\left[1 - \left(f - \frac{1}{2}\right)\right]$$

for $f > \frac{3}{2}$

$$Y(f) = 0$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$ LTI

$$y(t) = x(t) * h(t)$$

if $x(t)$ is periodic signal

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$y(t) = \sum_{-\infty}^{\infty} X_n e^{j2\pi f_0 t} * h(t)$$

$$Y(f) = \sum_{n=\infty}^{\infty} X_n \delta(f - nf_0) \cdot H(f)$$

$$= \sum_{n=-\infty}^{\infty} X_n H(f_0) \delta(f - nf_0)$$

$$= \sum_{n=-\infty}^{\infty} |X_n| e^{j\theta_{x_n}} |H(nf_0)| e^{j\theta_{H(nf_0)}} \delta(f - nf_0)$$

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| |H(nf_0)| e^{j2\pi n f_0 t} \cdot e^{j\theta_{x_n}} \cdot e^{j\theta_{H(nf_0)}}$$

$$= \sum_{n=-\infty}^{\infty} |X_n| |H(nf_0)| e^{(j2\pi n f_0 t + j\theta_{x_n} + j\theta_{H(nf_0)})}$$

Ex. Consider a system with amplitude and phase responce function given by

$$|H(f)| = \begin{cases} K & , \ |f| \le B \\ 0 & , \ o.w \end{cases}$$

$= K\Pi\left(\frac{f}{2B}\right)$, $\angle\theta_{H(f)} = -2\pi f_c t$ and $x(t) = A\cos(2\pi f t + \theta)$, find $y(t)$

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n|\,|H(nf_0)|\, e^{j(2\pi f_0 t + \theta_{H(nf_0)} + \theta_{X_n})}$$

$$x(t) = \frac{A}{2} e^{j2\pi f t} \cdot e^{j\theta} + \frac{A}{2} e^{-j2\pi f t} \cdot e^{\theta i}$$

$X_1 = \frac{A}{2} e^{j\theta}$ $\qquad |X_n| = \frac{A}{2}$

$X_{-1} = \frac{A}{2} e^{-j\theta}$ $\qquad \angle\theta_{X_n} = \theta$

To evaluate the spectrum of the output $Y(f)$

$$Y(f) = \sum_{n=-\infty}^{\infty} \frac{A}{2} K\Pi\left(\frac{f}{2B}\right) e^{j\theta} e^{-j\theta} e^{-j2\pi f_0 t}$$

To evaluate the output $y(t)$

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{A}{2} K\Pi\left(\frac{f}{2B}\right) e^{j\theta} e^{-j2\pi f_0 t} e^{j2\pi f_0 t}$$

Ex. Consider the following LTI system

$$x(t) = 3\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + 5y(t)$$

a) Evaluate the spectrum of response of the system, "H(f)"

$$\mathfrak{F}\left[3\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + 5y(t) = x(t)\right]$$

$$= X(f) = 3(j2\pi f)^2 Y(t) + 2(j2\pi f)Y(f) + 5Y(f)$$

$$X(f) = Y(f)\left[-3(2\pi f)^2 + j4\pi f + 5\right]$$

Since $H(f) = \frac{Y(f)}{X(f)}$

$$H(f) = \frac{1}{5-3(2\pi f)^2 + j\,4\pi f}$$

b) Evaluate the magnitude and phase of H(f)

$$|H(f)| = \frac{1}{\sqrt{(5-3(2\pi f)^2)^2 + (4\pi f)^2}} \quad , \quad \angle\theta_{Hf} = -\tan^{-1}\left(\frac{4\pi f}{5-3(2\pi f)^2}\right)$$

c) Determine the type of distortion of the system

we have two distortion

① Amp since Amplitude not constant

② phase since Phase nonlinear

d) Evaluate the energy spectral density.

$$G(f) = |H(f)|^2 = \frac{1}{(5-3(2\pi f)^2)^2 + (4\pi f)^2}$$

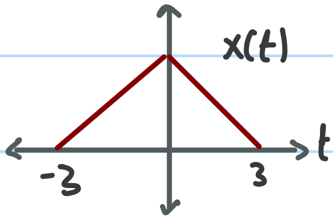
e) Evaluate the output of the system if the input $x(t) = 4\cos(20\pi t + \frac{\pi}{6})$

