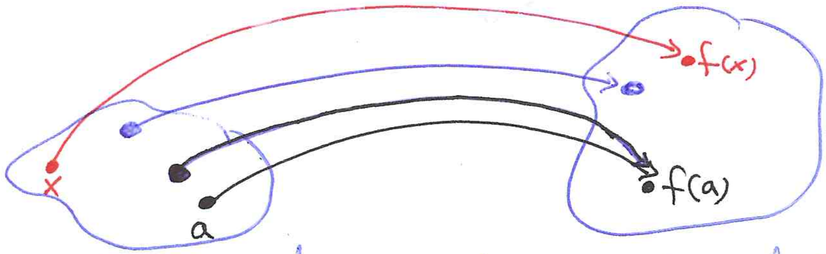
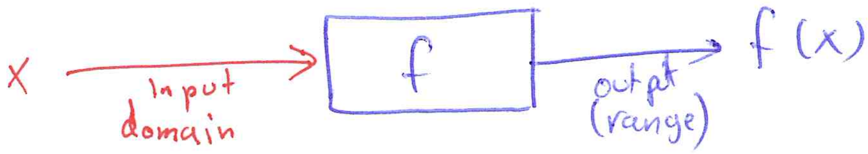
1.1 Functions and Their Graphs

Def: A function f from a set D to a set Y is a rule that assignes a unique (single) element $f(x) \in Y$ to each element $x \in D$.



D = domain set: the largest set of real x-values that gives real y-values.

Y = set contains the range



$y = f(x)$

x is independent variable
y is dependent variable

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* A function whose range is a set of real numbers is called real-valued function.

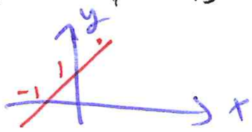
Example: Find the natural domain and range of the following functions

1. $y = x^2$ $\quad D = (-\infty, \infty) \quad R = [0, \infty)$
2. $y = \frac{1}{x}$ $\quad D = (-\infty, 0) \cup (0, \infty) = \mathbb{R}$
3. $y = \sqrt{x}$ $\quad D = [0, \infty)$ i.e $x \geq 0$ $\quad R = [0, \infty)$
4. $y = \sqrt{4-x}$ $\quad D = (-\infty, 4]$ i.e $4 - x \geq 0$ $\quad R = [0, \infty)$
5. $y = \sqrt{1-x^2}$ $\quad D = [-1, 1]$ i.e $1 - x^2 \geq 0$ $\quad R = [0, 1]$
   $1 \geq x^2$
   $1 \geq |x|$

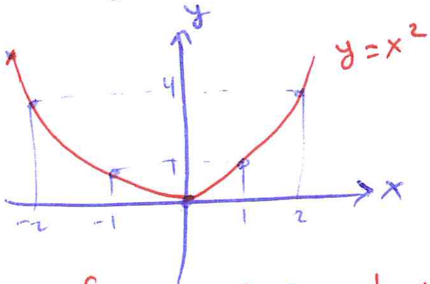
Graphs of functions

The graph of a function f whose domain is D, consistes of the points in the Cartesian plane whose coordinates are the (input, output) pairs for f i.e: $\{(x, f(x)) \mid x \in D\}$

Example: The graph of $f(x) = x+1$ is the set of points with coordinates $(x, x+1)$

