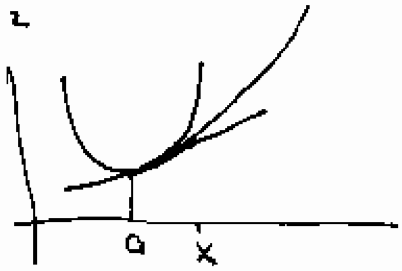
Tayler Theorem :-

$$F(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n + \cdots$$



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) \simeq f(a) + f'(a)(x-a)$ linear estimation.

$$\text{Error} = \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\text{Error} = \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

in general

$$f(x) \approx f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\text{Error} = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots \quad \text{(infinite Terms).}$$

Taylor :-

$$\text{Error} = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \quad c \text{ between } a, x$$

$$|\text{Error}| \leq \max_{\leq x \leq} \frac{|f^{(n+1)}(x)|\,(x-a)^{n+1}}{(n+1)!}$$

صار لـ Error حد أعلى
لكن المشكلة كيف نجد c

$$\Rightarrow \boxed{F(x) \cong \overbrace{f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}}^{P_n(x)} + \overbrace{\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}}^{E_n(x) \text{ Error}}}$$

- e^x, $a=0$

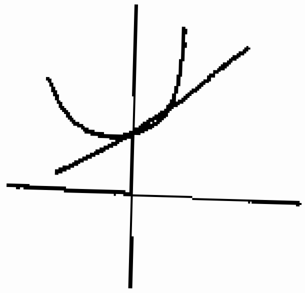
$$e^x = f(0) + f'(0)(x-0) + \frac{f''(c)}{2!}(x-0)^2$$

$$e^x = 1 + x + \frac{e^c x^2}{2!}$$

$e^x \approx 1+x$ with error $\frac{e^c x^2}{2!}$

$a^x = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(c)(x-0)^3}{3!}$

$e^x \approx 1 + x + \frac{x^2}{2}$ error $= \frac{e^c x^3}{6}$



$e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2}$ error $\frac{e^c (0.001)}{6} < 1 \times 10^{-3}$

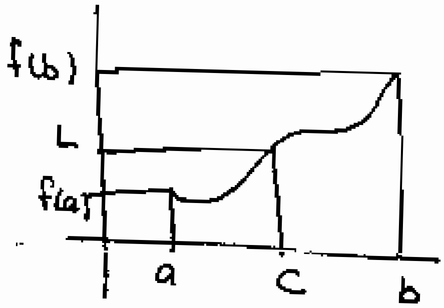
≈ 1.105

$c \in [0, 0.1]$

upper bound for error $\frac{e^c (0.001)}{6} \leqslant \frac{\overset{e^1}{3}(0.001)}{6} \leqslant 0.0005$.

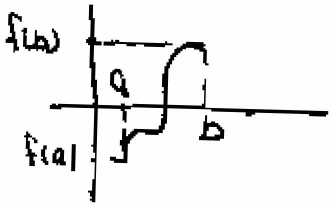
intermediate Valid Theorem (IVT)

- $f(x)$ is continous
- L between $f(a)$ and $f(b)$
- Then $\exists\, c \in (a,b)$ such that $f(c) = L$



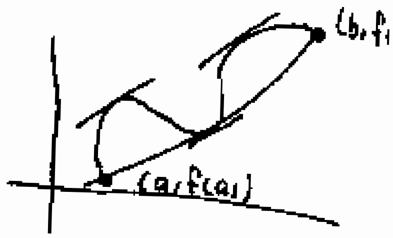
belzano

- $f(x)$ is continous
- $f(a) = f(b) < 0$
- Then $\exists\, c \in (a,b)$ such that $f(c) = 0$



mean value theorem (MVT)

- $f(x)$ is continous on $[a,b]$
- $f(x)$ is differentiable on (a,b)
- then $\exists\, c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



Section 1.3

Error analysis

Def:- suppose that $\hat{p}$ is an approximation to p

the error is $E_p = p - \hat{p}$

the relative error $R_p = \frac{E_p}{p} = \frac{p - \hat{p}}{p}$

ex:- 1. let $x = 3.141592$

$\hat{x} = 3.14$ ٣منازل متأكدين منهم

$Ex = 3.141592 - 3.14 = 0.001592$

$Rx = \frac{0.001592}{3.141592} = 0.000507$.

2. let $y = 1,000,000$

$\hat{y} = 999,996$ ٥ منازل متأكدين منهم الخطأ في ٦

$Ey = 4$

$Ry = \frac{4}{1,000,000} = 4 * 10^{-6}$

3. let $z = 0.000,012$

$\hat{z} = 0.000,009$ مش متأكدينه ولا من اي منزلة

$Ez = 0.000,003$

$Rz = 0.25$

normalized decimal Form:-

$\pm 0.d_1 d_2 d_3 \ldots \times 10^n$

$d_1 \neq 0$

- $x^2 = 2$

$x^2 - 2 = 0$ $\quad$ - + + + $\quad$ 1 1.25 1.5 2 $\quad$ حسب بلزانو

$C_0 = \frac{1+2}{2} = 1.5$

$C_1 = \frac{1+1.5}{2} = 1.25$

$C_2 = \frac{1+1.25}{2} = 1.125 = 0.1125 * 10^1$

2 signifigant digits $\Rightarrow$ Error $\leq 10^{-2}$

بعد اول منزلة غير صفرية

Def:- the number $\hat{p}$ is said to approximate p to d significant digits if d is the largest positive integer for which

$$\frac{|p-\hat{p}|}{|p|} < \frac{10^{-d}}{2}$$

i,e $2|R_{re}| \lessapprox 10^{-d}$

ex:-

1. $x = 3.141592$
   $x^{n} = 3.14$

$$E_x = 3.141592 - 3.14 = 0.001592$$

$$R_x = \frac{0.001592}{3.141592} = 0.000507$$

$$2|R_x| = 0.001014 \approx 10^{-3}$$
$$< 10^{-4}$$

المثال السابق 2. $2|R_y| = 8\times10^{-6} < 10^{-3}$

10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

3. $2|R_z| = 0.5 \not< 10^{-1}$
   no significant bits.

- if $P = \pm 0.d_1d_2 \cdots d_nd_{n+1} \cdots \times 10^{n}$ is the normalized decimal form of the number P, $d_1 \neq 0$, then the k^{th} digit chopped floating point representation of p is

$$fl_{chop}(P) = \pm 0.d_1d_2 \cdots d_k \times 10^{n}$$

the k^{th} digit round off Floating point representation of p is

$$fl_{round}(P) = \pm 0.d_1d_2 \cdots d_{k-1}r_k \times 10^{n}$$

where r_k is obtained by rounding $d_k, d_{k+1}, d_{k+2} \cdots$

- $P = 0.1234|444445$

4 digits chopped

$fl_{chop}(P) = 0.1234$

$fl(p)_{round} = 0.1235$

<u>Final</u> مهم

- use 4 digits arthmatic (round) ٤منازل بعداول منزلة غيرصفرية

$$\frac{\frac{3}{7}+\frac{5}{8}+\left(\frac{11}{15}\right)}{21} = ?? \quad \text{or} \quad \frac{\frac{3}{7}+0.5967+\frac{11}{15}}{21} = ??$$

$$\frac{(0.4286+0.5967)+0.7333}{21}$$

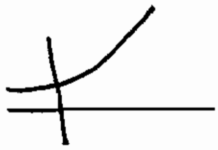
$0.4286+0.5967 = 1.0253 \approx 1.025$

$1.025 + 0.7333 = 1.7583 \approx 1.758$

$\frac{1.758}{21} = 0.08371$

- <u>order of estimation</u>

$e^x \cong 1+x$



$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

- $e^x \approx 1+x$
- $e^h \approx 1+h \qquad h \approx 0$

  $\text{Error} = \frac{h^2}{2!} \approx O(h^2)$ order of approximation.

  $e^{0.1} \approx 1+0.1 \approx 1.1 \qquad \text{error} = \underset{\text{Const}}{C}h^2$

  $e^{0.1} = 1.105170918 \qquad = C(0.1)^2$

  $= C(0.01)$

  $\leq 10^{-2}$

- $e^h = 1+h+\frac{h^2}{2!}$

  $\text{Error} = Ch^3 = O(h^3)$

  $e^{0.1} = 1+0.1+\frac{0.01}{2}$

  $= 1.105$

  $\text{Error} \approx C(0.1)^3$

  $\approx C(0.001) \leq 10^{-3}$

$\sin(0.1) \approx 0.1$

- $\sin h \approx h$ with error $O(h^3)$

  $\sin h \approx h - \frac{h^3}{3!}$ with error $O(h^5)$

suppose $e^h \simeq 1+h$ Error $= O(h^2)$ (0.01)

$\sin h = h - \frac{h^3}{3!}$ Error $= O(h^5)$ (0.00001)

$e^h + \sin h \approx 1 + 2h - \frac{h^3}{3!}$ with Error $O(h^2) + \cancel{O(h^5)}$ لاتؤثر

الاصغر هي التي تؤثر

$\approx 1 + 2h + O(h^2)$

def:- order of approximation

assum that $f(h)$ is approximated by $p(h)$ and there exists a real constant $M \geq 0$ and a positive integer n so that

$$\frac{|f(h) - P(h)|}{|h^n|} \leq M \quad \text{for small } h$$

we say $P(h)$ approximate $f(h)$ with order of approximation $O(h^n)$ and we write $f(h) = P(h) + O(h^n)$

$$|f(h) - P(h)| \leq M|h^n|$$
$$f(h) - P(h) \approx C h^n$$

Ex:- Show that $p(h) = 1+h$ estimate of $f(h) = e^h$ with order $O(h^2)$

or

show that $e^h = 1 + h + O(h^2)$

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$\frac{|e^h - (1+h)|}{|h^2|} = \frac{\frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots}{h^2} = \frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!} + \frac{h^3}{5!} + \cdots$$

harmonic series $\left(\sum \frac{1}{n}\right)$ divargens

$< \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$

$< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \rightarrow$ geomatric series $= \frac{1/2}{1 - 1/2} = 1$

$\leq$

$$e^h = 1 + h + O(h^2)$$

exercise

show that

1 - $\sin h = h - \frac{h^3}{3!} + O(h^5)$

2 - $f(h) = \sum_{k=0}^{n} f^{k}(h)\, x^k + O(h^{n+1})$

Theory:-

assume that $f(h) = p(h) + O(h^n)$

$g(h) = q(h) + O(h^m)$

and $r = \min[m, n]$

then

$f(h) \pm g(h) = p(h) + q(h) + O(h^r)$

$f(h) \cdot g(h) = p(h)\, q(h) + O(h^r)$

$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + O(h^r) \qquad q(h), g(h) \neq 0.$

Ex:-

$f(h) = p(h) + O(h^3)$

$g(h) = q(h) + O(h^2)$

$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + O(h^2)$

Ex:- (loss of significant)

$f(x) = x(\sqrt{x+1} - \sqrt{x})$

$g(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}}$

use 6 digits arthmatic and round to find $f(500)$, $g(500)$

$$f(500) = 500(\sqrt{501} - \sqrt{500})$$
$$= 500(22.3830 - 22.3607)$$
$$= 500(0.0223200) = 11.1500.$$

$$g(500) = \frac{500}{\sqrt{501}+\sqrt{500}} = \frac{500}{22.3830 + 22.3607} = 11.1748$$

exact answer $= 11.174755\ldots$

الطريقة الثانية أحسن لأن في الحالة الأولى
عملية الطرح خسرتنا significant digits

$$\frac{3}{17} = 0.176470588 + \epsilon$$

Note:-

$$P = \tilde{P} + \epsilon_P$$
$$g = \tilde{g} + \epsilon_g$$
$$P + g = \tilde{P} + \tilde{g} + \epsilon_P + \epsilon_g$$
$$= \tilde{P} + \tilde{g} + \epsilon_{P+g}$$
$$P.g = (\tilde{P} + \epsilon_P)(\tilde{g} + \epsilon_g)$$
$$= \tilde{P}\tilde{g} + \tilde{P}\epsilon_g + \tilde{g}\epsilon_P + \epsilon_P\epsilon_g$$
$$= \tilde{P}\tilde{g} + \epsilon_{Pg}$$

- $P = 9.8 \times 10^6 + 35 \times 10^{-9}$
  $\tilde{g} = 3.6 \times 10^7 + 2.4 \times 10^{-9}$

Chapter 2

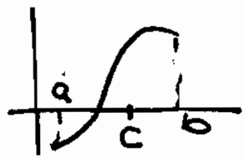
Section 2.2

$f(x)=0$

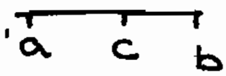
suppose $\exists c \in (a,b)$ such that $f(c)=0$
estimate c ??

section 2.2

- we estimate c by Bisection method
- we assume $f(a).f(b)<0$, f is continous



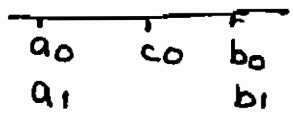
- $c=\frac{a+b}{2}$
- Find $f(c)$
- if $f(c)=0 \Rightarrow$ is Done
- if $f(a).f(c)<0 \Rightarrow r\in[a,c]$
  Else $r\in[c,b]$



Bisection

let $[a_0,b_0]=[a,b]$

$c_0=\frac{a_0+b_0}{2}$



find $f(c_0)$

if $f(c_0)=0$ Done

else if $f(a_0).f(c_0)<0$ then $[a_1,b_1]=[a_0,c_0]$

else $[a_1,b_1]=[c_0,b_0)$

⋮

nth step

n+1 step $f(c_n)$

if $f(c_n)=0$ Done

else if $f(a_n).f(c_n)<0$ then $[a_{n+1},b_{n+1}]=[a_n,c_n]$

Else $[a_{n+1},b_{n+1}]=[c_n,b_n]$

stop if $|c_{n+1}-c_n|<10^{-10}$

or

stop if $|b_n-a_n|<10^{-10}$

or stop if $|f(c_n)|<10^{-10}$

notes:-

$$b_1 - a_1 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0)$$

- $$b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$$

- $$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq r \leq \cdots \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$$

بتكبر a's
بتصغر b's

a_0 c_0 b_0
a_1 b_1
a_1 b_1

- $a_1 = a_0$ $\quad [a_1, b_1] = [a_0, b_0]$

or

$$a_1 = c_0 = \frac{a_0+b_0}{2} > \frac{a_0+a_0}{2} = \frac{2a_0}{2} = a_0 \qquad [a_1, b_1] = [c_0, b_0]$$

$a_1 > a_0$ $\quad b_0 > a_0$

- $b_1 = b_0$ $\quad [a_1, b_1] = [a_0, b_0]$

or

$a_0 < b_0$

$$b_1 = c_0 = \frac{a_0+b_0}{2} < \frac{b_0+b_0}{2} = b_0 \qquad [a_1, b_1] = [c_0, b_0]$$

$b_1 < b_0$

$a_n \uparrow r$

$b_n \downarrow r$

Theory Bisection theorem:-

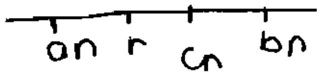
Assume that $f \in C[a,b]$ and that there exists a number $r \in [a,b]$ such that $f(r) = 0$, if $f(a) - f(b) < 0$ and $[c_n]_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection process then

$$|r - c_n| \leq \frac{b-a}{2^{n+1}}, \quad n = 0, 1, 2, \ldots$$

and

$$\lim_{n \to \infty} c_n = r$$

Proof:-



$$|r - c_n| \leq \frac{1}{2}|b_n - a_n| \leq \frac{1}{2} \cdot \frac{1}{2^n}(b-a) = \frac{1}{2^{n+1}}(b-a)$$

$$\lim_{n\to\infty} |r - c_n| = 0 \Rightarrow \underbrace{\lim_{n\to\infty} r}_{r} - \lim_{n\to\infty} c_n = 0$$

$$\lim_{n\to\infty} c_n = r$$

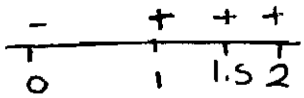
example:-

solve $x\sin x = 1 \Rightarrow x\sin x - 1 = 0$

$f(x) = x\sin x - 1$.

$f(0) = -1$

$f(2) = 0.818595$



$c_0 = \frac{0+2}{2} = 1$

$f(1) = \underset{\text{radian}}{\sin 1} - 1 = -0.158529$.

$[a_1, b_1] = [1, 2]$

$c_1 = \frac{1+2}{2} = 1.5$

$f(1.5) = \sin 1.5 - 1$.

$= 0.496243$

$[a_2, b_2] = [1, 1.5]$.

$c_2 = \frac{1+1.5}{2} = 1.25$.

$f(1.25) =$

$\vdots$

$c_7 = 1.1171875$

$c_8 = 1.11328125$

نزيد تثبيت
٣ منازل.

$\text{Error} \leq 10^{-5}$

$$\frac{b-a}{2^{n+1}} < 10^{-5} = \epsilon$$

$$\frac{b-a}{\epsilon} < 2^{n+1}$$

$$\ln \frac{b-a}{\epsilon} < (n+1)\ln 2$$

$$(n+1) \geq \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2}$$

$$n = \text{int}\left[\frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2} \right]$$

if

$n+1 \geq 16.7$

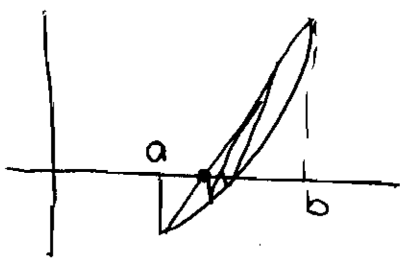
$n = 16$

in example

$$n = \text{int}\left[\frac{\ln \frac{2}{10^{-3}}}{\ln 2} \right] = 10$$

- False position method

- $f(a) \cdot f(b) < 0$
- F is continous

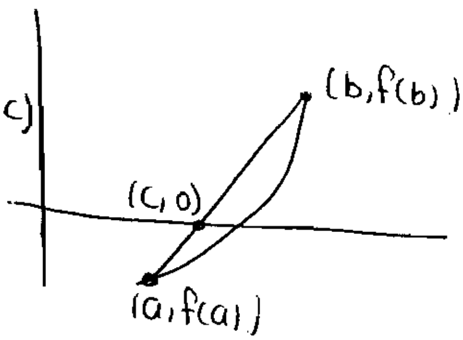


$$\text{slope} = \frac{f(b) - 0}{b - c} = \frac{f(b) - f(a)}{b - a}$$

$$f(b)\,(b-a) = (f(b) - f(a))(b-c)$$

$$b - c = \frac{f(b)\,(b-a)}{f(b) - f(a)}$$

$$c = b - \frac{f(b)\,(b-a)}{f(b) - f(a)}$$



Section 2.1

Fixed point itteration

• To solve $f(x)=0$ we solve $\underset{\text{[Fixed point]}}{x=g(x)}$ [where $f(x)=x-g(x)$]

i.e to Find the roots of $f \rightarrow$ we Find the Fixed point of $g(x)$.

Def:- p is a fixed point of g iff $g(p)=p$

1. $g(x)=\frac{1}{x}$.