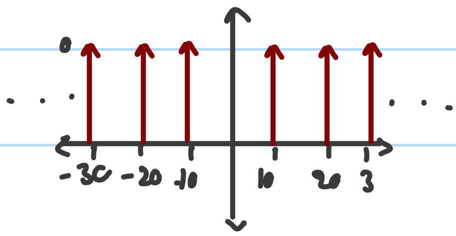
$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| \, |H(nf_0)| \, e^{j\theta x_n} e^{j\theta H(nf_0)} e^{j2\pi n f_0 t}$$

$$\sum_{n=\infty}^{\infty} \frac{4}{2} \frac{1}{\sqrt{(5-3(2\pi f)^2)^2 + (4\pi f)^2}} \cdot e^{\left(\frac{\pi}{6}\right)} \cdot e^{\left(\tan^{-1}\left(\frac{4\pi f}{5-3(2\pi f)^2}\right)\right)} e^{j20\pi t}$$

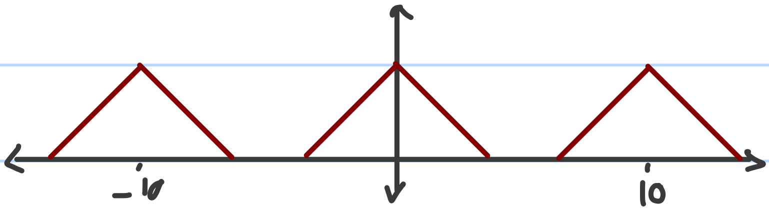
Ex. Consider LTI system where $x(t) = \Lambda\left(\frac{t}{3}\right)$ and $h(t) = \sum_{n=-\infty}^{\infty} \delta(t-10n)$

a) plot $x(t)$, $h(t)$ and $y(t)$





$y(t) \Rightarrow$



Since:- $y(t) = x(t) * h(t)$

$$= \Lambda\left(\frac{t}{3}\right) * \sum_{n=-\infty}^{\infty} \delta(t-10n)$$

$$= \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t-10n}{3}\right)$$

Remember $x(t) * \delta(t-3) = x(t-3)$

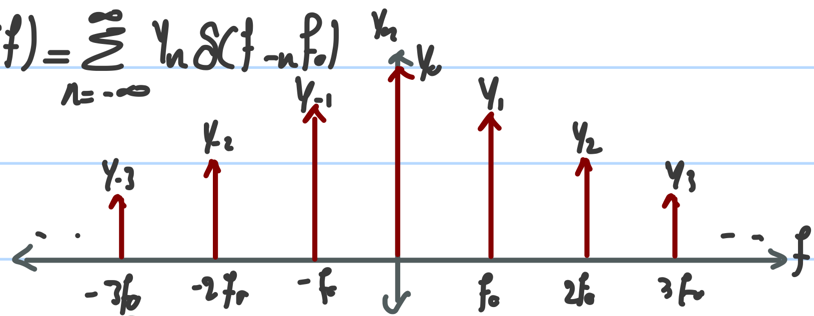
b) Evaluate and plot the spectrum of the output $y(t)$

$\because y(t)$ is periodic signal

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{j2\pi f_0 n(t)}$$

To evaluate Y_n:

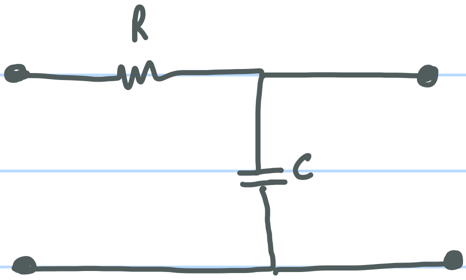
$$Y(f) = \sum_{n=-\infty}^{\infty} Y_n \delta(f-nf_0)$$



Filters

"Type of filters"