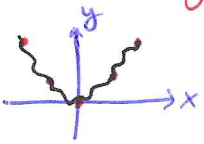


Example: Graph the function $y = x^2$ over the interval $[-2, 2]$

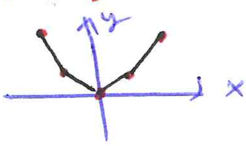


x	$y = x^2$
-2	4
-1	1
0	0
1	1
2	4

Question: why the graph of $y = x^2$ is not like

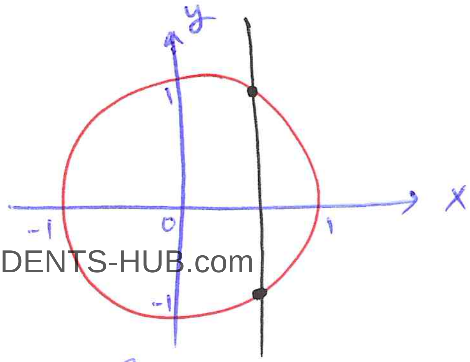


or



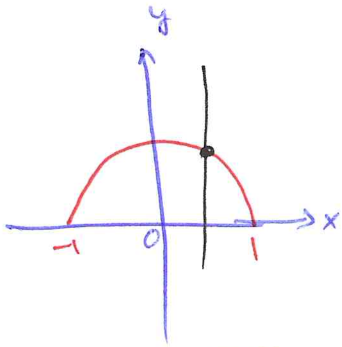
we will learn derivatives in Ch 3

Vertical line Test for a function



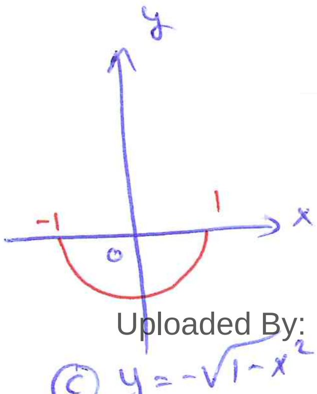
ⓐ $x^2 + y^2 = 1$

The circle is not a graph of a function. It fails the vertical line test.



ⓑ $y = \sqrt{1-x^2}$

The upper semicircle is the graph of a function

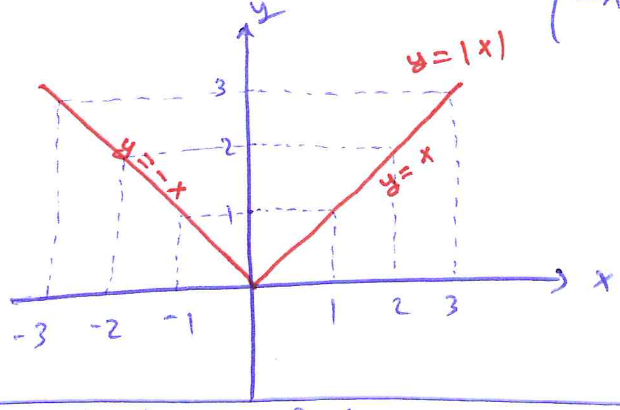


ⓒ $y = -\sqrt{1-x^2}$

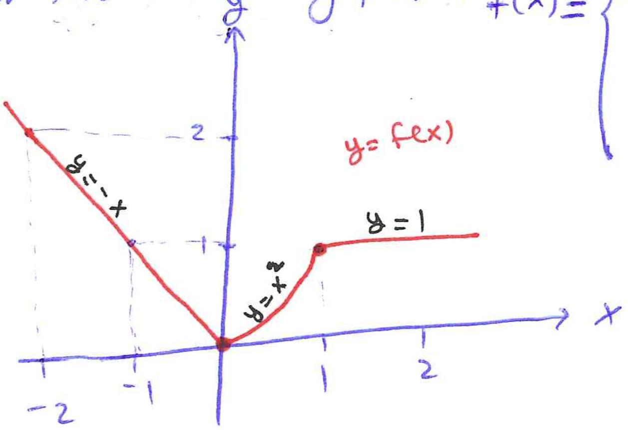
The lower semicircle is the graph of a function

Piecewise Defined Functions

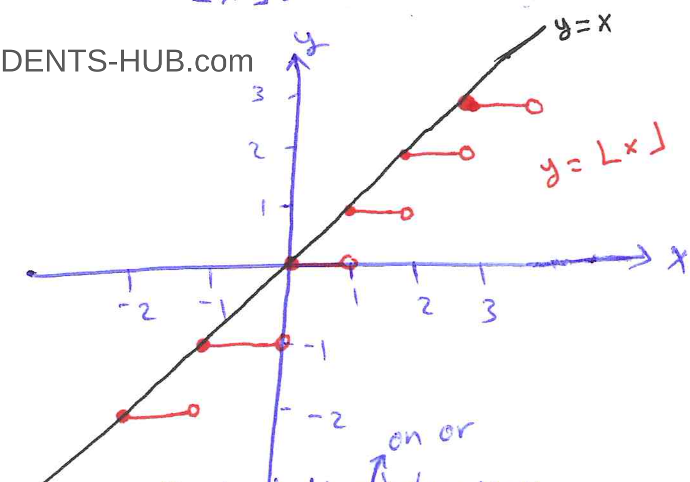
Examples [1] Absolute value function $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



[2] Consider the following function $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$



[3] The **greatest integer function**: is a function whose value at any number x is the greatest integer less than or equal to x. It is also called **the integer floor function**. It is denoted by $\lfloor x \rfloor$.