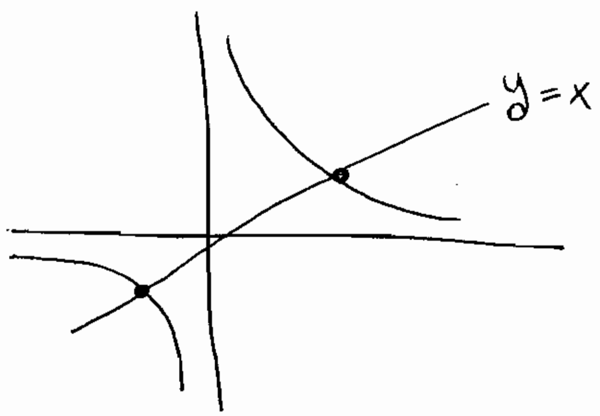
   Fixed points $1, -1$.

   $g(p)=p$.

   $\frac{1}{p}=p \Rightarrow p^2=1 \Rightarrow p=\pm 1$.

2. $g(x)=x+1$. No Fixed points.

3. $g(x)=x$. all points are Fixed points.



Def:- Fixed point iteration:-

start with P_0, $P_{n+1}=g(P_n)$, $n=0,1,2,3,\ldots$

$P_1=g(P_0)$

$P_2=g(P_1)$

$\vdots$

Theorem:-

if the Fixed point itteration converges to P, then P is the Fixed point of g.

algorithm

$[a_0, b_0] = [a, b]$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$f(c_0)$

if $f(c_0) = 0$ done.

else if $f(c_0) \cdot f(a_0) < 0 \Rightarrow [a_1, b_1] = [a_0, c_0]$

else $[a_1, b_1] = [c_0, b_0]$

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

- example

Solve $x \sin x = 1$.

$f(x) = x \sin x - 1$.

$f(0) = -1$.

$f(2) = 0.81859485$.

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$$= 2 - \frac{0.81859485(2-0)}{0.81859485 - (-1)} = 1.09975017.$$

$F(c_0) = 1.09975017 \sin(1.09975017) - 1$

$= -0.02001912$.

$[a_1, b_1] = [1.09975017, 2]$.

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 2 - \frac{0.81859485(2 - 1.09975017)}{0.81859485 - (-0.02001912)}$$

$= 1.12124074$

$f(c_1) = 0.00983461$

$[a_2, b_2] = [1.09975017, 1.12124074]$

شبتا ه منازلة $c_2 = 1.11416120$

$c_3 = 1.11415714$

proof:-

if $\lim P_n = P \Rightarrow \lim_{n\to\infty} P_{n+1} = \lim_{n\to\infty} g(P_n) = g(\lim_{n\to\infty} P_n) = g(P)$

since $P_{n+1} = g(P_n)$ $\quad \downarrow = P$.

example:-

Solve $x^2 - 2x - 3 = 0 \Rightarrow f(x) = 0$.

$(x-3)(x+1) = 0$.

$x = 3$.

$x = -1$.

$x^2 = 2x + 3$.

$x = \sqrt{2x+3} = g(x)$.

if $P_0 = 4$

$P_1 = g(4) = g(P_0) = \sqrt{11} = 3.31662$.

$P_2 = g(P_1) = g(3.31662) = \sqrt{9.63325} = 3.10375$

$P_3 = 3.03439$

$P_4 = 3.01184$ • Note that 3 is a fixed point of $g(x) = \sqrt{2x+3}$ because $g(3) = 3$

$P_n \to 3$

way2:- X تزيد divargance.

$2x = x^2 - 3$.

$x = \frac{x^2-3}{2} = g(x)$.

$P_0 = 4$

$P_1 = g(4) = 6.5$.

$P_2 = g(6.5) = 19.625$.

$P_3 = 191.07$.

way3:-

$x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} = g(x)$.

$P_0 = 4$

$P_1 = g(4) = \frac{3}{2} = 1.5$.

$P_2 = -6$

$P_3 = -0.375$

$P_4 = -1.26315$.

$P_5 = -0.919355$

$P_6 = -1.02762$

$P_7 = -0.990876$

Theorem:- (fixed poind Theorem I)

assume $g \in C[a,b]$ if $g(x) \in [a,b]$ for all $x \in [a,b]$ then g has a Fixed point in $[a,b]$ Furthermore if $|g'(x)| \leq k < 1$ for all $x \in (a,b)$ then g has a unique Fixed point.

Proof:-

if $g(a) = a$ or $g(b) = b$ Done.

if not $g(a) > a$ and $g(b) < b$

let $h(x) = g(x) - x$., h continous.

$h(a) = g(a) - a > 0$

$h(b) = g(b) - b < 0$.

by belzano $\exists c \in \mathbb{C}$ shuch that $h(c) = 0$.

$g(c) - c = 0$

$\boxed{g(c) = c}$

Uniqueness

Suppose $\exists P_1, P_2$ such that $g(P_1) = P_1$, $g(P_2) = P_2$.

Using mean Value theorem on (P_1, P_2)

$\exists c \in (P_1, P_2)$ such that $\left| \frac{g(P_2) - g(P_1)}{P_2 - P_1} \right| = |g'(c)| < 1$

$\frac{P_2 - P_1}{P_2 - P_1} = 1 \Rightarrow 1 < 1 \rightarrow$ Contradict تناقض

$P_1 = P_2$. X.

theorem:- (fixed point iteration theorem)

assume that $g(x)$ and $g'(x)$ are continous on a balanced interval $(a,b) = (P - \delta, P + \delta)$ that contains a Unique Fixed point P and that the started value P_0 is chosen in this interval.

1- if $g'(x) \leq k < 1$ for all $x \in (a,b)$ then the FPI Convarge $P_{n+1} = g(P_n)$ will Convarge (attractive Fixed point)

2. if $g'(x) > 1$ for all $x \in (a,b)$ then the Fixed point itteration divarges (we call it repulsive Fixed point).

Note :-

if p is given we can replace the above two conditions by

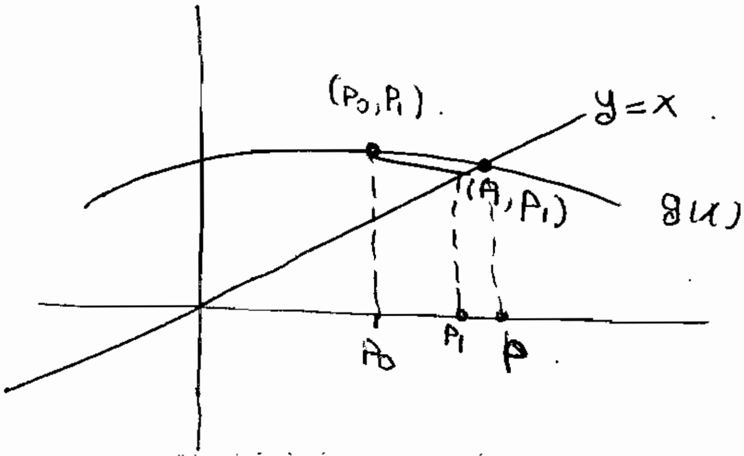
1. if $|g'(p)| < 1 \rightarrow$ the FPI converges.
2. if $|g'(p)| \geq 1 \rightarrow$ the FPI divarges.

<u>convargence</u> $|g'(x)| < 1$

$$-1 < g'(x) < 0 \qquad\qquad 0 < g'(x) < 1$$

$0 < g'(x) < 1$

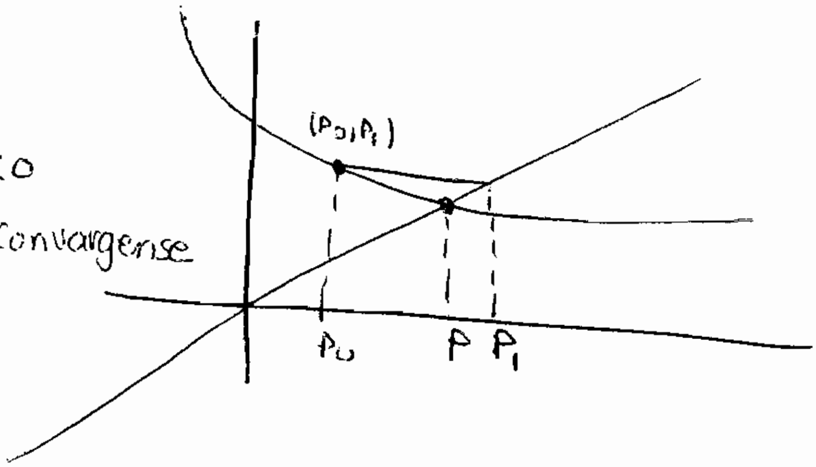
monotone convergense



$P_1 = g(P_0)$

$-1 < g'(x) < 0$

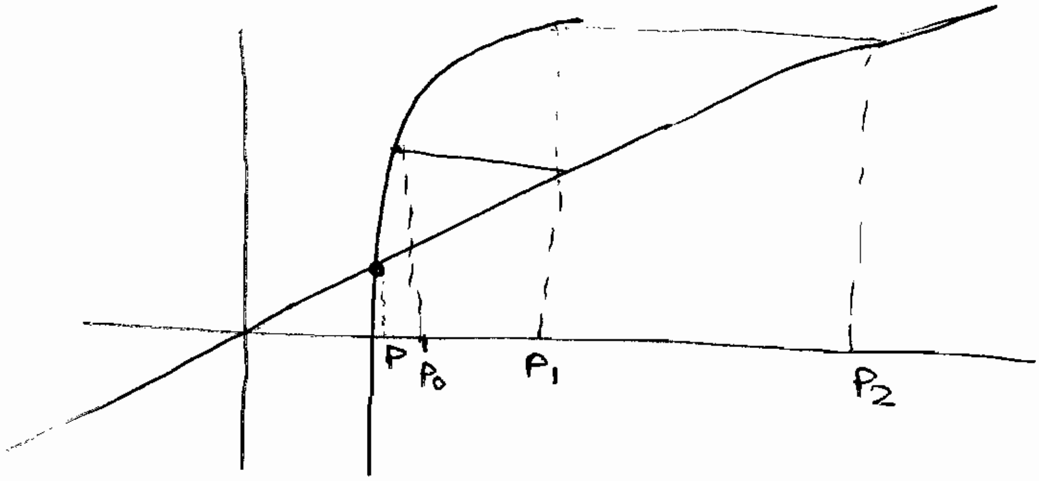
alternating convargense



divergence $|g'(x)| > 1$

$g'(x) > 1$ $\qquad$ $g'(x) < -1$.

$g'(x) > 1$



example

invastigate the nature of the FPI and show your answer by examples for

$$g(x) = 1 + x - \frac{x^2}{4}$$

Solution

$x = g(x)$.

$\cancel{x} = 1 + \cancel{x} - \frac{x^2}{4}$.

$x^2 = 4$

$x = \pm 2$ (Fixed points)

when $x = 2$.

$g'(x) = 1 - \frac{x}{2}$.

$|g'(2)| = 0 < 1 \rightarrow$ Convargance Fixed point. (attractive Fixed point)

to show that :-

let $P_0 = 1.6$.

$P_1 = g(1.6) = 1.96$.

$P_2 = g(1.96) = 1.996$.

$P_n \rightarrow 2$.

if $P_0 = 2.5$.

$P_1 = g(2.5)$

$\rightarrow P_n \rightarrow 2$.

at $x=-2$

~~+g'~~ $|g'(-2)| = 2 > 1$ divarge $\rightarrow$ FPI divarge. (Repulsive Fixed point).

$P_0 = -2.05$.

$P_1 = g(-2.05) = -2.1$ ---

$P_2 = g(-2.1) = -2.2$.

⋮

$P_n \rightarrow$ divargance.

Proof:-

by mean value.

$|P_1 - P| = |g(P_0) - g(P)| = |\underbrace{g'(c)}_{<1}| (P_0 - P) < (P_0 - P)$

$\rightarrow P_1$ is closer to P from P_0.

$P_0 \quad P_1 \quad P \quad P_1$.

P_1 اقرب لـ P من P_0

$|P_n - P| = |g(P_{n-1}) - g(P)| = |\underbrace{g'(c)}_{<k}| (P_{n-1} - P) \leq k (P_{n-1} - P) \leq k\,\underbrace{|P_{n-2} - P|}_{k}$

من النظرية

$\leq k \cdot k \cdot k\, |P_{n-3} - P|$

⋮

$\leq k^n |P_0 - P|$.

$\rightarrow |P_n - P| \leq k^n |P_0 - P|$

$\rightarrow \lim_{n \to \infty} |P_n - P| = 0$

$\rightarrow \lim_{n \to \infty} P_n = P$

② $|P_1 - P| = |\underbrace{g'(c)}_{>1}|\,|P_0 - P| > |P_0 - P|$.

Theorem:-

k is the upper bound للمشتقة

a. $\underbrace{|P_n - P|}_{\text{error}} \leq k^n |P_0 - P|$.

فنقدر نطلع upper bound for error

$\rightarrow$ we can found n

$\boxed{k = g'(P)} \rightarrow$ اذا منعرف P.

b. $|P_n - P| \leq \dfrac{k^n |P_1 - P_0|}{1 - k}$. (excersise).

example:-

$x^3 - x + 5 = 0$.

Use Fixed point itteration to find all the roots, Find k for each case.

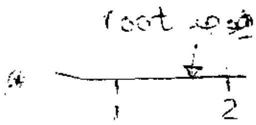
$F(x) = x^3 - x + 5$

$F(0) = 5$

$F(-1) = 5$

$F(-2) = -11$

$F(2) = 1$



$$x^3 = x + 5$$

$$x = \sqrt[3]{x+5} = (x+5)^{1/3}$$

$$g(x) = x$$

$$g'(x) = \frac{1}{3}(x+5)^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{(x+5)^2}} < 1 \qquad \text{for all } x \quad 0 < x$$

$P_0 = 1.5$

منوصل للـroot

for $x > 0$

$$x + 5 > 5$$

$$(x+5)^2 > 25$$

$$\sqrt[3]{(x+5)} > (25)^{\frac{1}{3}} > 2$$

$$\frac{1}{\sqrt[3]{(x+5)}} < \frac{1}{2}$$

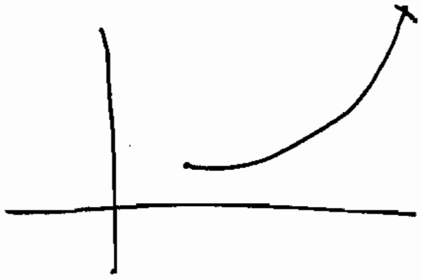
$$\frac{1}{3\sqrt[3]{(x+5)}} < \frac{1}{6} \qquad K = \frac{1}{6}$$

Discussion

- $f(x) = 1 + e^{-\cos(x-1)}$

$[1, 2]$.

max point. فهم كيف نطلع



5.

$x^4 - 3x^2 - 3 = 0$

10^{-2}

$P_0 = 1$

$[1, 2]$.

$x^4 = 3x^2 + 3$.

$x = \sqrt[4]{3x^2 + 3}$.

$P_1 = g(1) = \sqrt[4]{6} = 1.56508$

$P_2 = 1.79358$

$P_3 = 1.88595$

$P_4 = 1.92285$

$P_5 = 1.93751$

لزمنا 5 itteration حتى ثبتنا منزلتين

$P_6 = \cancel{1.94832}$

4/c. (12.2)

$$P_n = P_{n-1} - \frac{P_{n-1}^5 - 7}{5P_{n-1}^4}$$

$$g(x) = x - \frac{x^5 - 7}{5x^4}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g(x) = x - \frac{x}{5} + \frac{7}{5x^4}$$

$$g(x) = \frac{4x}{5} + \frac{7}{5x^4}$$

$$g'(x) = \frac{4}{5} - \frac{28}{5x^5}$$

$P = 7^{1/5}$

$P_n = g(P_{n-1})$

$x = 7^{1/5}$.

$x^5 = 7$

$x^5 - 7 = 0$.

$f(x) = x^5 - 7$

$$g'(7^{1/5}) = \frac{4}{5} - \frac{28}{5(7^{1/5})^5}$$

2.2, 2.4, ~~2.5~~ . اقرأ مادة
2.1.

$$= \frac{4}{5} - \frac{28}{5.7} = \frac{4}{5} - \frac{4}{5} = 0.$$

method أسرع
newton method.

مهم جداً
$\frac{14}{2.2}$

Solve $x = \tan x$ in $[4,5]$.

$g(x) = \sec^2 x > 1$.

$x = \tan^{-1} x$

$g(x) = \tan^{-1} x$

$g'(x) = \frac{1}{1+x^2} < 1$

$P_0 = 4.5$.

$P_1 = \tan^{-1}(4.5)$

$= 1.352127$

$P_2 = \tan^{-1}(P_1)$

$= 0.93$

$x = \tan x = \tan(x-\pi) = \tan(x+\pi)$

$x = \tan(x-\pi)$

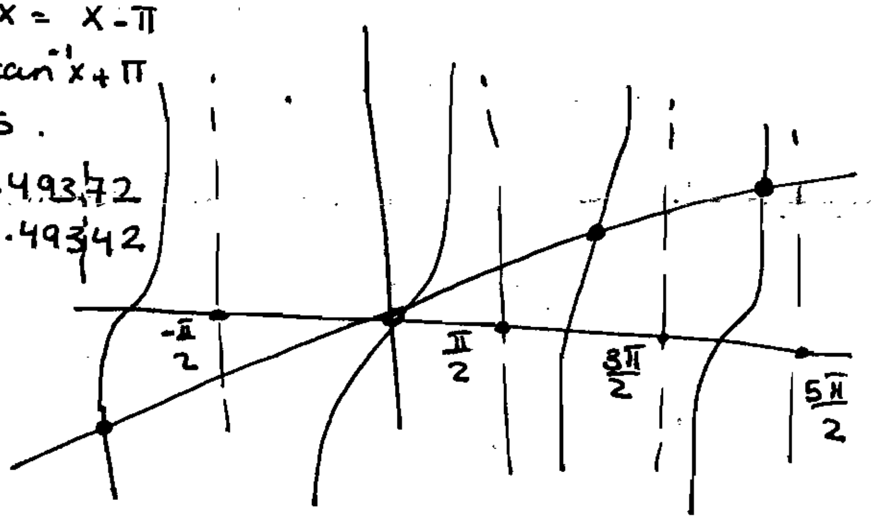
$\tan^{-1} x = x - \pi$

$x = \tan^{-1} x + \pi$

$P_0 = 4.5$.

$P_1 = 4.49372$

$P_2 = 4.49342$



$\frac{14}{2.1}$ Let $f(x) = (x-1)^{10}$

$P = 1$

$P_n = 1 + \frac{1}{n}$

Show that if $|f(P_n)| < 10^{-3}$ for $n > 1$

but $|P - P_n| < 10^{-3}$ requires $n > 1000$.

1. $|P - P_n| < \epsilon$
2. $|c_{n+1} - c_n| < \epsilon$
3. $|f(P_n)| < \epsilon$

$F(p_n) = \left(\frac{1}{n}\right)^{10} < 10^{-3}$ For $n > 1$.

$$|P - P_n| < 10^{-3} = \left|1 - 1 - \frac{1}{n}\right| < 10^{-3}$$
$$\left|-\frac{1}{n}\right| < 10^{-3}$$
$$\frac{1}{n} < 10^{-3} \Rightarrow n > 1000.$$

$\frac{15}{2.1}$ $P_n = \sum_{k=1}^{\infty} \frac{1}{k}$

show that P_n divarge even though $\lim_{n\to\infty}(P_n - P_{n-1}) = 0$.