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

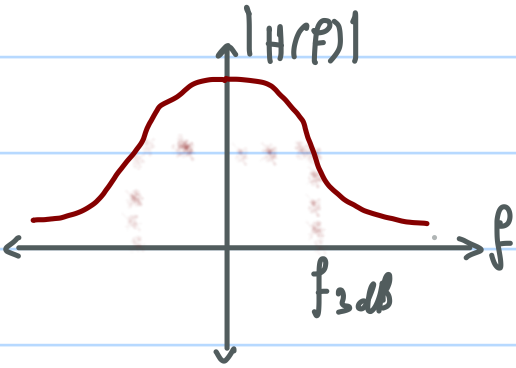


$$RC\frac{dy}{dt} + y(t) = x(t)$$

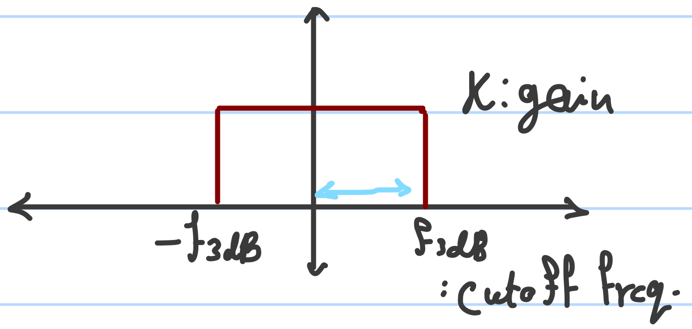
$\mathcal{F}$

$$RC(j2\pi f)Y(f) + Y(f) = X(f)$$

$$\frac{Y(f)}{X(f)} = H(f) = \frac{1}{1+j2\pi f RC}$$



1 "Low pass filters" (L.P.F)



Band width: $BW = f_{3dB}$ → في هاي الحالة

Ex. Consider the following signal $x(t) = \cos(20\pi t) \cdot 3\sin(40\pi t) + 6\cos(80\pi t)$ is applied to

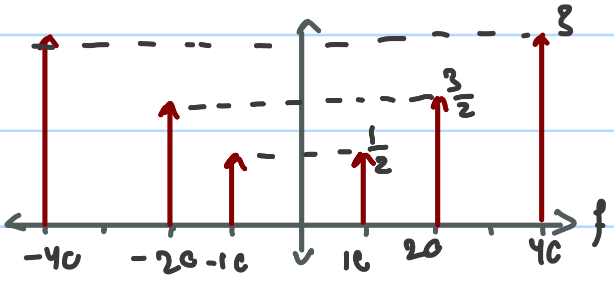
1) low pass filter with gain "2" and $BW = 15\,Hz$

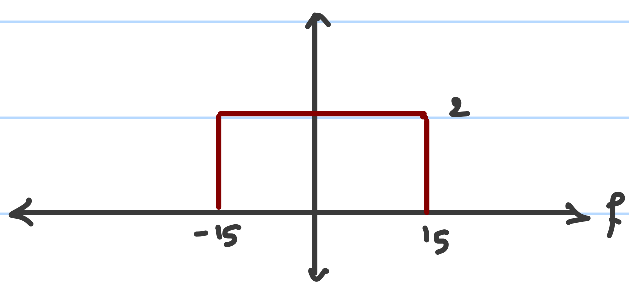
$x(t) \rightarrow \boxed{h(t)} \xrightarrow{y(t)}$, $\xrightarrow{X(f)} \boxed{H(f)} \rightarrow Y(f)$

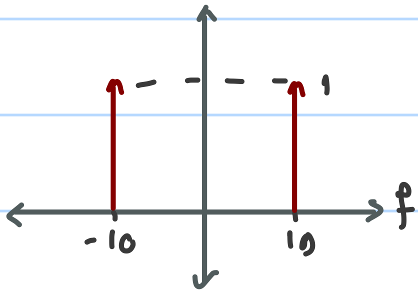
$y(t) = x(t) * h(t)$ $\qquad$ $Y(f) = X(f)H(f)$