$y = \lfloor x \rfloor$ lies on or below $y = x$

$\lfloor 1.5 \rfloor = 1$

$\lfloor 1.9 \rfloor = 1$

$\lfloor 1.3 \rfloor = 1$

$\lfloor 0.2 \rfloor = 0$

$\lfloor 0 \rfloor = 0$

$\lfloor -1.2 \rfloor = -2$

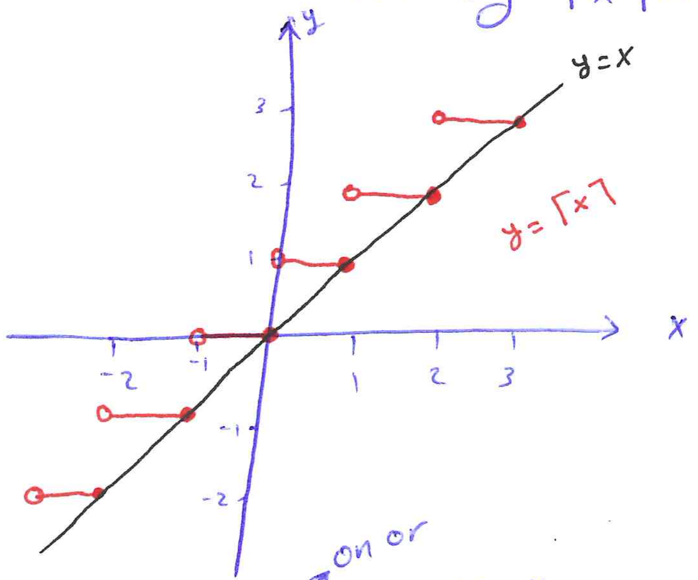
$\lfloor -0.3 \rfloor = -1$

$\lfloor 5 \rfloor = 5$

4 The least integer function : is a function whose value at any number x is the smallest integer greater than or equal to x.

$\Rightarrow$ It is also called the integer ceiling function.

$\Rightarrow$ It is denoted by $\lceil x \rceil$.



$\lceil 1.5 \rceil = 2$

$\lceil 1.9 \rceil = 2$

$\lceil 1.3 \rceil = 2$

$\lceil 0.2 \rceil = 1$

$\lceil 0 \rceil = 0$

$\lceil -1.2 \rceil = -1$

$\lceil -0.3 \rceil = 0$

$\lceil 5 \rceil = 5$

$y = \lceil x \rceil$ lies on or above $y = x$

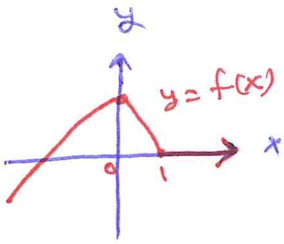
Increasing and decreasing functions

Def: Let f be a function defined on an interval I. Let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is an increasing on I

2. If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is a decreasing on I.

Example: The function y is an increasing on $(-\infty, 0]$ and decreasing on $[0, 1]$. The function is neither increasing nor decreasing on $[1, \infty)$.



Even and Odd Functions

* A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the domain of f

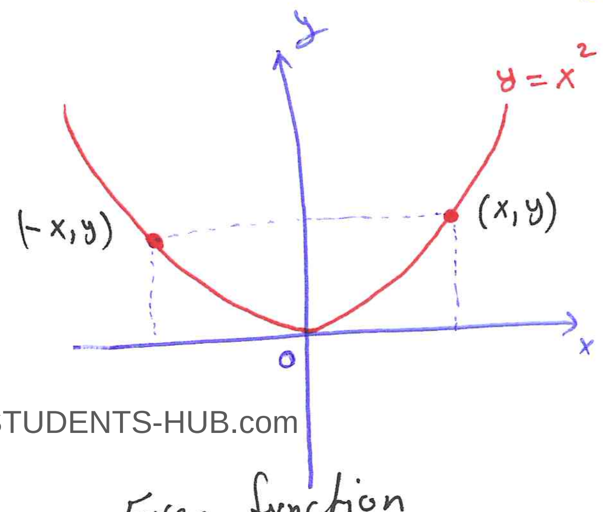
* A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the domain of f.