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\lim_{n\to\infty} P_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$$ harmonic series (divarge)

$$P_n - P_{n-1} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) = \frac{1}{n}$$

$$\lim_{n\to\infty}(P_n - P_{n-1}) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

$$c_n = \frac{a_n + b_n}{2} \quad \text{stop}.$$

$F(c_n) \leq \epsilon$ or $|c_n - c_{n-1}| \leq \epsilon$

stop if $F(c_n) \leq \epsilon$ and $\frac{c_n - c(n-1)}{c(n-1)} \leq 1 \times 10^{-6}$.

Solve this equ

$3x^2 - e^x = 0$

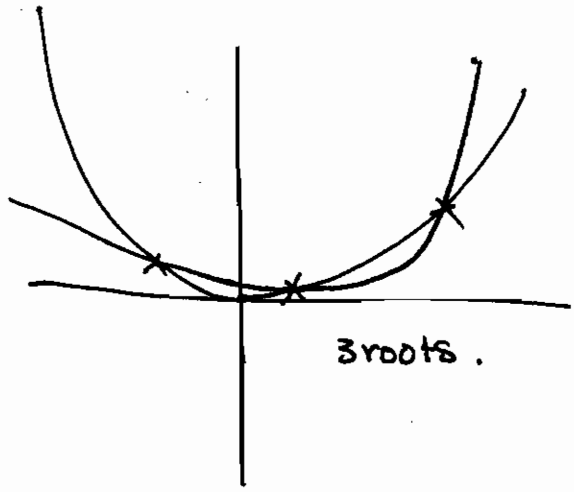
$f(0) = -1$

$f(1) = 0.28 > 0$

$f(2) = 12 - e^2 > 0$

$f(3) = 27 - e^3 > 0$

$f(4) = 48 - e^4 < 0$.



Newton method

$$f'(P_0) = \frac{f(P_0) - 0}{P_0 - P_1}$$

$$P_0 - P_1 = \frac{f(P_0)}{f'(P_0)}$$

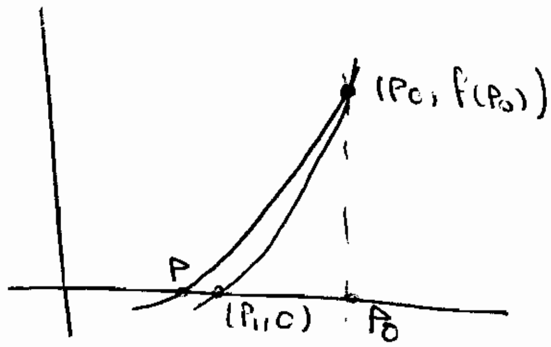
$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

$$\vdots$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$x = \underbrace{x - \frac{f(x)}{f'(x)}}_{g(x)} \leftarrow \text{Newton fixed point function}$$



Th:- Newton Raphson theorem

assume $f \in C^2[a,b]$ and $\exists\, P \in [a,b]$ such that $f(P) = 0$, if $f'(P) \neq 0$ then there exist a $\delta > 0$ such that the sequance $\{P_k\}_{k=0}^{\infty}$ which is defined by $P_k = g(P_{k-1}) = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$ will convarge to P for any initial approximation $P_0 \in [P-\delta, P+\delta]$

example:-
estimate $5^{\frac{3}{7}}$

$$x = 5^{3/7}$$

$$x^7 = 5^3$$

$$f(x) = x^7 - 125$$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$f'(x) = 7x^6$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$= P_n - \frac{P_n^7 - 125}{7P_n^6}$$

$$= \frac{6}{7} P_n + \frac{125}{7P_n^6}$$

$P_0 = 2$

$$P_1 = \frac{6}{7}(2) + \frac{125}{7(2)^6} = 1.71429$$

$$P_2 = \frac{6}{7}(1.71429) + \frac{125}{7(1.71429)^6} = 2.17$$

$\vdots$

Proof the theorem

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(P) = \frac{f(P)f''(P)}{(f'(P))^2} = 0$$

→ by Fixed point ~~iteration~~ theorey → the Fixed point itteration will convarge.

- if $e_{n+1} \approx A e_n$ where $e_n = P - P_n$ (error)

or $e_{n+1} \approx \frac{1}{100} e_n$ (error smaller than the first) (the best one)

$e_{n+1} \approx \frac{1}{2} e_n$ كدا صغر

كدا أصح / أحسن

كثير أحسن ← $e_{n+1} \approx A e_n^2$

if $e_n = 0.01$

$e_{n+1} = A(0.0001)$

Definition

P is a root of multiplicity M of $f(x)$ if $f(x) = (x-P)^M h(x)$,
$h(P) \neq 0$.

- $f(x) = x^3 - 3x + 2$
  1 is a root of $f(x)$
  what is the multiplicity of 1?

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x-1 & x^3 - 3x + 2 \\
 & -x^3 + x^2 \\
\hline
 & x^2 - 3x + 2 \\
 & -x^2 \pm x \\
\hline
 & -2x + 2 \\
 & +2x \mp 2 \\
\hline
 & 0
\end{array}
\qquad
\begin{array}{r|l}
 & x + 2 \\
\hline
x-1 & x^2 + x - 2 \\
 & -x^2 \pm x \\
\hline
 & 2x - 2 \\
 & 2x - 2 \\
\hline
 & 0
\end{array}
$$

$f(x) = (x-1)(x^2 + x - 2)$

1 has multiplicity 2 (quadratic root) $M=2$
-2 is a simple root ($M=1$)

Theory:-

P is a root of multiplicity M of $f(x)$ iff
$f(P)=0$, $f'(P)=0$, $\dots$ $f^{(m-1)}(P)=0$ but
$f^{(m)}(P) \neq 0$

Example:-

$f(x) = x^3 - 3x + 2$
$f(1) = 0$
$f'(x) = 3x^2 - 3$
$f'(1) = 0$

$f''(x) = 6x$
$f''(1) = 6$
$M = 2$

$e_{n+1} \cong A e_n$

$$\frac{e_{n+1}}{e_n} \approx A$$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = A \to \text{linear Convergence.}$$

if $e_{n+1} \approx A e_n^2$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} \approx A \to \text{quadratic convergence.}$$

Definition:- **Order of Convargence**

assume $P_n \to P$ and $e_n = P - P_n$, if there exists two positive numbers A, R such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^R} = A$$

Then the sequance is said to Converge to P with order of Convergence R, A is called the Asymptotic error Constant.

if $R=1$, we call it linear Convargance.
if $R=2$, we call it quadratic convargance.

example:-
show that $P_n = \frac{1}{n^3}$ Converges to $\underset{\downarrow}{\overset{P}{0}}$ linearly ??

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{(n+1)^3}\right|}{\left|0 - \frac{1}{n^3}\right|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}$$

$$= \left(\lim \frac{n}{n+1}\right)^3 = 1.$$

$\frac{1}{n^3} \underset{\text{convarge to}}{\longrightarrow} 0$ linearly

Example:-

$f(x) = x^{101} - x^{100} - x + 1$

$f(1) = 0$

$f'(x) = 101x^{100} - 100x^{99} - 1$

$f'(1) = 101 - 100 - 1 = 0$.

$f''(x) = (101)(100)x^{99} - (100)(99)x^{98}$

$f''(1) \neq 0$

$M = 2$.

Theorem:- Convergence of newton method

if we use newton itliration,

1. if p is a simple root, then

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \left|\frac{f''(p)}{2f'(p)}\right|$$

[p is a simple root
- Convergence is quadratic
$A = \left|\frac{f''(p)}{2f'(p)}\right|$, $R = 2$]

2. if p has moltiplicity $M > 1$, then

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \frac{M-1}{M}$$

[convargence is linear
$A = \frac{M-1}{M}$, $R = 1$]

example

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$.

$f(x) = (x-1)^2(x+2)$

$f''(x) = 6x$

-2 is a simple roots

convargance is fast $R = 2$ $\left(\frac{|e_{n+1}|}{|e_n|^2} = \left|\frac{f''(p)}{2f'(p)}\right|\right)$

$A = \left|\frac{f''(-2)}{2f'(-2)}\right| = \left|\frac{-12}{2(9)}\right| = \frac{2}{3}$.

$P = 1$, $M = 2$

linear convargance ($P = 1$)

$\lim_{n\to\infty} \frac{|e_{n+1}|}{\,} = \frac{1}{\,}$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$P_0 = -2.4$

n	P_n	e_n ($P - P_0$)	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^2}$
0	-2.4	0.4	
1	-2.07619047	0.0761904	0.4761 ....
2	-2.00359601	0.003596	0.6194...
3	-2.00000858	0.000008589	0.6642

$\downarrow \frac{2}{3} \approx A$

fast convargance.

$P_0 = 1.2$

n	P_n	e_n	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert}$
0	1.2	-0.2	
1	1.103030	-0.10303	0.515 ---
2	1.0523564	-0.052356	0.5081
3	1.0264008	-0.02640081	0.4962

$\downarrow A \approx \frac{1}{2}.$

slow convargance

$A \longrightarrow \frac{1}{2}.$

Theory:- accelarated newton method

if p is a root of multiplicity M then the iteration

$$p_{n+1} = p_n - \frac{M f(p_n)}{f'(p_n)}$$ will converge quadratically to p.

Ex:-

For the previous example. $f(x) = (x-1)^2 (x+2)$

1 has multiplicity 2, if we use the accelarated Newton itteration

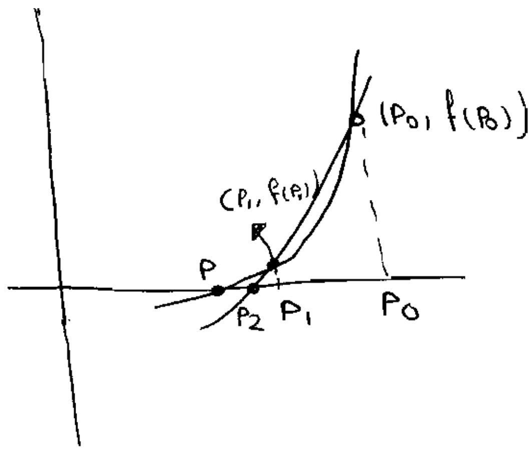
$$p_{n+1} = p_n - \frac{2 f(p_n)}{f'(p_n)}$$ will get quadratic convargance!

$p_0 = 1.2$

n	p_n	e_n	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^2}$
0	1.2	-0.2	
1	1.00606060.6	-0.00606	0.15
2	1.000006087	-0.000006087	0.15

Secant method:-

$$\frac{f(p_1)-0}{p_1-p_2} = \frac{f(p_1)-f(p_0)}{p_1-p_0}$$



$$p_1 - p_2 = \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)}$$

$$p_2 = p_1 - \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)}$$

$$p_3 = p_2 - \frac{f(p_2)(p_2-p_1)}{f(p_2)-f(p_1)}$$

$$\vdots$$

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1}-p_{n-2})}{f(p_{n-1})-f(p_{n-2})}}$$

Theorem:-

if we use secant method to get $p_n \to p$ then.

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{1.618}} = \left|\frac{f''(p)}{2f'(p)}\right|^{0.618}$$

$$\longrightarrow R = 1.618 = \frac{1+\sqrt{5}}{2}$$

Ex:-

$f(x) = (x+2)(x-1)^2$

$p_0 = -2.6$, $p_1 = -2.4$

and we use secant method.

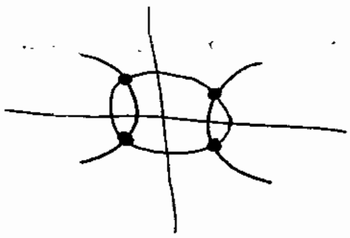
n	p_n	e_n	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^{1.618}}$
0	-2.6	0.6	
1	-2.4	0.4	
2	-2.106598	0.106598	
3	-2.02264	0.02264	
4	-2.00151	0.00151	

	False position method	Secant method	Newton method
Speed	1	1.6	2
• Coast	1	1	2
• Convorgance	very accurate	depends on P_0, P_1	depends on P_0.

• 2.6 Fixed point itteration for system of equation

$$x^2\cos y + y\sin x = 10$$
$$y\ln x + x^2\cos y = 5$$

$$x^2 - y^2 = 1$$
$$x^2 + y^2 = 2$$
$$2x^2 = 3.$$
$$x^2 = \frac{3}{2}.$$
$$x = \pm\sqrt{\frac{3}{2}}$$



$$x^2 - y^2 = x + 3$$
$$x^2 + y^2 = e^x - 1.$$
$$2x^2 = x + 3 + e^x - 1.$$
$$2x^2 - x - e^x - 2 = 0$$

$$x = g_1(x, y)$$
$$y = g_2(x, y)$$
$$(P_0, g_0)$$
$$P_1 = g_1(P_0, g_0) \qquad g_1 = g_2(P_0, g_0)$$
$$P_2 = g_2(P_1, g_1) \qquad g_2 = g_2(P_1, g_1) \qquad P_{n+1} = g_1(P_n, g_n)$$
$$g_{n+1} = g_2(P_n, g_n)$$

Definition:-

(p, q) is a Fixed Point of the system

$$x = g_1(x,y), \quad y = g_2(x,y) \quad \text{if} \quad p = g_1(p,q) \text{ and } q = g_2(p,q)$$

Def:-

Fixed point itteration for the system

$x = g_1(x,y)$, $y = g_2(x,y)$ is given (p_0, q_0) then

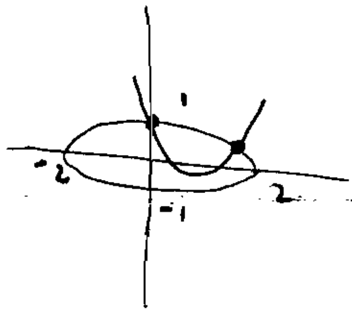
$$p_{n+1} = g_1(p_n, q_n)$$
$$q_{n+1} = g_2(p_n, q_n) \quad n = 1, 2, 3, \dots$$

Ex:-

$$f_1(x,y) = x^2 - 2x - y + 0.5 = 0$$
$$f_2(x,y) = x^2 + 4y^2 - 4 = 0 \quad \rightarrow x^2 + 4y^2 = 4$$

estimate the solutions ?

$$\frac{x^2}{4} + y^2 = 1$$



$$x = \frac{x^2 - y + 0.5}{2} = g_1(x,y)$$
$$y = \frac{-x^2 - 4y^2 + 8y + 4}{8} = g_2(x,y)$$

$(p_0, q_0) = (0,1)$.

$$p_1 = g_1(0,1) = \frac{0 - 1 + 0.5}{2} = -0.25$$
$$q_1 = g_2(0,1) = \frac{0 - 4 + 8 + 4}{8} = 1$$

⋮

$p_4 = -0.2221680$ $\qquad p_5 = -0.222194$

$q_4 = 0.9938121$ $\qquad q_5 = 0.9938095$

$(p_0, q_0) = (2,0)$ (divarges).

$p_1 = g_1(2,0) = 2.25$

$q_1 = g_2(2,0) = 0$

⋮

Let $g_1(x,y) = \dfrac{-x^2 + 4x + y - 0.5}{2}$

$g_2(x,y) = \dfrac{-x^2 - 4y^2 - 11x + 4}{11}$

$\downarrow$

$(p_0, q_0) = (2,1) \rightarrow$ Convergence

$(2,1) \rightarrow (1.900, 0.311)$

Th:- Fixed point itteration for system of equation:-

assume $g_1(x,y)$, $g_2(x,y)$ and their partial derivative are continous on a region that contains the fixed point (p,q), if the starting point (p_0,q_0) is chosing sufficiently closed to (p,q) and

$$\left|\frac{dg_1}{dx}\right| + \left|\frac{dg_1}{dy}\right| < 1 \quad \text{and} \quad \left|\frac{dg_2}{dx}\right| + \left|\frac{dg_2}{dy}\right| < 1 \quad \text{in that region}$$

then the FPI will convarge.

• Note:-
if (p,q) is given we apply the condition at (p,q) only.

Fixed point → we talk about g's
Newton → we talk about F.

__to proof__
المثال السابق

if $|x| < 0.5$ and $0.5 < y < 1.5$ أولاً نختار الفترة

$$\left|\frac{dg_1}{dx}\right| + \left|\frac{dg_1}{dy}\right| = |x| + 0.5 < 1$$

أكبر قيمة لـ 0.5

$$\left|\frac{dg_2}{dx}\right| + \left|\frac{dg_2}{dy}\right| = \frac{|x|}{4} + |1-y| < \frac{1}{8} + 0.5 < 1$$

أكبر قيمة لـ 0.5   أكبر قيمة لـ 1.5

حتى نثبت ان النقطة المختارة divergance ← نختار فترة لا يتحقق الشرطين السابقين او لا يتحقق شرط واحد على الأقل.

__example__ (linear system)

$$3x + 2y + 7z = 10 \rightarrow x = \frac{10 - 2y - 7z}{3} = g_1(x,y,z)$$

$$2x + 4y - z = 4 \rightarrow y = \frac{4 + z - 2x}{4} = g_2(x,y,z)$$

$$x + 5y + 10z = 15 \rightarrow \cancel{x = 15 - 5y}$$

$$z = \frac{15 - x - 5y}{10} = g_3(x,y,z).$$

$$p_1 = g_1(p_0, q_0, r_0)$$
$$q_1 = g_2(p_0, q_0, r_0)$$
$$r_1 = g_3(p_0, q_0, r_0).$$

$\rightarrow$

$$\left.\begin{aligned} p_1 &= g_1(p_1, q_0, r_0) \\ q_1 &= g_2(p_1, q_0, r_0) \\ r_1 &= g_3(p_1, q_0, r_0) \end{aligned}\right\} \text{Gauss-Sidel method}$$

- $$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$f_1(x,y) = 0$$
$$f_2(x,y) = 0$$

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix} - \underbrace{J^{-1}_{(p_n, q_n)}}_{\text{Jacubian}} \begin{pmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{pmatrix}$$

$$h : (x,y) \longrightarrow (f_1(x,y), f_2(x,y))$$

$$h' = J = \begin{pmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} \\ \frac{df_2}{dx} & \frac{df_2}{dy} \end{pmatrix}$$

$$\boxed{\vec{\mathbb{P}}_{n+1} = \vec{\mathbb{P}}_n - J^{-1}\,\mathbb{F}}$$

2.7 Newton method

given $F_1(x,y)=0,\ F_2(x,y)=0$

and $F_1(P,q)=0,\ F_2(P,q)=0$.

starting with (P_0,q_0) close to (P,q) then using taylor expansion in Two dimension at (P_0,q_0)

$$f_1(x,y) \cong f_1(P_0,q_0) + \left.\frac{df_1}{dx}\right|_{(P_0,q_0)}(x-P_0) + \left.\frac{dF_1}{dy}\right|_{(P_0,q_0)}(y-q_0)$$

$$F_2(x,y) \cong F_2(P_0,q_0) + \left.\frac{dF_2}{dx}\right|_{(P_0,q_0)}(x-P_0) + \left.\frac{dF_2}{dy}\right|_{(P_0,q_0)}(y-q_0).$$

subsitude (P,q) above

$$0 = F_1(P_0,q_0) + \left.\frac{dF_1}{dx}\right|_{(P_0,q_0)}(P-P_0) + \left.\frac{dF_1}{dy}\right|_{(P_0,q_0)}(q-q_0)$$

$$0 = F_2(P_0,q_0) + \left.\frac{dF_2}{dx}\right|_{(P_0,q_0)}(P-P_0) + \left.\frac{dF_2}{dy}\right|_{(P_0,q_0)}(q-q_0)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(P_0,q_0) \\ f_2(P_0,q_0) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} \\ \frac{df_2}{dx} & \frac{df_2}{dy} \end{bmatrix}_{(P_0,q_0)}}_{J} \begin{bmatrix} P-P_0 \\ q-q_0 \end{bmatrix}$$

$$\boxed{-\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{(P_0,q_0)} = J_{(P_0,q_0)} \begin{bmatrix} P-P_0 \\ q-q_0 \end{bmatrix}} \longrightarrow \text{Direct method.}$$

$$-J^{-1}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} P-P_0 \\ q-q_0 \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ q_0 \end{bmatrix} - J^{-1}_{(P_0,q_0)}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{(P_0,q_0)} = \begin{bmatrix} P_1 \\ q_1 \end{bmatrix} \quad \text{inverse way.}$$

- Inverse method.

$$\begin{bmatrix} P_{n+1} \\ g_{n+1} \end{bmatrix} = \begin{bmatrix} P_n \\ g_n \end{bmatrix} - J^{-1}_{(P_n, g_n)} \begin{bmatrix} F_1(P_n, g_n) \\ F_2(P_n, g_n) \end{bmatrix}$$

- Direct method

$$-\begin{bmatrix} F_1(P_n, g_n) \\ F_2(P_n, g_n) \end{bmatrix} = J_{(P_n, g_n)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\Delta x = P_{n+1} - P_n \longrightarrow P_{n+1} = \Delta x + P_n$$

$$\Delta y = g_{n+1} - g_n \longrightarrow g_{n+1} = \Delta y + g_n$$

- example

Solve used Newton method.

