exercises :

Q28 : Show that the Random variable X_1 and X_2 with joint p.d.f $f(x_1, x_2) = \begin{cases} 12 x_1 x_2 (1 - x_2) & , 0 < x_1 < 1 , 0 < x_2 < 1 \\ 0 & , \text{elsewhere} . \end{cases}$

are independent .

$$\Rightarrow f_1(x_1) = \int_0^1 12 x_1 x_2 (1 - x_2) \, dx_2 = \int_0^1 12 x_1 x_2 - 12 x_1 x_2^2 \, dx_2$$

$$= \frac{12 x_1 x_2^2}{2} - \frac{12 x_1 x_2^3}{3} \Big|_0^1$$

$$= 6 x_1 - 4 x_1 = \underline{2 x_1}$$

$$\Rightarrow f_2(x_2) = \int_0^1 (12 x_1 x_2 - 12 x_1 x_2^2) \, dx_1 = \frac{12 x_1^2 x_2}{2} - \frac{12 x_1^2 x_2^2}{2} \Big|_0^1$$

$$= 6 x_2 - 6 x_2^2 = 6 x_2 (1 - x_2) .$$

$$\Rightarrow f_1(x_1) . f_2(x_2) = \begin{cases} 2 x_1 (6 x_2 - 6 x_2^2) = 6 x_1 x_2 (1 - x_2) & , 0 < x_1 < 1 , 0 < x_2 < 1 . \\ 0 & , \text{elsewhere} . \end{cases}$$

$\Rightarrow f(x_1, x_2)$ independent

Q29: If the Random variables X_1 and X_2 have the joint p.d.f

$$f(X_1, X_2) = \begin{cases} 2e^{-X_1 - X_2} & , \ 0 < X_1 < X_2 \ , \ 0 < X_2 < \infty \\ 0 & , \ \text{elsewhere} \end{cases}$$

show that X_1 and X_2 are dependent.

$$\rightarrow f_1(X_1) = \int_{X_1}^{\infty} 2e^{-X_1 - X_2} \, dX_2 = -2e^{-X_1 - X_2} \Big|_{X_1}^{\infty}$$

$$= \lim_{b \to \infty} \cancel{-2e^{-X_1 - b}}^{\,0} - \left(-2e^{-X_1 - X_1}\right)$$

$$= 2e^{-2X_1}$$

$$\rightarrow f_2(X_2) = \int_{0}^{X_2} 2e^{-X_1 - X_2} \, dX_1 = -2e^{-X_1 - X_2} \Big|_{0}^{X_2} = -2e^{-2X_2}.$$

$$\rightarrow f_1(X_1) \, f_2(X_2) = -4e^{-2X_1 - 2X_2} \neq f(X_1, X_2) \Rightarrow \text{dependent}.$$

Q30: let

$$f(X_1, X_2) = \begin{cases} \frac{1}{16} & , \ X_1 = 1, 2, 3, 4 \ \text{and} \ X_2 = 1, 2, 3, 4. \\ 0 & , \ \text{elsewhere}. \end{cases}$$

be the joint p.d.f of X_1 and X_2, show X_1, X_2 are independent.

$$f_1(X_1) = \sum_{X=1}^{X=4} \frac{1}{16} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$$

$$f_2(X_2) = \sum_{X=1}^{X=4} \frac{1}{16} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$$

$$\rightarrow f_1(X_1) \, f_2(X_2) = \frac{1}{16} \rightarrow f(X_1, X_2) = f_1(X_1) \, f_2(X_2) \Rightarrow \text{indep.}$$

Q31: Find $Pr(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3})$ if the random variable X_1 and X_2 have the joint p.d.f. $f(x_1, x_2) = \begin{cases} 4x_1(1-x_2) & , 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$$Pr(0 < X_1 < \tfrac{1}{3}, 0 < X_2 < \tfrac{1}{3}) = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} (4x_1 - 4x_1x_2)\, dx_1\, dx_2$$

$$= \int_0^{\frac{1}{3}} 2x_1^2 - 2x_1^2 x_2 \Big|_0^{\frac{1}{3}} dx_2$$

$$= \int_0^{\frac{1}{3}} \frac{2}{9} - \frac{2}{9} x_2 \; dx_2$$

$$= \frac{2}{9} x_2 - \frac{\cancel{2}}{9} \frac{x_2^2}{\cancel{2}} \Big|_0^{\frac{1}{3}}$$

$$= \frac{2}{9}\left(\frac{1}{3}\right) - \frac{1}{9} \cdot \frac{1}{9}$$

$$= \frac{3 \times 2}{3 \times 27} - \frac{1}{81} = \frac{5}{81}$$

Q32: Find the probability of the union of the events $a < X_1 < b, -\infty < X_2 < \infty$ and $-\infty < X_1 < \infty$, $c < X_2 < d$, if X_1 and X_2 are two indep. variables with $Pr(a < X_1 < b) = \frac{2}{3}$ and $Pr(c < X_2 < d) = \frac{5}{8}$.

$$Pr[(a < x_1 < b, -\infty < X_2 < \infty) \cup (-\infty < X_1 < \infty, c < X_2 < d)]$$

$$= Pr[(a < X_1 < b) \cup (c < X_2 < d)]$$

$$= Pr(a < X_1 < b) + Pr(c < X_2 < d) - Pr((a < X_1 < b) \cap (c < X_2 < d))$$ since indep

$$= Pr(a < X_1 < b) + Pr(c < X_2 < d) - Pr(a < X_1 < b)\, Pr(c < X_2 < d)$$

$$= \frac{2}{3} + \frac{5}{8} - \left(\frac{2}{3}\right)\left(\frac{5}{8}\right)$$

$$= \frac{7}{8}$$