Example : $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$

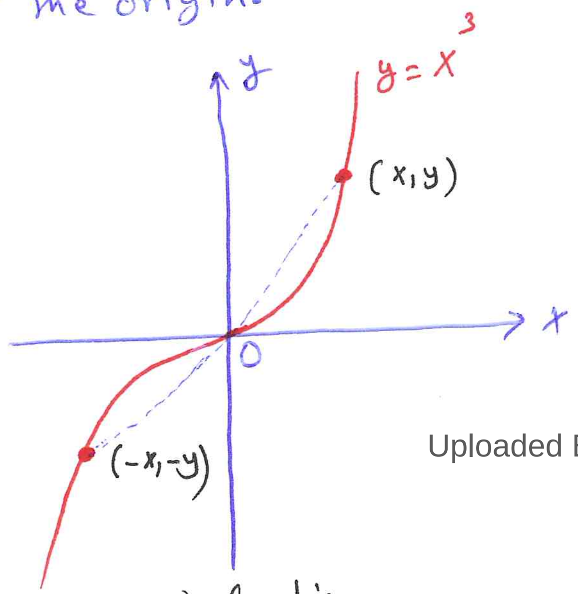
$f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$

Note that * the graph of an even function is symmetric about the y-axis

* the graph of an odd function is symmetric about the origin.



Even function symmetric about y-axis



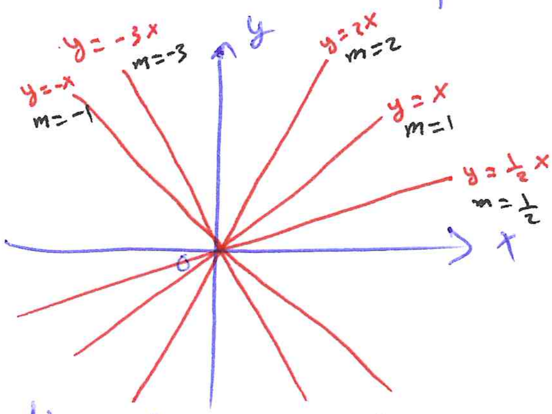
Odd function symmetric about the origin

Common Functions

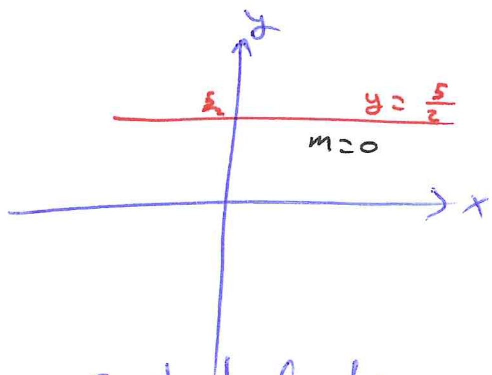
1 linear function : $f(x) = mx + b$ m, b are constants

slop (m)   y-intercept (b)

- when $m=1$ and $b=0 \Rightarrow f(x)=x$ is called the idenity function
- when $m=0 \Rightarrow f(x)=b$ is called constant function



lines through the origin



Constant fnction

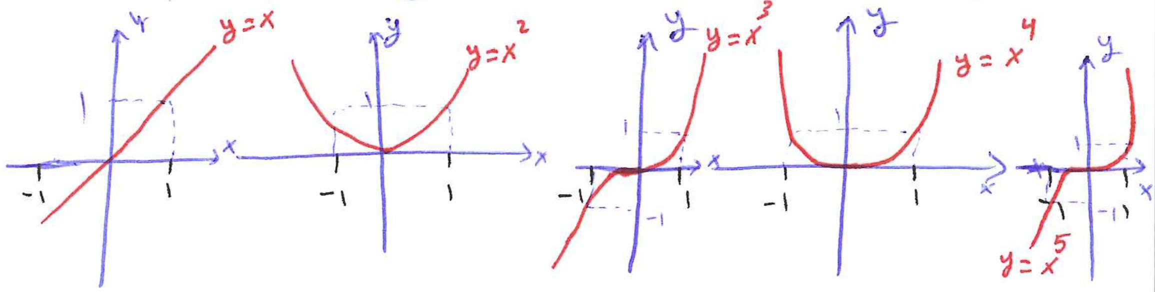
- A linear function with positive slope and passes through origin is called proportionality relationship.