- inverse method.

$$x^2 - 2x - y = 0.5 \longrightarrow x^2 - 2x - y - 0.5 = 0 = f_1(x,y)$$

$$x^2 + 4y^2 = 4. \longrightarrow x^2 + 4y^2 - 4 = 0 = f_2(x,y).$$

$$(P_0, g_0) = (2, 0.25)$$

$$J = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}_{(2, 0.25)} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} P_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$f_1(2, 0.25) = 0.25$

$f_2(2, 0.25) = 0.25.$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8} \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}.$$

$$\begin{pmatrix} P_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.008789 \\ 0.024414 \end{pmatrix}.$$

$$= \begin{pmatrix} 1.900691 \\ 0.311213 \end{pmatrix}$$

- Direct method

$$-\begin{pmatrix} f_1(2, 0.25) \\ f_2(2, 0.25) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$-\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}.$$

$$\Delta X = \frac{\begin{vmatrix} -0.25 & -1 \\ -0.25 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}} = \frac{-0.75}{8} = -0.09375$$

$$\begin{aligned} P_1 &= \Delta x + P_0. \\ &= -0.09375 + 2. \\ &= 1.90625. \end{aligned}$$

$$\Delta y = \frac{\cancel{\begin{matrix} -0.25 \\ -0.25 \end{matrix}}\begin{vmatrix} 2 & -0.25 \\ 4 & -0.25 \end{vmatrix}}{8} = \frac{-0.5+1}{8} = \frac{0.5}{8} = +0.0625$$

$$\Delta y = g_1 + g_0$$

$$\begin{aligned} g_1 &= \Delta y + g_0. \\ &= 0.0625 + 0.25 \\ &= 0.3125. \end{aligned}$$

discussion

3.4

$\boxed{10}$ $f(x) = (x-p)^m h(x)$.

$\leftrightarrow$ $f(p)=0$, $f'(p)=0$ ---- $f^{(m-1)}(p)=0$ but $f^{(m)}(p) \neq 0$.

$f(p)=0$

$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x)$.

$f'(p)=0$.

$f(p)=0$.

$(x-p)$ is a Factor of $f(p)$.

$(x-p)2$ is a factor of $f'(p)$.

$\boxed{8}$ $g(x) = x - \frac{mf(x)}{f'(x)}$. it will convarge quadrotically to p.

p is a root of multiplicity m for $f(x)$.

$g'(p)=0$ بدنا نثبت ان

$f(x) = (x-p)^m h(x)$, $h(p) \neq 0$.

$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x)$.

$$g(x) = x - \frac{m(x-p)^m h(x)}{m(x-p)^{m-1} h(x) + (x-p)^m h'(x)}$$

$$= x - \frac{m(x-p)h(x)}{m h(x) + (x-p)h'(x)}$$

$$g'(x) = 1 - \frac{(mh(x) + (x-p)h'(x))(mh(x) + (x-p)h'(x)) - m(x-p)h(x) \times \text{مشتقة المقام}}{[mh(x) + (x-p)h'(x)]^2}$$

$$g'(p) = 1 - \frac{(mh(p))^2}{mh(p)^2}$$

$= 0$.

$\boxed{6}$

a. $P_n = 10^{-2^n} \longrightarrow 0$ quadratically

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{\left(10^{-2^n}\right)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n}\cdot 2}} = 1$$

b. $P_n = 10^{-n^k}$ $\not\longrightarrow$ 0 quadratically.

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2(n^k)}}$$

$$= \lim_{n\to\infty} \frac{10^{2(n^k)}}{10^{(n+1)^k}} = \lim_{n\to\infty} \frac{10^{n^k} \cdot 10^{n^k}}{10^{(n+1)^k}} \longrightarrow \infty$$

$10 \leftarrow$ اسرع من البسط

$\boxed{20}$
$\boxed{2.3}$

$$1,564,000 = 1,000,000\, e^{\lambda} + \frac{435,000}{\lambda}(e^{\lambda}-1)$$

$$1564 = 1000\, e^{\lambda} + \frac{435}{\lambda}(e^{\lambda}-1)$$

$$f(\lambda) = 1000\, e^{\lambda} + \frac{435}{\lambda}(e^{\lambda}-1) - 1564 = 0$$

Chapter 3

linear systems:-

Iteratine methods:-

1. Fixed point itteration
2. Gauss - sidel method
3. Newton Method

- **Direct methods:-** (A is nonsingular).

1. Gaussian Elimenation $[A:b] \longrightarrow [U|c]$ + Back subsitution.
2. Gauss - Jordan $[A\backslash b] \longrightarrow [I\backslash x]$.
3. inverse method $x = A^{-1}b$
4. Gramer's :: $x_i = \frac{|A_i|}{|A|}$
5. L-U Factorization.

Section 3.3

- back subsitution

$$3x_1 + 2x_2 + 4x_3 = 9$$
$$4x_2 + 6x_3 = 10.$$
$$10x_3 = 10.$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 10 & 10 \end{array}\right]$$

$$10x_3 = 10 \longrightarrow \boxed{x_3 = 1}$$
$$4x_2 + 6x_3 = 10$$
$$4x_2 = 10 - 6.$$
$$4x_2 = 4 \longrightarrow \boxed{x_2 = 1}$$
$$3x_1 + 2x_2 + 4x_3 = 9.$$
$$3x_1 = 9 - 4 - 2.$$
$$3x_1 = 3 \longrightarrow \boxed{x_1 = 1}$$

$$
\left[\begin{array}{cccccc|c}
a_{11} & a_{12} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} & b_1 \\
\vdots & & & & & & \\
 & & & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} & b_{n-2} \\
0 & & & & a_{n-1,n-1} & a_{n-1,n} & b_{n-1} \\
 & & & & & a_{n,n} & b_n
\end{array}\right]
$$

$a_{n,n}x_n = b_n$.

$x_n = \dfrac{b_n}{a_{n,n}}$.

$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$.

$x_{n-1} = \dfrac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$.

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}.$$

$$x_k = \frac{b_k - a_{k,k+1}x_{k+1} - a_{k,k+2}x_{k+2} \cdots - a_{k,n}x_n}{a_{k,k}}$$

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{k,j}x_j}{a_{kk}} \qquad k = n, 1.$$

3.3 Cost

Steps	+/-	×/÷
1	0	1
2	1	2
3	2	3
k	k-1	k
n	n-1	n
Total	$\frac{(n-1)(n)}{2}$	$\frac{n(n+1)}{2}$

Total coast $= \dfrac{n^2-n}{2} + \dfrac{n^2+n}{2} = n^2$.

3.4 Gaussian Elimination

$$Ax = b$$

$$[A \backslash b] \longrightarrow [U \backslash C] + \text{back sub.}$$

Row operations:-

1. multiply any row by a nonzero constant
2. switch any two rows
3. Replace any row by adding to it a nonzero multiple of another row

$$\text{row } r := \text{row } r + C \text{ row } p$$
$$= \text{row} - m_{rp} \text{ row } p \quad ; \quad m_{r,p} = \frac{a_{r,p}}{a_{pp}} \quad r > p$$

Example

Solve:-

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$
$$2x_1 + 4x_3 + 3x_4 = 28$$
$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$
$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6.$$

Pivot element

pivot Row

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3$$

$$\begin{array}{l} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - 4R_1} \\ \xrightarrow{R_4 + 3R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right]$$

← pivot Row

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} = \frac{6}{-4} = 1.5$$

$$m_{42} = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} = \frac{7}{-4} = -\frac{7}{4} = -1.75$$

$$\begin{array}{l} \xrightarrow{R_3 - 1.5R_2} \\ \xrightarrow{R_4 + 1.75R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right]$$

← pivot Row

$$m_{4,3} = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} = \frac{9.5}{-5} = -1.9$$

$$\xrightarrow{R_4 - 1.9R_3} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right] \longrightarrow \begin{array}{l} x_4 = 2 \\ x_3 = 4 \\ x_2 = -1 \\ x_1 = 3 \end{array}$$

coast

step	+/-	x/÷
1	4x3	3 + 4x3
2	3x2	2 + 3x2
3	2x1	1 + 2x1
total	20	26

$\boxed{46}$

in general for nxn matrix

step	+/-	x/÷
1	$(n-1)n$	$(n-1)n + n-1$
2	$(n-2)(n-1)$	$(n-2)(n-1) + n-2$
3	$(n-3)(n-2)$	$(n-3)(n-2) + n-3$
⋮		
p	$(n-p)(n-p+1)$	$(n-p)(n-p+1) + n-p$
last step → $(n-1)$		

total +/- : $\sum_{p=1}^{n-1} (n-p)(n-p+1)$

x/÷ : $\sum_{p=1}^{n-1} (n-p)(n-p+1) + (n-p)$

$$\sum_{p=1}^{n-1} (n-p)(n-p+1) = \sum_{p=1}^{n-1} (n-p)^2 + (n-p)$$

Let $k = n - p$

if $p = 1 \longrightarrow k = n-1$

$p = n-1 \longrightarrow k = 1$

$$\therefore \sum_{1}^{n-1} k^2 + k$$

$$\sum_{n=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{n=1}^{n} k = \frac{(n-1)n}{2}.$$

$$\therefore \sum_{1}^{n-1} k^2 + k = \frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n}{2}$$

$$\text{total } \times / \div = \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$$

$\therefore$ Grand (الكلي) Total (المجموع) $= 2\left[\frac{(n^2-n)(2n-1)}{6} + \frac{n(n-1)}{2}\right] + \frac{n(n-1)}{2}$

$$= 2\left[\frac{2n^3-3n^2+n}{6} + \frac{3n^2-3n}{6}\right] + \frac{n^2-n}{2}$$

$$= \frac{2n^3-2n}{3} + \frac{n^2-n}{2}$$

$$= \frac{4n^3-4n+3n^2-3n}{6} = \frac{4n^3+3n^2-7n}{6}$$

$$\approx \frac{2}{3}n^3$$

- Coast for Gaussinan

$$\text{Coast} = \frac{4n^3+3n^2-7n}{6} + n^2$$

Coast For back subsitution.

$$= \frac{4n^3+9n^2-7n}{6}$$

$$\approx \frac{2}{3}n^3$$

- Algorithm

will store the Augumented matrix in $n+1$ coloumn.

$$\left[\begin{array}{cccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ a_{2,1}^{(1)} & a_{22}^{(1)} & \cdots & a_{2,n}^{(1)} & a_{2,n+1}^{(1)} \\ \vdots & & & \vdots & \vdots \\ a_{n,1}^{(1)} & a_{n,2}^{(1)} & \cdots & a_{nn}^{(1)} & a_{n,n+1}^{(1)} \end{array}\right]$$

and will construct an equivalent upper triangular. U

$$\left[\begin{array}{ccccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & & \cdots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & & \cdots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \cdots & a_{3,n}^{(3)} & a_{3,n+1}^{(3)} \\ & 0 & & & a_{n,n}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right]$$

Step 1 Store the coefficient in array

Step 2 Switch rows if necessary so that $a_{1,1}^{(1)} \neq 0$

find $m_{r,1} = \dfrac{a_{r,1}}{a_{1,1}^{(1)}}$ for $r=2$ to n.

For c From 2 to $n+1$.

Set $a_{r,c}^{(2)} = a_{r,c}^{(1)} - m_{r,1}\, a_{1,c}^{(1)}$

we get

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1,n}^{(1)} & a_{1(n+1)}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & a_{32}^{(2)} & \cdots & a_{3,n}^{(2)} & a_{3,n+1}^{(2)} \\ \vdots & \vdots & & & \vdots \\ 0 & a_{n,2}^{(2)} & \cdots & a_{n,n}^{(2)} & a_{n,n+1}^{(2)} \end{array}\right]$$

in general

P+1 step find $a_{p,p}^{(p)} \neq 0$ From $r = p+1$ to N

$m_{r,p} = \dfrac{a_{r,p}^{(p)}}{a_{p,p}^{(p)}}$ and $a_{r,p}^{(p+1)} = 0$

For $c = p+1$ to $n+1$

$a_{r,c}^{(p+1)} = a_{r,c}^{(p)} - m_{r,p} * a_{p,c}^{(p)}$

we have 3 loop

- Error

$0.37205 * (7) = 2.60435 \approx 2.6044$ نضرب بعدد صغير امن

$0.12345 * (7) = 0.86415 \approx 0.86415$ من الضرب بعدد كبير

- Gaussian elimination with pivoting:-

to avoid propagation of error we use the pivot element to be the largest in the remaining of the coloumn in $|a_{k,p}| = \max[|a_{pp}|, |a_{p+1,p}|$ $\cdots |a_{n-1,p}|, |a_{n,p}|]$ and switch row p with row k if $k > p$

Example:-

$(1.000, 1.000)$ is a solution to
$$1.133x_1 + 5.281x_2 = 6.414$$
$$24.14x_1 - 1.210x_2 = 22.93$$

solve the above by Gaussian with pivoting and without poviting.

- without pivoting

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & 1.210 & 22.93 \end{array}\right] \quad m_{21} = \frac{24.14}{1.133} = 21.31$$

$$\rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right] \quad \begin{array}{l} x_2 = 1.001 \\ x_1 = 0.9956 \end{array}$$

with pivoting

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & \cancel{22.93}\ 6.414 \end{array}\right] \quad m_{21} = \frac{1.133}{24.14} = 0.0464$$

$$\rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right] \quad \begin{array}{l} x_1 = 1.000 \\ x_2 = 1.000 \end{array}$$

$Ax = b$

1. Gaussion $[A \backslash b] \longrightarrow [U \backslash c]$ + backsubsitution

2. Gauss - Jordan Elemination

$$\left[\begin{array}{ccc|c} *^{1} & *^{(1)} & *^{(1)} & *^{(1)} \\ *^{0} & *^{(1)} & *^{(1)} & *^{(1)} \\ *^{0} & *^{(1)} & *^{(1)} & *^{(1)} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & *^{(3)} & *^{(3)} \\ 0 & 1 & *^{(2)} & *^{(3)} \\ 0 & 0 & *^{(3)} & *^{(3)} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & *^{(4)} \\ 0 & 1 & 0 & *^{(4)} \\ 0 & 0 & 1 & *^{(4)} \end{array}\right]$$

Step	+/-	*/÷
1	3×2	$3 + 3 \times 2$
2	2×2	$2 + 2 \times 2$
3	1×2	$1 + 1 \times 2$
⋮ n		

Solve

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ x_1 - 2x_2 + 3x_3 &= 2 \\ 3x_1 + 4x_2 - x_3 &= 6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & -8/3 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & 1 & -5/8 & 3/8 \\ 0 & 2 & -5 & -3 \end{array}\right]$$

Exercise

Find the total coast for Gauss Jord elimination.

3. Inverse method.

$[A \backslash I] \rightarrow [I \backslash A^{-1}]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} * & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \end{array}\right]$$

$Ax = b$

$x = A^{-1} b$

multiplication coast $= 2n^2 - n$.

Coast

Step	+/-	*/÷
1	5x2	5 + 5x2
2	4x2	4 + 4x2
3	3x2	3 + 3x2
⋮	$(2n-p)(n-1)$	$(2n-p)(n-1) + (2n-p)$

$\rightarrow$

Step	+/-	*/÷
1	$(2n-1)\times(n-1)$	$(2n-1)+(2n-1)(n-1)$
2	$(2n-2)(n-1)$	$(2n-2)+(2n-2)(n-1)$
p	$(2n-p)(n-1)$	$(2n-p)+(2n-p)(n-1)$
n	$n(n-1)$	$n+n(n-1)$

coast $= \frac{16n^3 - 9n^2 - n}{6} \approx 2\frac{2}{3}n^3$

4. Cramer's method

$x_i = \frac{|A_i|}{|A|}$

Find the coast of Cramer's method for 3*3 matrix.

$x_1 = \frac{|A_1|}{|A|}$

$x_2 = \frac{|A_2|}{|A|}$

$x_3 = \frac{|A_3|}{|A|}$

$4 \underset{\text{determinant}}{|\ |_{3\times3}} + \underset{قسمة}{3}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$4 \times 3 + \underset{ضرب}{2} = 14$

Coast $= 4 \times (14) + 3$
$= 59$

3.6 L-U Factorization

$$Ax = b$$

$$LUx = b$$

1. $LY = b \longrightarrow$ Forward Subsitution
2. $UX = Y \longrightarrow$ backward subsitution (coast $= n^2$).

$$[A] \longrightarrow \underbrace{\begin{bmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots a_{2n}^{(2)} \\ 0 & \cdots & a_{nn}^{(n)} \end{bmatrix}}_{U} \quad L = \begin{bmatrix} 1 & & & \\ m_{12} & 1 & 0 & \\ m_{31} & m_{32} & 1 & \\ \ddots & & & 1 \end{bmatrix}$$

Ex:-

Solve Using L-U Factorization.

$$4x_1 + 3x_2 - x_3 = -2$$
$$-2x_1 + 4x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + 6x_3 = 7$$

no switch in Row

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & -2 \\ -2 & -4 & 5 & \\ 1 & 2 & 6 & \end{array}\right] \quad m_{21} = \frac{-2}{4} = -\frac{1}{2}, \quad m_{31} = \frac{1}{4} = 0.25$$

$$\xrightarrow[R_3 - 0.25R_1]{R_2 + 0.5R_1} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} \quad m_{32} = \frac{1.25}{-2.5} = -0.5$$

$$\xrightarrow{R_3 + 0.5R_1} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} = U \qquad \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 0 \end{bmatrix} = L$$

$$LY = b.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \\ 7 \end{bmatrix} \qquad \begin{aligned} y_1 &= -2 \\ y_2 &= 19 \\ y_3 &= 17 \end{aligned}$$

Forward subsitution

$Ux = y$.