$x(t) = \cos(20\pi t) \cdot 3\sin(40\pi t) + 6\cos(80\pi t)$

$f_1 = 10$ ($n=1$) $\quad f_2 = 20$ ($n=2$) $\quad f_3 = 40$ ($n=4$) $\quad GCD(10, 20, 40) \rightarrow f_0 = 10$





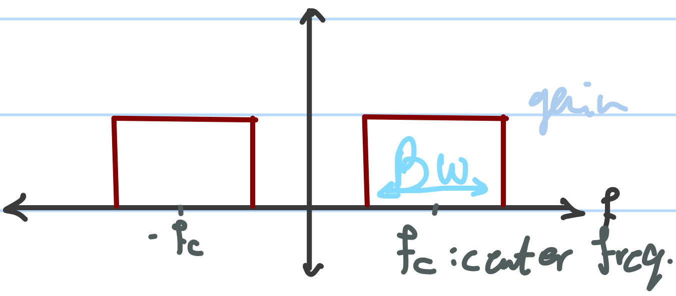


$$Y(f) = \delta(f+10) + \delta(f-10)$$

$$y(t) = e^{-j2\pi 10} + e^{j2\pi 10} \quad \cdot \frac{2}{2}$$

$$y(t) = 2\cos(20\pi t)$$

$\boxed{2}$ Band pass filter (B.P.F)

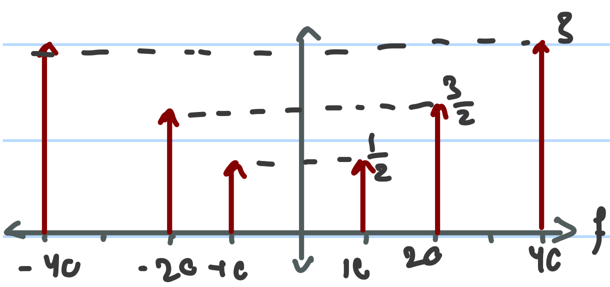


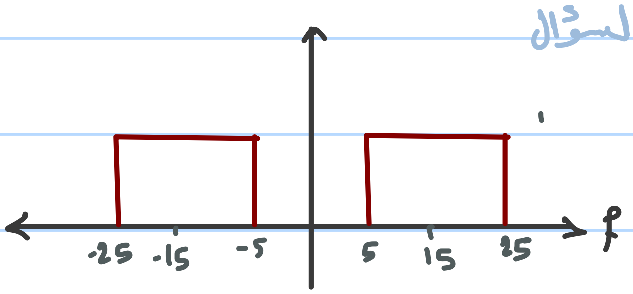
Ex. Consider the following signal $x(t) = \cos(20\pi t) - 3\sin(40\pi t) + 6\cos(80\pi t)$ is applied to

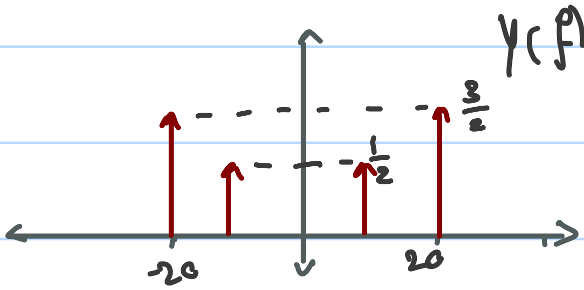
$\boxed{2}$ band pass filter center at 15 Hz and BW = 20 Hz

* قيمة ال gain = ١ في حال

ما ذكرها في السؤال



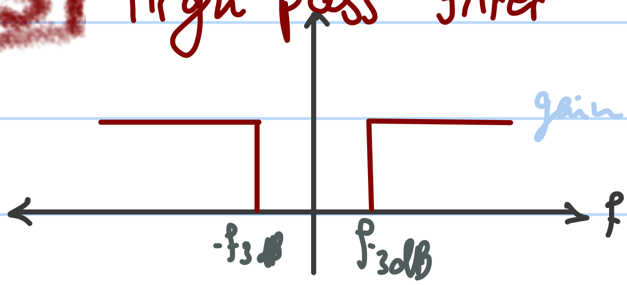




$$Y(f) = \frac{3}{2}\delta(f+20) + \frac{3}{2}\delta(f-20) + \frac{1}{2}\delta(f+10) + \frac{1}{2}\delta(f-10)$$