Def : Two variables y and x are proportional (one to another) if one is always a constant multiple of the other, i.e : $y = kx$ for some nonzero constant k

If the variable is proportional to the reciprocal $\frac{1}{x}$, then y is inversely proportional to x.

2 Power functions $f(x) = x^a$ a is constant

a a is positive integer $f(x) = x^n$, $n = 1, 2, 3, 4, 5$



- as the power $n \uparrow$, the curves get more flat toward x-axis on the interval $(-1, 1)$ and more steeply for $|x| > 1$
- all curves passes through $(1,1)$ and origin.
- functions with even power are symmetric about y-axis
- functions with odd power are symmetric about the origin
- Even functions are $\downarrow$ on the interval $(-\infty, 0]$ and $\uparrow$ on $[0, \infty)$
- Odd functions are $\uparrow$ over the entire real line $(-\infty, \infty)$

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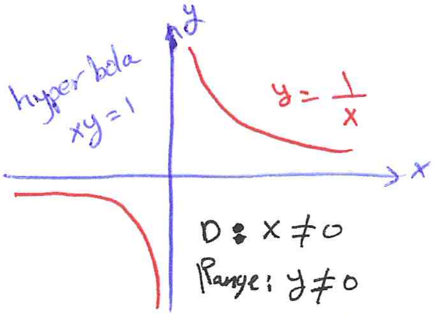
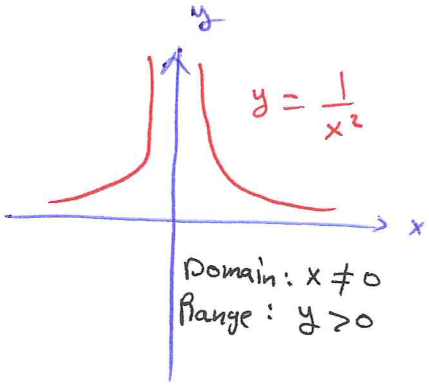
b $a = -1$ or $a = -2$





a $f(x) = x^{-1} = \frac{1}{x}$

$f \downarrow$ on $(-\infty, 0)$ and $(0, \infty)$

f is symmetric about origin

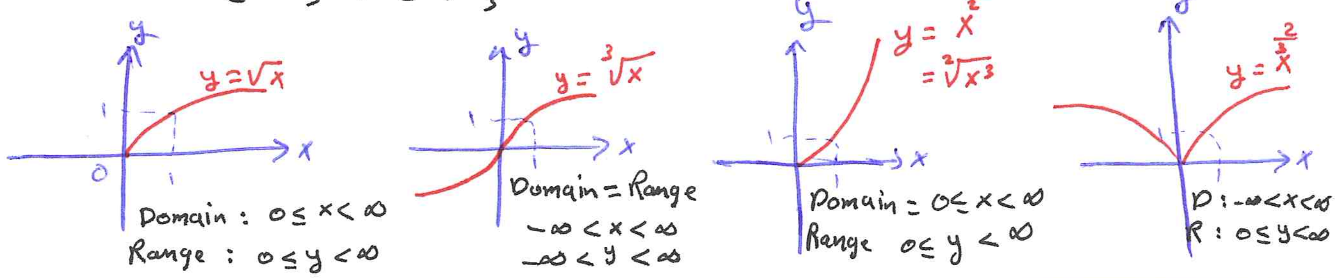
b $g(x) = x^{-2} = \frac{1}{x^2}$

$g \uparrow$ on $(-\infty, 0)$

$g \downarrow$ on $(0, \infty)$

g is symmetric about y-axis

[c] $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



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[3] **Polynomials :** A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real constants called the coefficients of p.

* All polynomials have Domain $(-\infty, \infty)$
* If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the degree of the polynomial p.

- Linear functions $p(x) = mx + b$ with $m \neq 0$ are polynomials of degree 1.
- Quadratic functions $p(x) = ax^2 + bx + c$ with $a \neq 0$ are polynomials of degree 2.
- Cubic functions $p(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ are polynomials of degree 3. ...