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 19 \\ 17 \end{bmatrix}$$

$$x_3 = \frac{17}{8.5} = 2$$

$$x_2 = -4$$

$$x_1 = 3$$

• total Coast =

$$\underbrace{\frac{n^3-n}{3}}_{\text{Row operations (multiplication)}} + \underbrace{n^2}_{\text{backsubsitution}} + \underbrace{n^2-n}_{\text{Forward subsitution}} + \underbrace{\frac{2n^3-3n^2+n}{6}}_{\text{Row operation (addition)}}$$

$$= \frac{2n^3-2n}{6} + \frac{6n^2}{6} + \frac{6n^2-6n}{6} + \frac{2n^3-3n^2+n}{6}$$

$$= \frac{4n^3+9n^2-7n}{6}$$

step	+/-	* / ÷
1	(n-1)(n-1)	(n-1) + (n-1)(n-1)
2	(n-2)(n-2)	(n-2) + (n-2)(n-2)
p	$(n-p)^2$	$(n-p) + (n-p)^2$

$$\text{total} = \sum (n-p)^2 + \sum (n-p) + (n-p)^2$$

$$= 2\sum (n-p)^2 + \sum (n-p)$$

$$= \frac{2(n-1)(n)(2n-1)}{6} + \frac{(n-1)(n)}{2}$$

$$= \frac{2(2n^3-3n^2+n)}{6} + \frac{n^2-n}{2}$$

$$= \frac{4n^3-6n^2+2n+3n^2-3n}{6}$$

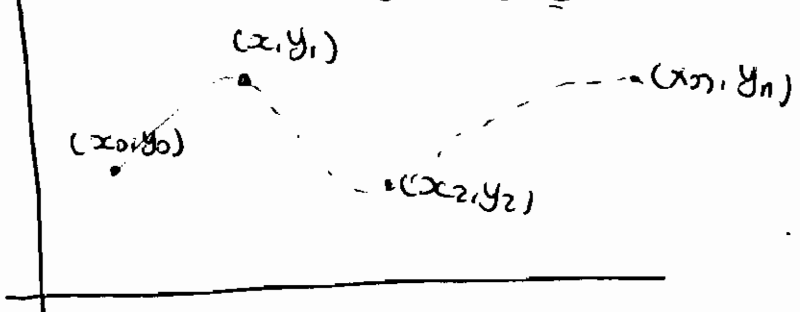
$$= \frac{4n^3-3n^2-n}{6}$$

Interpolation by polynomials:-

given (x_0, y_0) (x_1, y_1) (x_2, y_2) ---- (x_n, y_n)

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
⋮	⋮
x_n	y_n

ماهي العلاقة (الاقتران) بين هذه النقاط



حتى نعرف الاقتران نعمله approximation

Curve Fitting أحسن خط يمر بين النقط بحيث تكون المسافة بينه النقط اصغر ما يمكن

interpolation :- is estimation of the unknown Function by poly which passes through all given points

$$\boxed{P_n(x_i) = f(x_i)}$$

P_n is the approximation polynomial
f is the unknown Function

n من الدرجة Polonomial ← $(n+1)$ اذا عندنا points

Example $(1,2)$, $(3,5)$, $(7,10)$

ساقطا الاقتران تربيعي ←

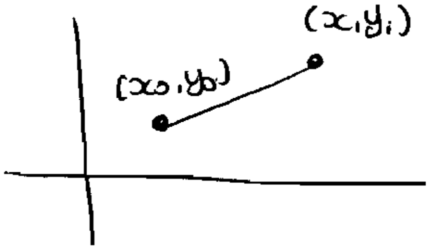
$P_2(x) = Ax^2 + Bx + C$
$P_2(1) = A + B + C = 2$
$P_2(3) = 9A + 3B + C = 5$
$P_2(7) = 49A + 7B + C = 10$

- given $(x_0, y_0)\ (x_1, y_1)\ \dots\ (x_n, y_n)$.

  we need to find the polynomial $P_n(x)$ which satisfies

$$\boxed{P_n(x_i) = y_i \ , \quad i = 0, \dots, n}$$

- given $(x_0, y_0), (x_1, y_1)$.



$$\text{slope} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \longrightarrow y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$P_n(x) = \underbrace{\frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}}_{L_{n,0}} y_0 + \underbrace{\frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}}_{L_{n,1}} y_1$$
$$- - - - + \underbrace{\frac{(x - x_0)(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}}_{L_{n,k}} y_k$$

$L_{n,k}$ → Lagrange coefficient polynomial

$$P_1(x) = \sum_{k=0}^{1} L_{1,k}(x)\, y_k$$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k \leftarrow \text{Lagrange Polynomial}$$

Proof $\rightarrow$ $\boxed{P_n(x_i) = y_i}$

$n=1 \rightarrow \quad P_1(x_0) \overset{??}{=} y_0$

$P_1(x_0) = y_0$

$P_1(x_1) = y_1$

$n=2 \rightarrow$

$P_2(x_0) = y_0$

$P_2(x_1) = y_1$

$P_2(x_2) = y_2$

828

85

$n=k \rightarrow \boxed{P_n(x_k) = y_k}$

Examples:-

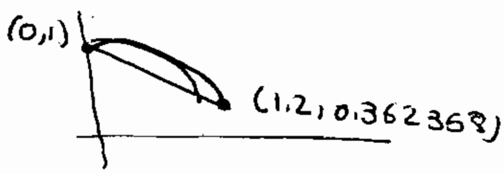
given $f(x) = \cos x$ on $[0, 1.2]$

Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and Compare the answers $P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ to the exact.

to Find $P_1(x)$

$(x_0, y_0) = (0, \cos 0) = (0, 1)$

$(x_1, y_1) = (1.2, \cos 1.2) = (1.2, 0.362358)$



$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$= \frac{x - 1.2}{0 - 1.2}(1) + \frac{x - 0}{1.2 - 0}(0.362358)$$

$$P_1(x) = -0.833333(x - 1.2) + 0.301965\,x$$

$$P_1(0.35) = -0.833333(0.35 - 1.2) + 0.301965(0.35)$$

$$= 0.8140208$$

exact $= \cos(0.35) = 0.9393727$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$(0, \cos 0)$, $(0.6, \cos 0.6)$, $(1.2, \cos(1.2))$

$(0, 1)$, $(0.6, 0.825336)$, $(1.2, 0.362258)$

$$P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)}(1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)}(0.825336) + \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}(0.362258)$$

$$= 1.38889(x-0.6)(x-1.2) - 2.292599x(x-1.2) + 0.503275x(x-0.6)$$

$P_2(0.35) = 0.923315 0528$

$h = \frac{1.2-0}{3} = 0.4$

$(0, 1), (0.4, 0.921061), (0.8, 0.696707), (1.2, 0.362258)$

$$P_3(x) = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)}(1) + \frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)}(0.921061)$$

$$+ \frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)}(0.696707) + \frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}(0.362258)$$

$$= -2.60417(x-0.4)(x-0.8)(x-1.2) \ldots$$

$$= 0.939607167$$

4.3 Langrange interpolating polynomial.

given $(x_0, y_0)\ (x_1, y_1)\ ----, (x_n, y_n)$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k = L_{n,0}(x)\, y_0 + L_{n,1}(x)\, y_1 + \cdots + L_{n,n}(x)\, y_n$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)(x-x_2) \cdots (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_n)}{(x_k-x_0)(x_k-x_1)(x_k-x_2) \cdots (x_k-x_{k-1})(x_k-x_{k+1}) \cdots (x_k-x_n)}$$

Theory :-

if $f(x) = P_n(x) + E_n(x)$

$$E_n(x) = \frac{(x-x_0)(x-x_1) \cdots (x-x_n)}{(n+1)!} f^{(n+1)}(c)$$

for the previous example

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

$(0,1), (0.6, 0.825336), (1.2, 0.362326)$.

$$E_2(x) = \frac{x(x-0.6)(x-1.2)}{6} f^{(3)}(c).$$

$$E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(c)$$

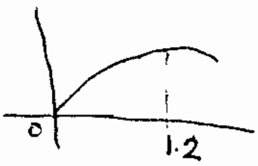
$$|E_2(x)| \leq \left|\frac{x(x-0.6)(x-1.2)}{6}\right| \max_{x_0 \leq x \leq x_n} |f^{3}(c)|$$

$f(x) = \cos x \quad [0, 1.2]$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$.



$$\max_{0 \leq x \leq 1.2} |f'''(x)| = \sin(1.2) = 0.9320$$

$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6} (0.9320).$$

$$|E_2(0.35)| \leq \left\{ \frac{0.35(0.25)(0.85)}{6} (0.9320)\right.$$

$$= 0.01155$$

$P_2(0.35) = 0.93315$.

Exact $= 0.9393727$.

Error $= 0.0062$

Find an upperbound for $E_2(x)$ for all x.

نشتق و نعوض المشتقة بالصفر ← نوجد x_{max}

ونعوض ← max upperbound.

$g'(x) = 0 \longrightarrow$ uniform bound.

<u>Theorem:-</u> <u>Uniform bound</u>

For <u>uniform partition</u>. $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$ n: integer

$$\longrightarrow x_k = x_0 + kh$$
$$x_n = x_0 + nh$$

Let $M_n = \max\limits_{a \leq x \leq b} |f^{(n)}(x)|$

then 1. $|E_1(x)| \leq \frac{h^2 M_2}{8}$ for all $x \in [x_0, x_1]$

2. $|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$ for all $x \in [x_0, x_2]$

3. $|E_3(x)| \leq \frac{h^4 M_4}{24}$ for all $x \in [x_0, x_3]$

Ex:-

Using the theorem for the previous example

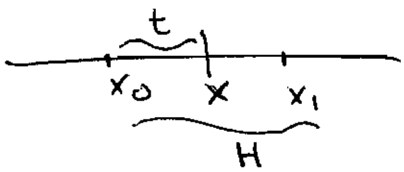
$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.6)^3(0.9320)}{9\sqrt{3}} = 0.03587$$

$\downarrow$

upper bounded for all x

Proof $\quad |E_1(x)| \leq \dfrac{h^2 M_2}{8}$ for all $x \in [x_0, x_1]$

to show $|E_1(x)| \leq \dfrac{h^2 M_2}{8}$.



$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2!} M_2$$

$$h(x) = (x-x_0)(x-x_1).$$

let $t = x - x_0$

$$H(x) = t(x-x_1)$$
$$= t(t-h)$$

$x - x_1$ *
$= (x_0+t) - (x_0+h)$

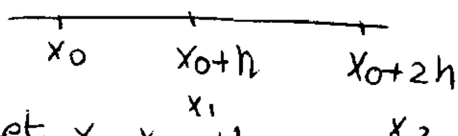
$$H(t) = t(t-h).$$
$$H(t) = t^2 - th$$
$$H'(t) = 2t - h = 0$$
$$t = \frac{h}{2} \longrightarrow \text{critical point}$$

$$H(\cancel{t}\,x) = \frac{h}{2}\left(\frac{h}{2} - h\right)$$
$$= \frac{h}{2}\left(-\frac{h}{2}\right) = -\frac{h^2}{4}$$

$$\max|H(x)| = \frac{h^2}{4}$$

$$E_1(x) \leq \frac{h^2}{4} \cdot \frac{M_2}{2} = \frac{h^2 M_2}{8}$$

proof $\quad |E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$



let $x = x_0 + th$

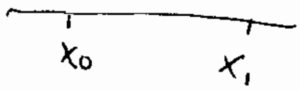
$0 \leq t \leq 2$

Theorem

$f(x) = P_n(x) + E_n(x)$.

then

$$E_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(c).$$

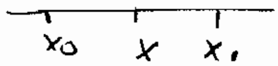


proof: for $n=1$.

To ~~should~~ show that the error.

$$E_n(x) = f(x) - P_n(x).$$

is equal to $\dfrac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c)$.



let $h(t) = f(t) - P_1(t) - E_1(x)\dfrac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$.

$h(x)$ are continous and differentiable.

$$h(x_0) = f(x_0) - P_1(x_0) - E_1(x)\frac{\cancel{(x_0-x_0)}^{0}(x_0-x_1)}{\cancel{(x-x_0)}(x-x_1)}$$

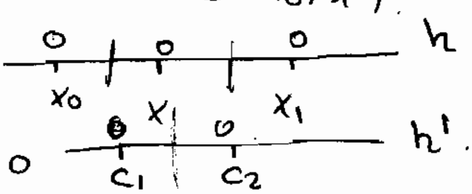
$$h(x_1) = f(x_1) - P_1(x_1) - 0 = 0$$

$$h(x) = f(x) - P_1(x) - E_1(x)\frac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)}$$

$$= \underbrace{f(x) - P_1(x)}_{E_1(x)} - E_1(x)$$

$$h(x) = 0.$$

Using MVT on (x_0, x), $\exists\, c \in (x_0, x)$ such that



$$h'(c_1) = \frac{h(x) - h(x_0)}{x - x_0} = 0$$

Similarly $\exists\, c_2 \in (x, x_1)$ such that

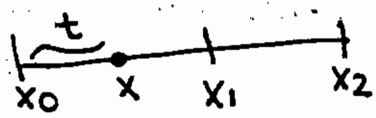
$$h'(c_2) = \frac{h(x_1) - h(x)}{x_1 - x} = 0$$

Similarly $\exists\, c \in (c_1, c_2)$ such that

$$h''(c) = \frac{h'(c_2) - h'(c_1)}{c_2 - c_1} = \frac{0-0}{c_2 - c_1} = 0.$$

- proof $\quad |E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$

$$x = x_0 + t$$
$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h$$



$$|E_2(x)| = \frac{(x-x_0)(x-x_1)(x-x_2) f^{3}(c)}{3!}$$

$$|E_2(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2) M_3}{6}$$

$$x - x_0 = t$$
$$x - x_1 = t - h$$
$$x - x_2 = t - 2h$$

$$|E_2(x)| \leq \frac{(t)(t-h)(t-2h) M_3}{6}$$

$$\phi(t) = t(t-h)(t-2h)$$
$$= (t^2 - th)(t-2h)$$
$$= t^3 - 2ht^2 - ht^2 + 2h^2 t = t^3 - 3ht^2 + 2h^2 t$$
$$\phi'(t) = 3t^2 - 6ht + 2h^2$$
$$\phi'(t) = 0$$
$$t = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 2}}{6} h$$
$$t = 0.4226497 3 h$$
$$t = 1.577350269 h$$

$$\phi(t) = 0.384900179 h^3$$

$$|E_2(x)| \leq \frac{0.384900179 h^3 M_3}{6}$$

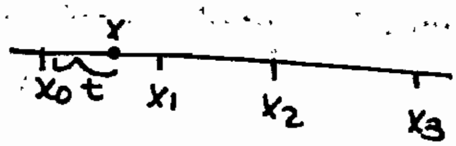
$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$$

• proof $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{4}(c)}{4!}$$

$$|E_3(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, M_4}{24}$$

$x = x_0 + t$
$x_1 = x_0 + h$
$x_2 = x_0 + 2h$
$x_3 = x_0 + 3h$



$x - x_0 = t$
$x - x_1 = t - h$
$x - x_2 = t - 2h$
$x - x_3 = t - 3h$

$$|E_3(x)| \leq \frac{(t)(t-h)(t-2h)(t-3h)\, M_4}{24}$$

Let $\phi(t) = t(t-h)(t-2h)(t-3h)$
$= (t^2 - th)(t-2h)(t-3h)$
$= (t^3 - 2t^2h - t^2h + 2th^2)(t-3h)$
$= (t^3 - 3t^2h + 2th^2)(t-3h)$
$= t^4 - 6ht^3 + 11h^2t^2 - 6h^3t$

$\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$

$\phi'(t) = 0$

$t = 2.618033989h$
$= 0.381966011h$
$= 0.05h$

$\phi(t) = 1 - 11 = 1$

For $\phi(t)$ the max $= h^4$ @ $t = 2.618033989h$

$$|E_3(x)| \leq \frac{h^4 M_4}{24}$$

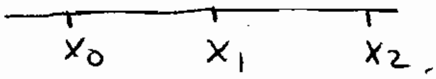
$$h'(t) = f'(t) - P_1'(t) - E_1(x)\left(\frac{(t-x_0)+(t-x_1)}{(x_0-x_1)(x-x_1)}\right)$$

$$h''(t) = f''(t) - \underset{\substack{P_1''(t) \text{ because} \\ \text{the function} \\ \text{is linear}}}{0} - \frac{E_1(x)(2)}{(x_0-x_1)(x-x_1)}$$

$$\rightarrow \quad h''(c) = f''(c) - \frac{2E_1(x)}{(x-x_0)(x-x_1)} = 0.$$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2} f''(c).$$

<u>for $n=2$</u>



exercise

$$h(t) = f(t) - P_2(t) - E_2(x)\frac{(t-x_0)(t-x_1)(t-x_2)}{(x-x_0)(x-x_1)(x-x_2)}.$$

$$h(x_0) = f(x_0) - P_2(x_0) - 0$$

$$h(x_1) = f(x_1) - P_2(x_1) - 0.$$

$$h(x_2) = f(x_2) - P_2(x_2) - 0.$$

$$h(x) = f(x) - P_2(x) - E_2(x).(1).$$

$$= 0$$

4.4 Newton interpolation polynomial

given $x_0, x_1, x_2, \dots, x_n$.

$(x_0, y_0), (x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$.

$$P_n(x_i) = f(x_i)$$

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1).$$

$$P_3(x) = P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2).$$

$$\vdots$$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}).$$

$$P_1(x_0) = a_0 + a_1(x_0 - x_0).$$

$$P_1(x_0) = a_0 = f(x_0) = y_0 \longrightarrow \boxed{a_0 = y_0}$$

$$P_1(x_1) = \underbrace{f(x_0)}_{= a_0} + a_1(x_1 - x_0) = f(x_1).$$

$$\boxed{a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}} = F[x_0, x_1]$$ First divided differance.

$$f(x_2) = P_2(x_2) = f(x_0) + F[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1).$$

$$a_2 = \frac{f(x_2) - f(x_0) - f[x_0, x_1)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}.$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = F[x_0, x_1, x_2].$$

Then

$$a_k = f[x_0, x_1, x_2 \dots x_k]$$ i'st k^{th} divided difference.

Definition

$F[x_k] = f(x_k)$ zeroth divided difference.

$$F[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$ 1st divided differance.