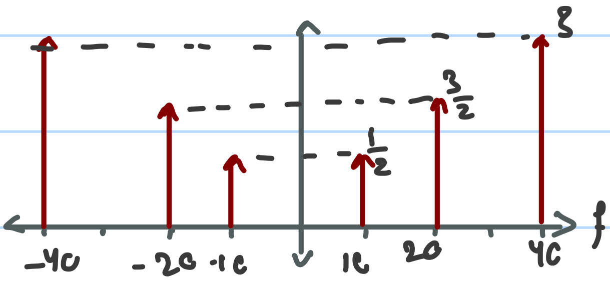
$$= 3\cos(40\pi t) + \cos(20\pi t)$$

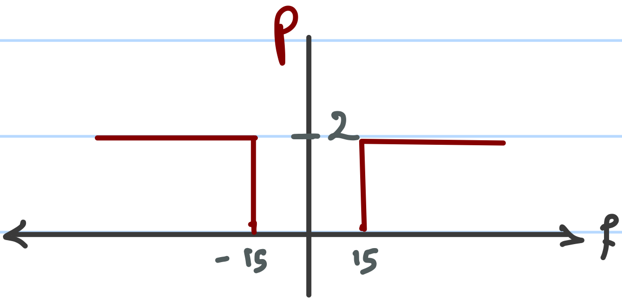
3 High pass filter

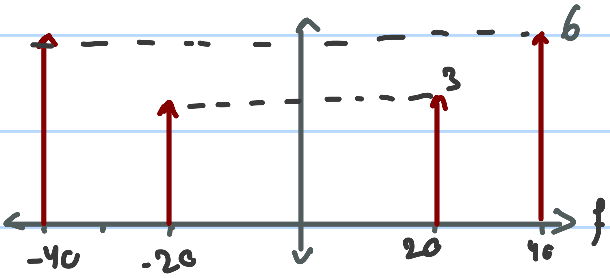


Ex. Consider the following signal $x(t) = \cos(20\pi t) \cdot 3\sin(40\pi t) + 6\cos(80\pi t)$ is applied to

$\boxed{3}$ high pass filter with cutoff freq. ($f_{3dB} = 15\ Hz$) and gain "2"