⋮

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[4] **Rational functions:** are a quotient or ratio

$$f(x) = \frac{p(x)}{q(x)}$$, where p, q are polynomials

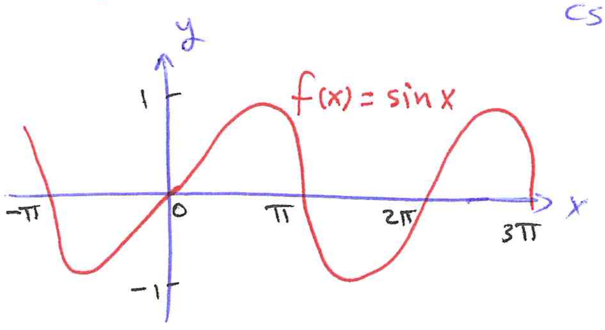
Example: $f(x) = \frac{x^2 - 3}{2x + 1}$

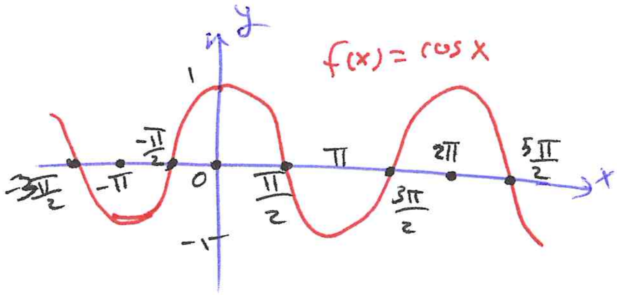
[5] **Algebraic functions:** are any function constructed from polynomials using algebraic operation ($+, -, \times, \div$ and taking roots)

Example [1] All rational functions are algebraic.

[2] $y = x^2 (1 - x)^{\frac{3}{4}}$

6 Trigonometric functions: sin, cos, tan, csc, sec, cot (section 1.3)

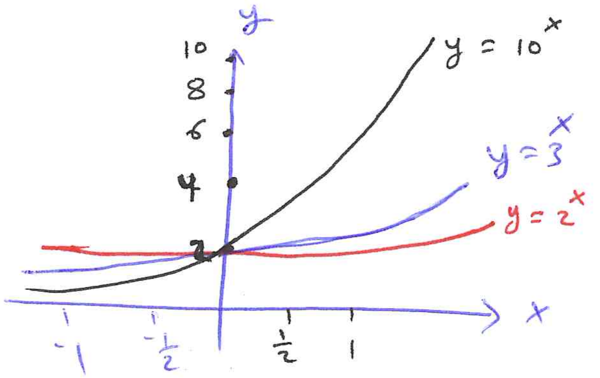


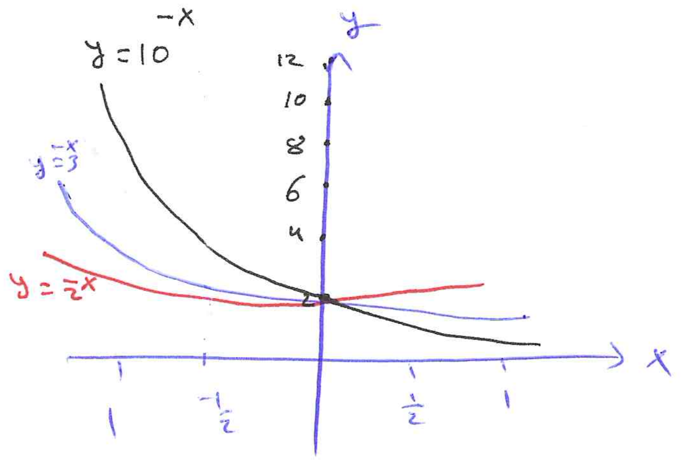


7 Exponential functions

$f(x) = a^x$, $a > 0$ and $a \neq 1$

Domain : $(-\infty, \infty)$
Range : $(0, \infty)$
} always





8 logarithmic functions

$f(x) = \log_a x$ the base $a \neq 1$ and $a > 0$

They are the inverse functions of the exponential functions.

Domain: $(0, \infty)$
Range : $(-\infty, \infty)$
} always