2nd divided difference.

$$F[x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-1}, x_k] - f[x_{k-2}, x_k]}{x_k - x_{k-2}}$$

3rd divided difference

$$F[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-2}, x_{k-1}, x_k] - F[x_{k-3}, x_{k-2}, x_k]}{x_k - x_{k-3}}$$

example:

$(1,3), (2,5), (4,7), (8,11), (9,15)$

$$f[2,4,8] = \frac{f[4,8] - f[2,4]}{8-2} = \frac{\frac{f(8)-f(4)}{8-4} - \frac{f(4)-f(2)}{4-2}}{6}$$

$$= \frac{\frac{11-7}{4} - \frac{7-5}{2}}{6} = 0$$

$$f[1,2,4,8] = \frac{f[2,4,8] - f[1,2,4]}{8-1}$$

$$= \frac{\frac{f(4,8) - f(2,4)}{8-2} - \left[\frac{f(2,4) - f(1,2)}{4-1}\right]}{8-1}$$

another way

	$f(x_k)$	1'st divided	2nd divided	3rd divided
x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$	$F[x_{k-3}, x_{k-2}, x_{k-1}, x_k]$
x_0	$f(x_0)$ a_0			
x_1	$f(x_1)$	$f[x_0, x_1]$ a_1		
x_2	$f(x_2)$	$f[x_1, x_2]$	$f[x_0, x_1 x_2]$ a_2	
x_3	$f(x_3)$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	a_3 $F[x_0, x_1, x_2, x_3]$
x_4	$f(x_4)$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$F[x_1, x_2, x_3, x_4]$
x_5	$f(x_5)$	$f[x_4, x_5]$	$f[x_3, x_4, x_5]$	$F[x_2, x_3, x_4, x_5]$
x_6	$f(x_6)$	$f[x_5, x_6]$	$f[x_4, x_5, x_6]$	$F[x_3, x_4, x_5, x_6]$
⋮				

example

Find Newton interp $P_1, P_2, P_3, P_4, \ldots$ for the following table

x_k	$f(x_k)$	1'st	2'nd	3rd.	4th
1	-3 a_0				
2	0	3 a_1			
3	15	15	6 a_2		
4	48	33	9	1 a_3	
5	105	57	12	1	0
6	192	87	15	1	0

$P_1(x) = a_0 + a_1(x - x_0)$
$= -3 + 3(x-1)$

$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$
$= -3 + 3(x-1) + 6(x - x_0)(x-2)$

$P_3(x) = P_2(x) + 1(x-1)(x-2)(x -$

$P_4 = P_3$

$P_5 = P_4 = P_3$

note:-

Error for newton interpolation polynomial equal to the Error for laggrage because they uses the same Polynomial.

Example:-

Estimate $f(5.5)$ using Newton int polynomial P_1, P_2, P_3, P_4 for the Following table.

x_k	$f(x_k)$	1st	2nd	3rd	4th
1	3				
3	4.5	0.75			
4.25	6		0.138462		
5.75	7.25			0.05722	
6	8				0.10287

$P_1(x) = a_0 + a_1(x - x_0)$
$= 3 + 0.75(x-1)$.

$f(5.5) \approx P_1(5.5)$
$= 3 + 0.75(5.5-1)$
$= 6.375$.

$P_2(x) = P_1(x) + a_2(x-1)(x-3)$
$= 0.13846(x-1)(x-3) + P_1(x)$

$f_2(5.5) \approx P_2(5.5)$
$\approx 6.375 + 0.13846(5.5-1)(5.5-3)$
≈ 7.93267

$P_3(x) = P_2(x) + 0.05722(x-1)(x-3)(x-4.25)$

$f_3(5.5) \approx P_3(5.5)$
$\approx 7.93267 + 0.05722(5.5-1)(5.5-3)(5.5-4.25)$
≈ 8.7373

$$P_4(x) = P_3 + 0.10287\,(x-1)(x-3)(x-4.25)(x-5.75)$$

$$
\begin{aligned}
f(5.5) &\approx P_4(5.5) \\
&\approx 8.7373 + 0.10287\,(4.5)(2.5)(1.25)(-0.25) \\
&\approx 8.375.
\end{aligned}
$$

Chapter Five

5.1+5.2

Best Fit

given $(x_0,y_0),(x_1,y_1)\ \dots\ (x_n,y_n)$ to Find the best Fitting curve

i.e

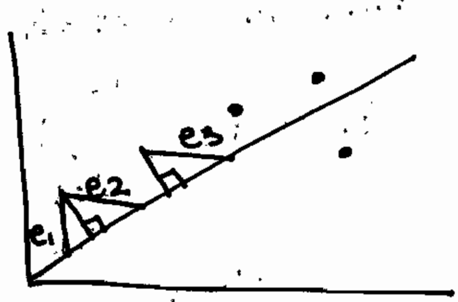
the curve with smallest distance to the given point

if $e_k=f(x_k)-y_k$

max error $E_\infty(F)=\|f\|_\infty=\max\limits_{0\le k\le 1}|e_k|$

Avarge error $=E_1(f)=\|f\|_1=(\sum|e_k|)/n$

Root Mean Square error $=E_2(f)=\|f\|_2=(\sum|e_k|^2/n)^{1/2}$



Example 5.1

Compare the max error, Avarge error and RMS error for the linear approximation $f(x)=-1.6x+8.6$ to the data $(-1,10)\ (0,9)\ (1,7)\ (2,5)\ (3,4)\ (4,3)\ (5,0)\ (6,-1)$

x_k	y_k	$f(x_k)$	$\lvert e_k\rvert$	e_k^2
-1	10	10.2	0.2	0.04
0	9	8.6	0.4	0.16
1	7	7	0	0
2	5	5.4	0.4	0.16
3	4	3.8	0.2	0.04
4	3	2.2	0.8	0.64
5	0	0.6	0.6	0.36
6	-1	-1	0	0

$\sum e_k=2.6$

$E_\infty(F)=0.8$

$E_1(F)=\dfrac{\sum|e_k|}{n}=\dfrac{2.6}{8}=0.325$

$\sum e_k^2=1.4$

$E_2(f)=\left(\dfrac{\sum e_k^2}{n}\right)^{1/2}=0.42$

- to Find the best Fitting curve we need to minimize the least square error (RMS)

$$E_2(f) = \left(\frac{\sum_{k=1}^{n} |f(x_k) - y_k|^2}{n} \right)^{1/2}$$

$$n\, E_2^2(f) = \sum_{k=1}^{n} (f(x_k) - y_k)^2$$

$$E(?) = \sum_{k=1}^{n} (f(x_k) - y_k)^2$$

1. To Find The best Fitting line $f(x) = Ax + B$

$$E(A,B) = \sum_{k=1}^{n} |(Ax_k + B) - y_k|^2$$

$$\frac{dE}{dA} = \sum_{k=1}^{n} 2|(Ax_k + B) - y_k| \cdot x_k = 0 \quad --- (1)$$

$$\frac{dE}{dB} = \sum_{k=1}^{n} 2|(Ax_k + B) - y_k| \cdot 1 = 0 \quad --- (2)$$

(1) من $\quad A\sum_{k=1}^{n} x_k^2 + B\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k x_k \; --$

(2) من $\quad A\sum_{k=1}^{n} x_k + nB = \sum_{k=1}^{n} y_k \; ---$

$\rightarrow$ Normal equations

Example :-

Find the best Fitting line $F(x) = Ax + B$ For the data

$(-1,10)$ $(0,9)$ $(1,7)$ $(2,5)$ $(3,4)$ $(4,3)$ $(5,0)$ $(6,-1)$.

	x_k	y_k	x_k^2	$x_k y_k$
	-1	10	1	-10
	0	9	0	0
	1	7	1	7
	2	5	4	10
	3	4	9	12
	4	3	16	12
	5	0	25	0
	6	-1	36	-6
Σ	20	37	92	25

$$92A + 20B = 25$$
$$20A + 8B = 37$$

$$A = \frac{\begin{vmatrix} 25 & 20 \\ 37 & 8 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \cong -1.61$$

$$B = \frac{\begin{vmatrix} 92 & 25 \\ 20 & 37 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \cong 8.64$$

Example

for the following Data Find the best curve of the Form

$$y = Ax^2$$

$$E(A) = \sum_{k=1}^{n} (A x_k^2 - y_k)^2$$

$$\frac{dE}{dA} = 2\sum_{k=1}^{n} (A x_k^2 - y_k) \cdot x_k^2 = 0$$

$$A = \frac{\sum_{k=1}^{n} y_k x_k^2}{\sum_{k=1}^{n} x_k^4}$$

التكملة مثل السابق :

$$A = \frac{85}{2276} = 0.037346$$

Example

Find the best Fitting parrabola $f(x) = Ax^2 + Bx + C$

$$E(A,B,C) = \sum_{k=1}^{n} \left[(Ax_k^2 + Bx_k + C) - y_k \right]^2$$

$$\frac{dE}{dA} = 0 = 2\sum_{k=1}^{n} \left[(Ax_k^2 - Bx_k + C) - y_k \right] \cdot x_k^2$$

$$\frac{dE}{dB} = 0 = 2\sum_{k=1}^{n} \left[(Ax_k^2 + Bx_k + C) - y_k \right] \cdot x_k$$

$$\frac{dE}{dC} = 0 = 2\sum_{k=1}^{2} \left[(Ax_k^2 + Bx_k + C) - y_k \right] \cdot 1$$

5.2

linearization

$$f(x) \longrightarrow Ax+B$$

Examples:-

Find the best Fitting curve of the form $f(x) = Ce^{Dx}$ for the following table. $(0,1.5), (1,2.5), (2,3.5), (3,5), (4,7.5)$

$$y = ce^{Dx}$$

$$\ln y = \ln c + Dx$$

$$\ln y = Dx + \ln c$$

$$Y = AX + B.$$

$$Y = \ln y$$

$$X = x$$

$$D = A$$

$$c = e^{B}$$

x_k	y_k	X_k	$Y_k = \ln y_k$	X_k^2	$X_k Y_k$
0	1.5	0	0.405465	0	0
1	2.5	1	0.916291	1	0.916291
2	3.5	2	1.25..	4	2.5
3	5	3	1.6	9	4.82813
4	7.5	4	2.01	16	8.059..
Σ		10	6.198860	30	16.309742

table 5.4 From the text book.

$$30A + 10B = 16.309742$$

$$10A + 5B = 6.198860$$

$$A = \frac{\begin{vmatrix} 16.309742 & 10 \\ 6.198860 & 5 \end{vmatrix}}{\begin{vmatrix} 30 & 10 \\ 10 & 5 \end{vmatrix}} = 0.3912023$$

$$B = 0.457367.$$

$$D = A \cong 0.39$$

$$C = e^{B} = e^{0.457367} \cong 1.58$$

$$f(x) = 1.58\, e^{0.39x} = Ce^{Dx}.$$

Example:-

1. $y = \dfrac{D}{x+c}$

$$\frac{1}{y} = \frac{x}{D} + \frac{c}{D}.$$

$$Y = AX + B$$

$Y = \dfrac{1}{y}$, $X = x$, $A = \dfrac{1}{D}$, $B = \dfrac{c}{D}.$

$A = \dfrac{1}{D} \downarrow D = \dfrac{1}{A}$ $\quad$ $B = \dfrac{c}{D} \downarrow c = BD$

$$y = \frac{D}{x+c}$$

$$y = \frac{1}{x} D + \frac{D}{c}$$

$A = D/x$

$Ax = D/x$

2. $y = \dfrac{x}{A+Bx}$

$$\frac{1}{y} = \frac{A}{x} + B.$$

$$Y = AX + B.$$

$Y = \dfrac{1}{y}$

$X = \dfrac{1}{x}.$

$A = A$

$B = B.$

3. $y = cxe^{-Dx}$

$$\frac{y}{x} = c\,e^{-Dx}$$

$$\ln\left(\frac{y}{x}\right) = \ln c - Dx$$

$$\ln\left(\frac{y}{x}\right) = -Dx + \ln c$$

$$Y = A\cancel{X} + B$$

$Y = \ln\left(\dfrac{y}{x}\right)$

$X = x$

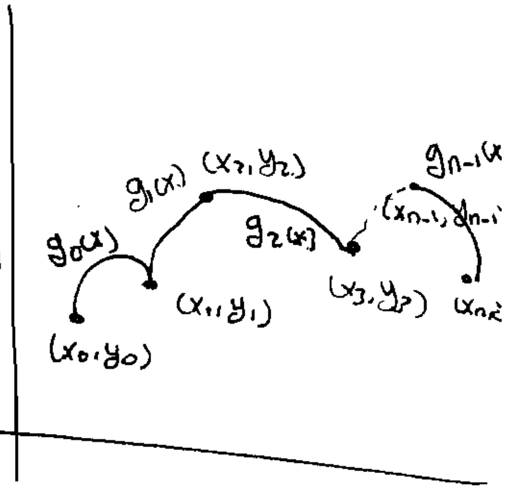
$A = -D \rightarrow D = -A$

$B = \ln c \rightarrow c = e^{B}.$

Section 5.3
Cubic spline

given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

The cubic spline is a Function $g(x)$ such that it is a cubic polynomial between every two nodes and its of this Form $g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$ on $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, n-1$ and that satisfies



1. $g_i(x_i) = y_i \quad i = 0, 1, \dots, n-1 \;,\; g_{n-1}(x_n) = y_n$

   $(n+1)$ conditions.

if I have n Function $\longrightarrow$ $4n$ unknowns.

2. $g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \quad i = 0, \dots, n-2$

   $(n-1)$ conditions

   $g_0(x_1) = g_1(x_1)$

   $g_1(x_2) = g_2(x_2)$

   $g_{n-2}(x_{n-1}) = g_{n-2}(x_n)$.

- continous
- نفس المشتقة
- نفس التقعر

3. $g_i'(x_{i+1}) = g_{i+1}'(x_{i+1}) \quad i = 0, \dots, n-2 \rightarrow (n-1)$ condition

4. $g_i''(x_{i+1}) = g_{i+1}''(x_{i+1}) \quad i = 0, \dots, n-2 \longrightarrow (n-1)$ condition.

so we have $(n+1) + (3(n-1)) = 4n-2$ conditions.

$\rightarrow$ ~~eq~~ Since $g_i(x_i) = y_i \Rightarrow$
$d_i = y_i$

$\longrightarrow$ equation (2) gives

$$y_{i+1} = g_{i+1}(x_{i+1}) = a_i(x_{i+1}-x_i)^3 + b_i(x_{i+1}-x_i)^2 + c_i(x_{i+1}-x_i) + d_i$$

$$* \quad = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$$

where $h_i = (x_{i+1} - x_i)$

$g_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i$

$$g_i''(x) = 6a_i(x - x_i) + 2b_i$$

if $\boxed{S_i = g_i''(x_i)}$

Subsitute $\longrightarrow$ $\boxed{b_i = \frac{S_i}{2}}$

Using the same equation.

$$g_i''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i$$

$$S_{i+1} = 6a_i h_i + S_i$$

$$\boxed{a_i = \frac{S_{i+1} - S_i}{6h_i}}$$

Subsitute in $*$

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_iS_i + h_iS_{i+1}}{6}$$

- Considering the equation

$$g_i''(x_i) = g_{i+1}''(x) \text{ we get}$$

$$h_{i-1}S_{i-1} + 2(h_{i-1} + h_i)S_i + h_iS_{i+1} = 6\left[f(x_i, x_{i+1}) - f[x_{i-1}, x_i]\right)$$

for $i = 1, \dots, n-1$

$$\begin{array}{c c}
 & \begin{array}{cccccccc} S_0 & S_1 & S_2 & S_3 & \cdots & S_{n-2} & S_{n-1} & S_n \end{array} \\
\begin{array}{c} i=1 \\ i=2 \\ i=3 \\ \\ \\ \\ \end{array} &
\left[\begin{array}{cccccccc}
h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & h_2 & 2(h_2+h_3) & \cdots & 0 & 0 & 0 \\
 & & & \ddots & & & & \\
 & & & & h_{n-3} & 2(h_{n-3}, h_{n-2}) & h_{n-2} & 0 \\
 & & & & & h(n-2) & 2(h_{n-2}, h_{n-1}) & h_{n-1}
\end{array}\right]
\end{array}
\left[\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{array}\right]
= 6
\left[\begin{array}{c} F[x_1, x_2] - F[x_0, x_1] \\ F[x_2, x_3] - F[x_1, x_2] \\ F[x_3, x_4] - F[x_2, x_3] \\ \vdots \\ F[x_n, x_{n+1}] - F[x_{n-1}, x_n] \end{array}\right]$$

$(n-1)$ equations X $(n+1)$ unknown, we need two more condition.

1. Natural Spline $S_0 = S_n = 0$

we get $(n-1)$ equations with $(n-1)$ Unknowns

When $n=1$ X ما بينفع لأنه مفروض على الجهتين

$$\begin{array}{cc} 0 & 0 \\ \hline x_0 & x_1 \end{array}$$

لا يوجد matrix لكن يوجد معادلة.

when $n=2$

$$\begin{array}{ccc} S_0=0 & S_1 & S_2=0 \\ \hline x_0 & x_1 & x_2 \end{array}$$

$$2(h_0+h_1)S_1 = 6[f[x_1,x_2] - f[x_0,x_1]].$$

when $n=3$

$$\begin{array}{cccc} S_0=0 & S_1 & S_2 & S_3=0. \\ \hline x_0 & x_1 & x_2 & x_3. \end{array}$$

$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2] - f[x_0,x_1] \\ f[x_2,x_3] - f[x_1,x_2] \end{bmatrix}$$

when $n=4$

$$\begin{array}{ccccc} S_0=0 & S_1 & S_2 & S_3 & S_4 \\ \hline x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_1+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2] - f[x_0,x_1]. \\ f[x_2,x_3] - f[x_1,x_2] \\ f[x_3,x_4] - f[x_2,x_3] \end{bmatrix}$$

Example

Find the natural spline for the given table.

x_i	y_i
0	2
1	4.4366
1.5	6.7134
2.25	13.9130

h_0, h_1, h_2

$h_0 = 1, \quad h_1 = 0.5, \quad h_2 = 0.75$

$f[0,1] = 2.4366$

$f[1,1.5] = 4.5536$

$f[1.5, 2.5] = 9.5995$

$$\begin{bmatrix} 2(1.5) & 0.5 \\ 0.5 & 2(1.25) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} 4.5536 - 2.4366 \\ 9.5995 - 4.5536 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 12.7020 \\ 30.2754 \end{bmatrix}$$

$S_0 = 0$
$S_3 = 0$.

$S_1 = 2.292, \quad S_2 = 11.6618$

$$a_i = \frac{S_{i+1} - S_i}{6h_i}$$

$$a_0 = \frac{S_1 - S_0}{6h_0} = \frac{2.292 - 0}{6(1)} = 0.3820$$

$a_1 = ??$

$a_2 = ??$

$$b_i = \frac{S_i}{2}$$

$$b_0 = \frac{S_0}{2} = 0$$

$$b_1 = \frac{S_1}{2} = 1.146$$

$$b_2 = \frac{S_2}{2} = 5.8259$$

$c_i = \cdots$

$c_0 = 2.0546$

$c_1 = 3.2005$

$c_2 = 6.6866$

$d_i = y_i$

$d_0 = 2$

$d_1 = 4.4215$

$d_2 = 6.7130$

$$g_0(x) = 0.3820(x-0)^3 + 0(x-0)^2 + 2.054(x-0) + 2.000 \quad \text{on } [0,1]$$

$$g_1(x) = 3.1199(x-1)^3 + 1.146(x-1)^2 + 5.205(x-1) + 4.4366 \quad \text{on } [1, 1.5]$$

$$g_2(x) = -2.5895(x-1.5)^3 + 5.8259(x-1.5)^2 + 6.6866(x-1.5) + 6.7134 \quad \text{on } [1.5, 2.2]$$

$$f(0.66) = 3.4659 \qquad \text{Exact} = 3.4343$$

$$f(1.75) = 8.7087 \qquad \text{Exact} = 8.4467$$

natural Spline

$S_0 = 0 \qquad S_n = 0$

<u>$n=3$</u>

$$\begin{array}{cccc} S_0=0 & S_1 & S_2 & S_3=0 \\ \bullet & \bullet & \bullet & \bullet \\ x_0 & x_1 & x_2 & x_3 \end{array}$$

$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_1,x_2) - F(x_0,x_1) \\ F(x_2,x_3) - F(x_1,x_2) \end{bmatrix}$$

<u>$n=4$</u>

$$\begin{array}{ccccc} S_0=0 & S_1 & S_2 & S_3 & S_4=0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_1,x_2) - F(x_0,x_1) \\ F(x_2,x_3) - F(x_1,x_2) \\ F(x_3,x_4) - F(x_2,x_3) \end{bmatrix}$$

- **Clambed Spline**

$F'(x_0) = A$

$F'(x_n) = B$

(1) $\longrightarrow$ $2h_0S_0 + h_0S_1 = 6[F(x_0, x_1) - A]$

(2) $\longrightarrow$ $h_{n-1}S_{n-1} + 2h_{n-1}S_n = 6[B - F(x_{n-1}, x_n)]$

$n=1$

$$\begin{bmatrix} 2h_0 & h_0 \\ h_0 & 2h_0 \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \end{bmatrix} = 6\begin{bmatrix} f[x_0, x_1] - A(x_0) \\ B - F[x_0, x_1] \end{bmatrix}$$

$g_0(x) = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + b_0$

نعوض في النقاط وكذلك المشتقة عند الأطراف وبالتالي نعرف المجاهيل .