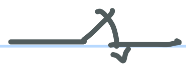


$$y(f) = 3\delta(f+20) + 3\delta(f-20) + 6\delta(f+40) + 6\delta(f-40)$$

$$y(t) = 6\cos(40\pi t) + 12\cos(80\pi t)$$

* Discrete Signals and Discrete system

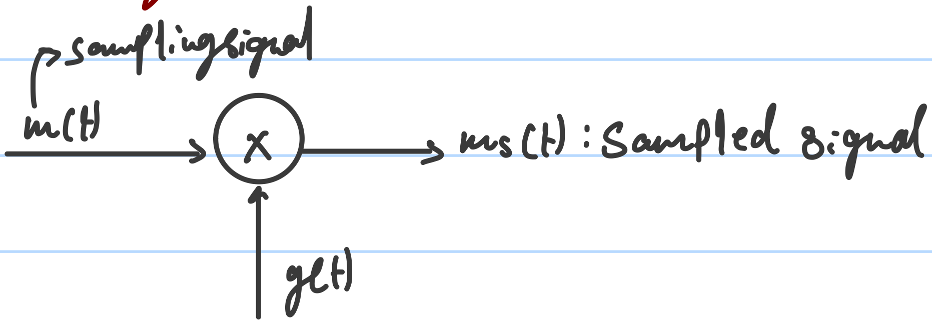
Cont. time
Cont. Amp

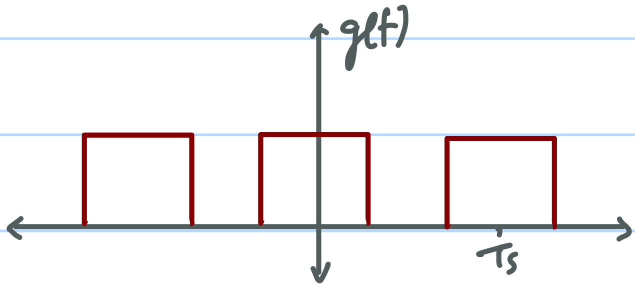


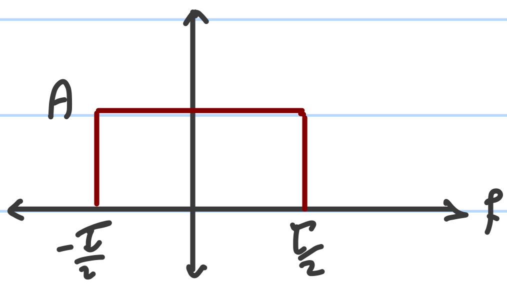
$t = nT_s$

Discrete Amp
cont. Time

Analog → Discrete







Evaluate g_n

$$g_n = f_s\, p(nf_s)$$

$$A\,\pi\left(\frac{t}{\tau}\right)$$

$$p(f) = A\tau \,\mathrm{sinc}(\tau f)$$

$$p(nf_s) = A\tau \,\mathrm{sinc}(\tau n f_s)$$

$$m_s(t) = m(t)\cdot g(t)$$

$$= m(t) \sum_{n=-\infty}^{\infty} g_n\, e^{j2\pi n f_s t}$$

$$M_s(f) = M(f) * \sum_{n=-\infty}^{\infty} g_n \delta(f - nf_s)$$
$$= \sum_{n=-\infty}^{\infty} g_n M(f - nf_s)$$

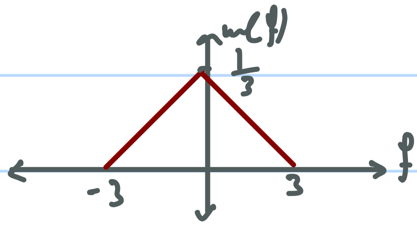
Since $g_n = f_s P(nf_s) = A\tau \operatorname{sinc}(\tau n f_s)$

$$M_s(f) = \sum_{n=-\infty}^{\infty} A\tau f_s \operatorname{sinc}(\tau n f_s) M(f - f_0)$$

let $m(t) = \operatorname{sinc}^2(3t)$

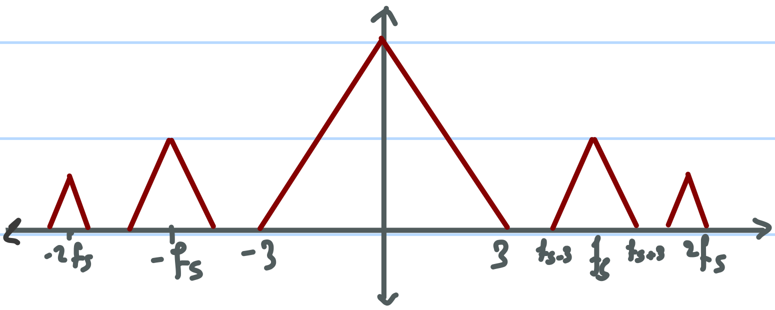
a) Evaluate and plot the fourier transform of $m(t)$

$$m(f) = \frac{1}{3} \Lambda\left(\frac{f}{3}\right)$$



b) Evaluate and plot the spectrum of sampled signal

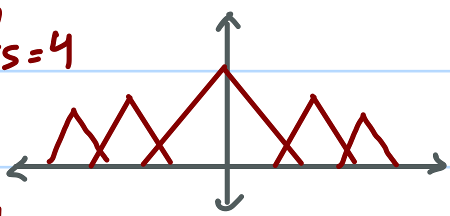
$$M_s(f) = \sum_{n=-\infty}^{\infty} A\tau \operatorname{sinc}(\tau n f_s) \cdot \frac{1}{3} \Lambda\left(\frac{f - f_s}{3}\right)$$



$f_s - f_m \geq f_m$

$f_s \geq 2f_m$ → يعني اني بقدر ارجع الاشارة لأصلها

if $f_s = 4$



aliasing.