$n=2$

$$\begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = 6\begin{bmatrix} F(x_0, x_1) - A \\ F(x_1, x_2) - F(x_0, x_1) \\ B - F(x_1, x_2) \end{bmatrix}$$

$$g(x) = \begin{cases} a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0 & x_0 \leq x \leq x_1 \\ a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1 & x_1 \leq x \leq x_2 \end{cases}$$

$f'(x_0) = A$

$f'(x_1) = B$

$f(x_0) = d_0 = y_0$

$f(x_1) = d_1 = y_1$

$g_0(x_1) = g_1(x_1)$

$g_0'(x_1) = g_1'(x_1)$

$g_0''(x_1) = g_1''(x_1)$

$g_1(x_2) = y_2$

$g_0(x_0) = y_0$

$g_1(x_1) = y_1$

$g_1(x_2) = y_2$

For $n=3$

$$\begin{bmatrix} 2h_0 & h_0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & h_2 & 2h_2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_0,x_1) - A \\ F(x_1,x_2) - F(x_0,x_1) \\ F(x_2;x_3) - F(x_1,x_2) \\ B - F(x_2,x_3) \end{bmatrix}$$

$\boxed{Q2}$ Clamped spline

$(0,0)$ $(1,1)$ $(2,2)$

$S'(0)=1$, $S'(2)=1$

$$g(x) = \begin{cases} g_0(x) = a_0(x-0)^3 + b_0(x-0)^2 + c_0(x-0) + d_0 & \text{on } [0,1] \\ g_1(x) = a_1(x-1)^3 + b_1(x-1)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$$g(x) = \begin{cases} g_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & \text{on } [0,2] \\ g_1(x) = a_1(x-1)^3 + b_1(x-2)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$g_0[0] = d_0 = 0$

$g_1(2) = d_1 = 1$

$g_0'(x) = 3a_0x^2 + 2b_0x + c_0$

$g_0'(0) = c_0 = 1$

$g_1'(x) = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$

$= 3a_1 + 2b_1 + c_1 = 2$

$g_0'(1) = g_1'(1)$

$3a_0x^2 + 2b_0x + c_0 = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$

$3a_0 + 2b_0 + 1 = 3a_1(0) + 2b_1(0) + c_1$

$g_0''(1) = g_1''(1)$

$6a_0x + 2b_0 = 6a_1(x-1)^2 + 2b_1$

$6a_0 + 2b_0 = 2b_1$

$g_1(2) = 2$

$a_1 + b_1 + c_1 = 2 - 1$

$a_1 + b_1 + c_1 = 1$

$g_0'(1) = g_1(1)$

$a_0 + b_0 + c_0 + d_0 = d_1$

$a_0 + b_0 + c_0 = 1$

0	0	//	///
1	1	1	///
1	2	1	0

$h_0 = 1$

$h_1 = 1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix}$$

$S_0 = 0$

$S_1 = 0$

$S_2 = 0.$

Q15 cubic-poly $[a, b]$

$f(x)$ its own clamped spline $\overset{+}{a}$ $\overset{+}{b}$ but it cannot be its own Free spline ?

$f(a) =$

cubic $S_0, S_1 \neq$ zero

المشتقة الثانية ≠ صفر its not natural

المشتقة الأولى ≠ صفر

$a_3 \neq 0$

$g(x) = a_3 x^3 + b x^2 + C_2 x + d_2$

$g'(x) = 3a_3 x^2 + 2bx + C_2$

$g''(x) = 6a_3 x + 2b_2$

$= a_3 / b_2$ صفر عند الطرفين

$a_3 = 0$ ولكن

4 - Unknowns

4 equ (condition)

$y_0 = a_0 (x - x_0)^3 + b_0 (x - x_0)^2 + C_0 (x - x_0) + d_0$

أربع نقاط مشتركة وبالتالي poly نفسها

Th:- Central differance Formula of order $O(h^2)$ (f_1).

assum that $f \in C^2[a,b]$, and $x-h, x, x+h \in [a,b]$ then f

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

furthermore there exists a number $c \in [a,b]$ such that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

where the error then $-\frac{h^2 f^{(3)}(c)}{6}$ is called the trancation error and is denoted by

$$E_{tranc}(f,h) \neq \text{ i,e } E_{tranc}(f,h) = \frac{h^2 f^{(3)}(c)}{6}$$

Let

t	d
0.1	13.21
0.2	20.55
0.3	24.12
0.4	29.79

$$V(0.2) = \frac{d(0.3) - d(0.1)}{2(0.1)} = \frac{24.12 - 13.21}{0.2} = 54.55$$

$$\text{error} = C(0.1)^2$$
$$= C(0.01) \rightarrow \text{error in the } 4^{th} \text{ digit.}$$

$$V(0.3) = \frac{d(0.4) - d(0.2)}{0.02}$$

V(0.4) = لا أعرف

V(0.1) = لا نعرف

- $f(x) = \cos x$

$f'(0.8) = ??$ $\quad h = 0.01$

$$f'(0.8) \cong \frac{f(0.8+0.01) - f(0.8-0.01)}{2(0.01)} \cong \frac{\cos(0.81) - \cos(0.79)}{0.02}$$

$.001$

$$\cong \frac{0.6894998933 - 0.7303895326}{0.02} = -\underbrace{0.7173}44160.$$

Exact $= f'(0.8) = \sin(0.8) = -\underbrace{0.7173}56091$ ٤منازل تقريبا صحيحة

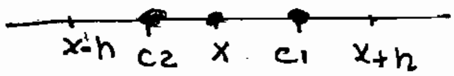
by Theorem $c(h^2)^2 = \cancel{\ldots} = C(0.01)^2 = C(\underbrace{0.0001})$

(0.01)

ابع منازل صحيحة وفقا "للنظرية".

- <u>Derivation</u>

Using Taylor expansion at x,



$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(c_1), \quad c_1 \in (x, x+h).$$

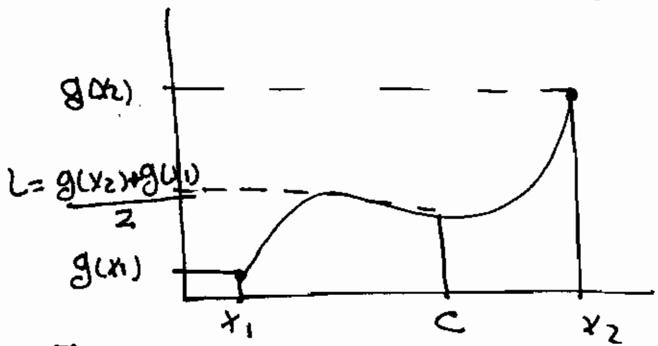
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(c_2), \quad c_2 \in (x-h, x).$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(f'''(c_1) - f'''(c_2)\right)$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(2f'''(c)\right) \quad c \in (c_1, c_2).$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{F_1} \cong \underbrace{\frac{h^2 f'''(c)}{6}}_{\text{Error.}} = f'(x).$$

IVP (Intermediate value property)



$\Rightarrow \exists\, c \in (x_1, x_2)$ such that $g(c) = \frac{g(x_1)+g(x_2)}{2}$

$\rightarrow g(x_1) + g(x_2) = 2g(c)$

ction 6.1

Central differance Formula of $O(h^4)$

assume $f \in C^5[a,b]$ and $x-2h, x-h, x+h, x+2h \in [a,b]$ then

$$f'(x) = \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$

with error

$$E_{tranc}(f,h) = \frac{h^4 f^{(5)}(c)}{30} \approx ch^4$$

Example 1
Let

t	d
0.1	13.25
0.2	18.53
0.3	21.25
0.4	24.30
0.5	27.12

فقط يمكن
استخراط
عند 0.3

$$V(0.3) = \frac{-d(0.5)+8d(0.4)-8d(0.2)+d(0.1)}{12(0.1)}$$

$$= \frac{-27.12+8(24.30)-8(18.53)+13.2}{12}$$

Example 2

$f(x) = \cos x$

$f'(0.8)$ using $h=0.01$

$$f'(0.8) = \frac{-\cos(0.82)+8\cos(0.81)-8\cos(0.79)+\cos(0.78)}{0.12}$$

$f'(0.8) = -0.717356108$

Compare to exact $-\sin(0.8) = -0.717356091$

error $c(0.01)^4 = c(10^{-8})$

error $= 1 * 10^{-7}$ ٧ منازل ثبتنا

near to 10^{-8}

- Derivation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(c)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(3) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(c)$$

نطرحهم

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f^{(5)}(c)$$

(1) $-8(f(x+h) - f(x-h)) = 16hf'(x) + \frac{16h^3}{3!}f'''(x) + \frac{16h^5}{5!}f^{(5)}(c)$

(2) -- $f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16h^3}{3!}f'''(x) + \frac{64h^5}{5!}f^{(5)}(c)$

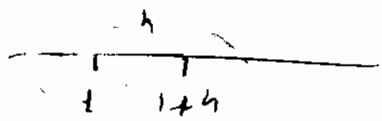
نطرح (١) من (٢)

$$-f(x+2h) + 8\{f(x+h) - 8f(x+h) + f(x-2h) = 12f'(x) - \frac{48h^5}{120}f^{(5)}(c)$$

$$\underbrace{\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}}_{F2} + \underbrace{\frac{1}{30}h^4 f^{(6)}(c)}_{E_{trac}(f,h)} = f'(x)$$

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h}$$



when h is smaller we get best estimation for $f'(x)$. ?

Example

$f(x) = e^x$

$f'(x) = e^x$

$f'(1) = e$

$$f'(1) \cong \frac{f(1+h) - f(1)}{h} = \frac{e^{1+h} - e^h}{h} \xrightarrow{??} e$$

هل هذا صحيح يقترب
من e عندما $h \to 0$

h	$D_n = e^{1+h} - e / h$
0.1	2.858841960
0.01	2.731918700
0.001	2.719642000
0.0001	2.718420000
10^{-5}	2.718300000 $\longrightarrow$ the best h
10^{-6}	2.719000000
10^{-7}	⋮
10^{-10}	0000000

- Notation

$$f(x+h) = y_1 + e_1$$
$$f(x-h) = y_{-1} + e_{-1}$$
$$\vdots$$
$$f(x+kh) = y_k + e_k$$

$$f(x+h) = \cos(0.81) = \underbrace{0.6894948433}_{y_1} \text{ (is not exact (have error))}$$
$$= y_1 + e_1$$
$$|e_1| < 0.5 * 10^{-10}$$
$$< 0.5 * 10^{-9}$$

$$F_1 = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$
$$= \frac{(y_1 + e_1) - (y_{-1} + e_{-1})}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$
$$= \frac{y_1 - y_{-1}}{2h} + \underbrace{\frac{e_1 - e_{-1}}{2h}}_{\substack{\text{Round off error} \\ E_{round}(f,h)}} - \underbrace{\frac{h^2 f^{(3)}(c)}{6}}_{\substack{\text{trancation error} \\ E_{tranc}(f,h)}}$$

$$\text{Total error} = E_{tot}(f,h) = E_{round}(f,h) + E_{tranc}(f,h)$$

$$= \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

$$\left|E_{tot}(f,h)\right| = \left|\frac{e_1 - e_{-1}}{2h}\right| + \left|\frac{h^2 f^{(3)}(c)}{6}\right| \quad \text{if } |e_k| < \epsilon$$

$$\leqslant \underbrace{\frac{2\epsilon}{2h} + \frac{h^2 M_3}{6}}_{g(n)} \qquad M_3 = \max|f^{(3)}(x)|$$

$$g(n) = \frac{\epsilon}{h} + \frac{h^2}{6}M$$

$$g'(h) = -\frac{\epsilon}{h^2} + \frac{h}{3}M = 0$$

$$\frac{h}{3}M = \frac{\epsilon}{h^2}$$

$$h^3 = \frac{3\epsilon}{M}$$

$$h = \left(\frac{3\epsilon}{M}\right)^{1/3} \quad \text{best } h$$

- $f(x) = \cos x$, $\epsilon = 0.5 * 10^{-9}$

$$h = \left(\frac{3 * 0.5 * 10^{-9}}{1}\right)^{1/3} = 0.001144714$$

1 max for $|f^{(3)}(x)|$

$$h = 0.001 \quad \text{best } h$$

- Find best h for F2.

F2

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$E_{tot} = \frac{-e_2 + 8e_1 - 8e_{-1} + \cancel{8}e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$|E(f,h)| \leqslant \frac{|e_2| + 8|e_1| + 8|e_{-1}| + |e_{-2}|}{12h} + \frac{h^4 M}{30} \qquad M = \max_{a \leqslant x \leqslant b}|f^{(5)}(x)|$$

$$\leqslant \frac{18\epsilon}{12h} + \frac{h^4 M}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M}{30} = g(n) \qquad |e_n| < \epsilon$$

$$g'(h) = \frac{-3\epsilon}{2h^2} + \frac{4h^3M}{30} = 0$$

$$\frac{2}{15}h^3M = \frac{3\epsilon}{2h^2}$$

$$h^5 = \frac{45\epsilon}{4M}$$

$$\text{Optimal } h = \left(\frac{45\epsilon}{4M}\right)^{1/5}$$

- $f(x) = \cos x$

$\epsilon = 0.5 * 10^{-9}$ $\quad\quad M = 1$

$$h = \left(\frac{45 * 0.5 * 10^{-9}}{4 * 1}\right) = 0.022\ldots$$

$$\text{Optimal } h = 0.01$$

Section 6.2

High order derivations

- $O(h^2)$

1. $f''(x) \cong \dfrac{f_1 - 2f_0 + f_{-1}}{h^2}$ $\qquad f_k = f(x+kh)$

2. $f'''(x) \cong \dfrac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3}$

   ⋮

- $O(h^4)$

1. $f''(x) \cong \dfrac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2}$

2. $f'''(x) \cong \ldots$

3. $f''''(x) \cong \ldots$

حسب المعلومات نحدد ماذا نستخدم.

$$f_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(c)$$

$$f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(c)$$

$$f_1 + f_{-1} = 2f_0 + h^2 f''(x) + \frac{h^4 f^{(4)}(c)}{12} \qquad \text{where } f_0 = f(x)$$

$$\underbrace{\frac{f_1 - 2f_0 + f_{-1}}{h^2}}_{\text{Formula}} - \underbrace{\frac{h^2 f^{(4)}(c)}{12}}_{\text{trancation error}} = f''(x)$$

Best h:-

$$E_{tot}(f,h) = E_{round}(f,h) + E_{trunc}(f,h)$$

$$E_{tot}(f,h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$$

if $|e_n| < \epsilon$, and $M = \max_{a \le x \le b} |f^{(4)}(x)|$

then

$$|E_{tot}| \le \frac{4\epsilon}{h^2} + \frac{h^2 M}{12} = g(h)$$

$$g'(h) = \frac{-8\epsilon}{h^3} + \frac{hM}{6} = 0$$

$$\frac{hM}{6} = \frac{8\epsilon}{h^3}$$

$$h^4 = \frac{48\cancel{M}\epsilon}{M}$$

$$h = \left(\frac{48\epsilon}{M}\right)^{1/4}$$

- Example

$f(x) = \cos x$

$f''(0.8)$ using $h = 0.01$.

$$f''(0.8) \approx \frac{\cos(0.81) - 2\cos(0.8) + \cos(0.79)}{(0.01)^2} \cong -0.69669000\,6$$

Exact $= -\cos(0.8) = -0.697067$.

EXAMPLE

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.987480

$V(0) = ??$

$V(0.1) \cong \checkmark$

$V(0.2) = \checkmark$

$V(0.3) = ??$

$a(0) = ??$

$$a(0.1) = \frac{d(0.2) - 2d(0.1) + d(0.0)}{(0.1)^2}$$

$$= \frac{0.998295 - 2(0.999135) + 0.989992}{0.01}$$

$$a(0.2) = \checkmark = \frac{d(0.3) - 2d(0.2) + d(0.1)}{(0.1)^2}$$

- Forward differance formula's of $O(h^2)$

$$f'(x) \cong \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$f''(x) \cong \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}$$

- Backward differance formula's of $O(h^2)$

$$f'(x_0) \cong \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

$$f''(x_0) \cong \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$$

$$f'(x_2) = \frac{3f_2 + 4f_1 + f_0}{2h}$$

$$f''(x_2) \cong \frac{2f_2 - 5f_1 + 4f_0 - f_{-1}}{h^2}$$

Example

$f(x) = \cos x$

$h = 0.01$

- Forward

$$f'(0.8) = \frac{-3\cos(0.8) + 4\cos(0.81) - \cos(0.82)}{2(0.01)}$$

- Backward

$$f'(0.8) = \frac{3\cos(0.8) - 4\cos(0.79) + \cos(0.78)}{2(0.01)}$$

- Forward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.81) + 4\cos(0.82) - \cos(0.83)}{(0.01)^2}$$

- Backward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.79) + 4\cos(0.78) - \cos(0.77)}{(0.01)^2}$$

- Using the table

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.998...

$$v(0) = \frac{-3d(0) + 4d(0.1) - d(0.2)}{2(0.1)}$$

Forward يمكن استخدام + central

$$v(0.1) = \frac{-3d(0.1) + 4d(0.2) - d(0.3)}{2(0.1)}$$

$v(0.2)$ يمكن استخدام central Forward

$$v(0.3) = \frac{3d(0.3) - 4d(0.2) + d(0.1)}{2(0.1)} \quad \text{backward}$$

$$a(0) \cong \frac{2d(0) - 5d(0.1) + 4d(0.2) - d(0.3)}{(0.1)^2}$$

$a(0.1) =$ Central

$a(0.2) =$ Central

$$a(0.3) = \frac{2d(0.3) - 5d(0.2) + 4d(0.1) - d(0)}{(0.1)^2}$$

- derive $f'(x_2) = \dfrac{3f_2 - 4f_1 + f_0}{2h}$ $\quad O(h^2)$

$f_1 = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c)$

$f_2 = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{6} h^3 f'''(c)$

$3f_2 = 3f(x) + 6hf'(x) + 6h^2 f''(x) + 4h^3 f'''(c)$

$4f_1 = 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{2h^3}{3} f'''(c)$

$3f_2 - 4f_1 = -f(x) + 2hf'(x)$

$-f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3 f'''(c)}{3!}$

$f_{-2} = f(x-2h) = f(x) - 2h f'(x) + 2h^2 f''(x) - \frac{8h^3 f'''(c)}{6}$ $\qquad f(x) = f_0$

$-4f_{-1} = -4f_0 + 4hf'(x) - 2h^2 f''(x) + \frac{4h^3 f'''(c)}{6}$

$-4f_{-1} + f_{-2} = -3f_0 + 2hf'(x) + 0 - \frac{4}{6} h^3 f'''(c)$

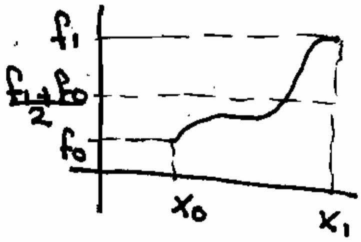
$$\underbrace{\frac{3f_0 - 4f_{-1} + f_{-2}}{2h}}_{\text{Formula}} + \underbrace{\frac{2}{3} h^2 f'''(c)}_{\text{Error}} = f'(x)$$

7.1 Newton Cotes Formula's :-

1. trapazoidal Rule :-

$x_0 \quad x_1$

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) \quad \text{with error } -\frac{h^3 f^{(2)}(c)}{12}$$



2. Simpsion's $\frac{1}{3}$ Rule :-

$x_0 \quad x_1 \quad x_2$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) \quad \text{with error} = -\frac{h^5 f^{(4)}(c)}{90}$$

3. Simpsion's $\frac{3}{8}$ Rule :-

$x_0 \quad x_1 \quad x_2 \quad x_3$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \quad \text{with error} = -\frac{3h^5 f^{(4)}(c)}{80}$$

Example 1 :-

Estimate $\int_0^1 (1 + e^{-x}\sin x)\,dx$ using the three rules.

1. trapozoidal $= \frac{h}{2}(f_0 + f_1) = \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}(1 + 0.72159) = 0.86079$

2. Simpson

$0 \quad \frac{1}{2} \quad 1$

$$h = \frac{x_n - x_0}{n} = \frac{x_2 - x_0}{2} = \frac{1-0}{2} = \frac{1}{2}$$

$$\int_0^1 f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) = \frac{1/2}{3}(f(0) + 4f(1/2) + f(1))$$

$$= \frac{1}{6}(1 + 4(1.55152) + 0.72159) = 1.32128$$

3. simpson $\frac{3}{8}$

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

$$h = \frac{x_n - x_0}{n} = \frac{x_3 - x_0}{3} = \frac{1-0}{3} = \frac{1}{3}$$

$$\int_0^1 f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$

$$= \frac{3(1/3)}{8}(f(0) + 3f(1/3) + 3f(2/3) + f(1))$$

$= 1.31440$ أقرب واحدة للجواب

Exact answer $= 1.3082506$

Example 2

t x	V(t) f(x)
1	20.1
2	22.5
3	26.6
4	28.9

$$\int_1^4 f(x)dx = ??$$

by simpson 3/8 Rule (because we have 4 points)

$$\int_1^4 f(x)\,dx = \frac{3(1)}{8}\left(f(1)+3f(2)+3f(3)+f(4)\right)$$

or by trapzoidal

$$\int_1^4 f(x)dx = \frac{4}{2}\left(f(1)+f(4)\right)$$

Example

Derive trapzoidal error or Rule.

we use $P_1(x)$ and $\int_{x_0}^{x_1} f(x)\,dx \approx \int_{x_0}^{x_1} P_1(x)\,dx$

$$= \int_{x_0}^{x_1} \left(\frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1\right) dx \qquad x = x_0+ht \qquad dx = h\,dt$$

$$x_k = x_0 + kh$$

$$= \int_0^1 \left(\frac{h(t-1)}{-h} y_0 + \frac{ht}{h} y_1\right) h\,dt = -y_0 h\int_0^1 (t-1)\,dt + hy_1\int_0^1 t\,dt$$

$$= \frac{y_0 h}{2} + \frac{hy_1}{2} = \frac{h}{2}(y_0+y_1)$$

$$\text{Error} = \int_{x_0}^{x_1} E_1(x) = \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c)\,dx$$

$$= \int_0^1 h(t)h(t-1)\frac{f^{(2)}(c)}{2}\,h\,dt = \frac{f^{(2)}(c)}{2}\int_0^1 h(t)\,h(t-1)\,h\,dt$$

$$= \frac{h^3 f^{(2)}(c)}{2}\int_0^1 (t^2-t)\,dt = \frac{-h^3 f^{(2)}(c)}{12}$$

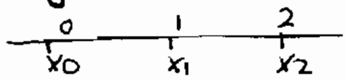
Def:-

The degree of precision or accuracy of a quadrature Formula is the largest positive integer ~~m~~ is such that the Formula is exact For x^k, $k = 0, 1, 2, \ldots$

Example:-

Find the degree of accuracy of simpson's method:-

$\frac{h}{3}(f_0 + 4f_1 + f_2)$



$F(x)$	Formula	Exact	Error
$x^0 = 1$	$\frac{1}{3}(f(0) + 4f(1) + f(2))$ $\frac{1}{3}(1 + 4(1) + 1) = 2$	$\int_0^2 1\,dx = 2$	0
$x^1 = x$	$\frac{1}{3}(0 + 4(1) + 2) = 2$	$\int_0^2 x\,dx = \frac{x^2}{2}\Big\vert_0^2 = 2$	0
x^2	$\frac{1}{3}(0 + 4(1) + 4) = \frac{8}{3}$	$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big\vert_0^2 = \frac{8}{3}$	0
x^3	$\frac{1}{3}(0 + 4(1) + 8) = 4$	$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big\vert_0^2 = 4$	0
x^4	$\frac{1}{3}(0 + 4(1) + 16) = \frac{20}{3}$	$\int_0^2 x^4\,dx = \frac{x^5}{5}\Big\vert_0^2 = \frac{32}{5}$	$\frac{32}{5} - \frac{20}{3} \neq 0$

degree of accuracy of Simpson's is 3

Note:-

degree of accurarcy of trapzoidal is 1

degree of accuracy of simpson $\frac{1}{3}$ is 3

degree of accuracy of simpson 3/8 is 3

Theorey:-

Error $= k f^{(n+1)}(\xi)$, k is the degree of accuracy

Example

For simpson's $\frac{1}{3}$ method

$$\text{Error} = k f^{(4)}(\xi)$$

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f'''(x) = 24x$

$f^{(4)}(x) = 24$

$\text{Error} = k f^{(4)}(\xi)$

$\frac{32}{5} - \frac{20}{3} = k(24)$

$\frac{96-100}{15} = 24k$

$k = -\frac{\cancel{4}^{1}}{15 \times \cancel{24}_{6}} = -\frac{1}{90}$

- if $f(x) = (x-x_0)^4$

$$\int_{x_0}^{x_2} f(x)\,dx$$

Error = Exact - Formula

$$\text{Exact} = \int_{x_0}^{x_2} F(x)\,dx = \int_{x_0}^{x_2} (x-x_0)^4\,dx = \left.\frac{(x-x_0)^5}{5}\right|_{x_0}^{x_2} = \frac{32h^5}{5}$$

$$\text{Formula} = \frac{h}{3}\left[f(x_0) + 4f(x+h) + f(x_2)\right] = \frac{20}{3}h^5$$

$$\text{Error} = -\frac{1}{90}h^5$$

$\frac{7}{6.1}$

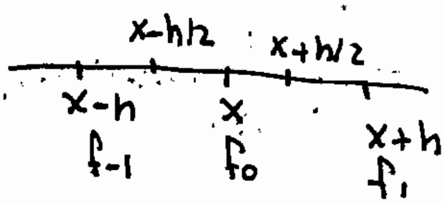
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{2h}$$

$$f_x(x,y) = \frac{f(x+h,y) - f(x-h,y)}{2h}$$

$$f_y(x,y) = \frac{f(x,y+h) - f(x,y-h)}{2h}$$

$\frac{10}{6.2}$

$$f'\left(x+\frac{h}{2}\right) = \frac{f_1 - f_0}{h}$$



$$f'\left(x-\frac{h}{2}\right) = \frac{f_0 - f_{-1}}{h}$$

$$f''(x) = \left(f'(x)\right)' = \frac{f'(x+h/2) - f'(x-h/2)}{2(h/2)}$$

$$= \frac{\frac{f_1 - f_0}{h} - \frac{f_0 - f_{-1}}{h}}{h}$$

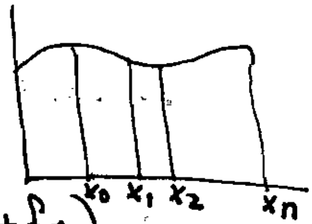
$$f''(x) = \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$f'''(x) = \left(f'(x)\right)'' = \left(f''(x)\right)'$$

7.2 Composite Rules

1. Composite trapozoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$



$$= \frac{h_1}{2}(f_0+f_1) + \frac{h_2}{2}(f_1+f_2) + \cdots + \frac{h_n}{2}(f_{n-1}+f_n)$$

$h_k = h$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right)$$

$$= \frac{h}{2}\sum_{k=1}^{n}(f_{k-1}+f_k) = T(f,h)$$

EXAMPle

t	v(t)
0	10
1	12
2	13
3	15

$$D(3) = \frac{1}{2}\left(f(0) + 2f(1) + 2f(2) + f(3)\right)$$

EXAMPle

t	v(t)
1	10
3	15
4	20
5	21

$$D(5) = \frac{2}{2}\left(f(1)+f(3)\right) + \frac{1}{2}\left(f(3)+f(4)\right) + \frac{1}{2}\left(f(4)+f(5)\right).$$

- Error For Composit trapozoidal.

$$\text{Error} = -\frac{h^3 f^{(2)}(c_1)}{12} - \frac{h^3 f^{(2)}(c_2)}{12} + \cdots - \frac{h^3 f^{(2)}(c_n)}{12}$$

$$= \frac{-h^3}{12}\left(f^{(2)}(c_1) - f^{(2)}(c_2) - \cdots - f^{(2)}(c_n)\right)$$

$$= -\frac{h^3}{12}\left(n f^{(2)}(c)\right) \qquad h = \frac{b-a}{n}$$

$$= -\frac{h^3}{12}\left(\frac{b-a}{n} f^{(2)}(c)\right)$$

$$E_T(f,h) = \frac{-(b-a)\, f^{(2)}(c)\, h^2}{12} \approx O(h^2)$$

- EXAMPLE

Find the number m at step size h so that $|E_T(f,h)| \leq 5 * 10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = T(f,h)$

where m is the number of trapozoidal composite $m = n$

$$|E_T(f,h)| \leq 5 * 10^{-9}$$

$$\frac{(b-a)\, f^{(2)}(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 * 10^{-9}$$

- $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

عوضنا في 2 لان الاقتران متناقص اعلى قيمة له عند 2 $\quad \max|f''(x)| = \frac{2}{8} = \frac{1}{4}$

$2 \leq x \leq 7$

$$\frac{(b-a)\, f^{(2)}(c) \left(\frac{b-a}{h}\right)^2}{12} \leq 5 * 10^{-9}$$

$$\frac{5\,(0.25)\left(\frac{5}{n}\right)^2}{12} \leq 5 * 10^{-9}$$

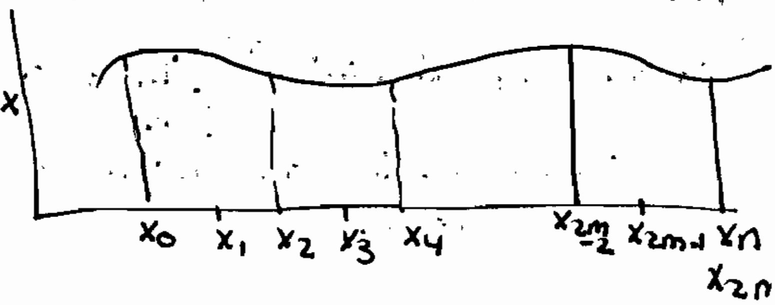
$$n \geq \sqrt{\frac{5 * 0.25 * 25}{12 * 5 * 10^{-9}}} = 22821.77$$

$$n = 22822$$

$$h = \frac{b-a}{n} = \frac{5}{22822} = 0.000219$$

2. Composite Simpson's $\frac{1}{3}$ Rule.

$$\int_{x_0}^{x_n = x_{2m}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + \dots + \int_{x_{2m-2}}^{x_{2m}} f(x)\,dx$$



$n = 2m$

$m = \frac{n}{2}$

$$= \frac{h_1}{3}(f_0 + 4f_1 + f_2) + \frac{h_2}{3}(f_2 + 4f_3 + f_4) + \dots + \frac{h_n}{3}(f_{2m-2} + 4f_{2m-1} + f_{2m})$$

$h_k = h$

$$= \frac{h}{3}(f_0 + 4f_1 + f_2 + f_2 + 4f_3 + 2f_4 + \dots + 2f_{2m-2} + 4f_{2m-1} + f_{2m})$$

$$= \frac{h}{3} \sum_{k=1}^{m} (f_{2k-2} + 4f_{2k-1} + f_{2k}) = S(f,h)$$

- Error For Composit Simpson

$$E_S(f,h) = -\frac{h^5 f^{(4)}(c_1)}{90} - \frac{h^5 f^{(4)}(c_2)}{90} - \dots - \frac{h^5 f^{(4)}(c_m)}{90}$$

$$= -\frac{h^5}{90}\left(f^{(4)}(c_1) + f^{(4)}(c_2) + \dots + f^{(4)}(c_m)\right)$$

$$= -\frac{h^5}{90}\left(m f^{(4)}(c)\right)$$

$$= \frac{-h^5}{90}\left(\frac{b-a}{2h} f^{(4)}(c)\right) \qquad m = \frac{b-a}{2h}$$

$$= \frac{-(b-a)h^4 f^{(4)}(c)}{180}$$

$$\approx Ch^4$$

EXAMPLE

Find the number $\overset{n}{\underline{m}}$ and step size h that $|E_S(f,h)| \leq 5*10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = S(f,h)$

$$|E_S(f,h)| \leq 5*10^{-9}$$

$$\left|\frac{b-a\left(\frac{b-a}{2m}\right)^4 f^{(4)}(c)}{180}\right| < 5*10^{-9}$$

$$\frac{5\cdot\left(\frac{5}{2m}\right)^4 0.75}{180} < 5*10^{-9}$$

$$m > \sqrt[4]{\frac{5*5^4*0.75}{2^4*180*5*10^{-9}}} = 112.9$$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = \frac{-1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = \frac{-6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$

$\max\limits_{2 \leq x \leq 7} |f^{(4)}(x)| = \left.\frac{24}{x^5}\right|_{x=}$

$= \frac{24}{32} = 0.7.$

7.5 Gauss - legendre Formulas

2 points Formula

$$\int_a^b F(x)\,dx \cong \omega_1 f(x_1) + \omega_2 f(x_2)$$

عندنا ٤ مجاهيل بدنا ٤ معادلات .

we assume degree of presecion 3

1. $E(\phi)=0 \rightarrow \int_{-1}^{1} 1\,dx = 2 \rightarrow$ Formula $= \omega_1(1)+\omega_2(1) \rightarrow \omega_1+\omega_2=2$
2. $E(x)=0 \rightarrow \int_{-1}^{1} x\,dx = 0 \rightarrow$ Formula $= \omega_1 x_1 + \omega_2 x_2 \rightarrow \omega_1 x_1+\omega_2 x_2=0$
3. $E(x^2)=0 \rightarrow \int_{-1}^{1} x^2\,dx = 2/3 \rightarrow$ Formula $= \omega_1 x_1^2 + \omega_2 x_2^2 \rightarrow \omega_1 x_1^2+\omega_2 x_2^2=2/3$
4. $E(x^3)=0 \rightarrow \int_{-1}^{1} x^3\,dx = 0 \rightarrow$ Formula $= \omega_1 x_1^3 + \omega_2 x_2^3 \rightarrow \omega_1 x_1^3+\omega_2 x_2^3=0$

$$\text{Exact} = \int_{-1}^{1} f(x)\,dx$$

$$\text{Formula} = \omega_1 f(x_1) + \omega_2 f(x_2)$$

$$\text{Exact} = \text{Formula}$$

حل المعادلات

$$\omega_1 x_1^3 = -\omega_2 x_2^3$$
$$\omega_1 x_1 = -\omega_2 x_2$$
$$x_1^2 = x_2^2$$

مرفوضة $x_1 = x_2$ or $\boxed{x_1 = -x_2}$

$$\omega_1 x_1 + \omega_2(-x_1) = 0$$
$$x_1(\omega_1 - \omega_2) = 0$$
$$\omega_1 - \omega_2 = 0$$
$$\boxed{\omega_1 = \omega_2}$$

$$\omega_1 + \omega_2 = 2$$
$$2\omega_1 = 2$$
$$\boxed{\omega_1 = 1},\ \boxed{\omega_2 = 1}$$

$$1(x_1)^2 + 1(x_1)^2 = \frac{2}{3}$$
$$2x_1^2 = \frac{2}{3}$$
$$x_1^2 = \frac{1}{3} \rightarrow \boxed{x_1 = \pm\frac{1}{\sqrt{3}}}$$

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= G_2(F)$$

Gauss - lagendre 2 points Formula

EXAMPle

Estimate $\int_{-1}^{1} \frac{1}{x+2}\, dx$ using

$\frac{1}{(x+2)}$

1. $T(f,2) = \frac{2}{2}\left(f(-1)+f(1)\right) = 1.3333$
2. $S(f,2) = \frac{1}{3}\left(f(-1)+4f(0)+f(1)\right) = 1.11111$
3. $G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$f = (x+2)^{-1}$

$f' = -(x+2)^{-2}$

$f'' = 2(x+2)^{-3}$

$f''' = -6(x+3)^{-4}$

$f^{(4)} = 24(x+3)^{-5}$

$= 1.09091$

Exact $= 1.09861$

$E =$ Exact $-$ Formula $= \frac{24}{(x+2)^5}$

$= 1.09091 - 1.09861$

The Error For $G_2(F) = \frac{f^{(4)}(c)}{135}$ $= K f^{4}(c)$

Error $= k f^{(4)}(c)$

$f(x) = x^4$

- Gauss legendre 3 points Formula should have 5 degree of accuracy

$$\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$G_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$\text{Error} = \frac{f^{(6)}(c)}{15.750}$$

- Gauss - legendre Formulas

- Gauss legendre two pts Formula

$G_2(F) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ with error $= \frac{1}{135} f^{(4)}(c)$

has 3 - degree of acuracy

- Gauss legender three pts Formula

$$G_3(F) = \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) \text{ with error} = \frac{1}{15750} f^{(6)}$$

has 5 - degree of accuracy

$$G_n(F) = w_1 f(x_1) + w_2 f(x_2) + \dots + w_n f(x_n)$$

has $2n-1$ degree of accuracy

- $G_8(F) \rightarrow$ error $= \dfrac{f^{(16)}(c)\, 2^{17} (8!)^4}{(16!)^3\, 17!}$ Very accurate Formula

- <u>Theorem</u>

if x_n's are the pts of Gauss legnder Formula, and w_n's are the weights in $[-1,1]$ to apply the Formula on $[a,b]$ we use the transformation.

$$t = \frac{a+b}{2} + \frac{b-a}{2} x \quad : \quad [-1,1] \longrightarrow [a,b]$$

$$dt = \frac{b-a}{2} dx.$$

$$\int_a^b f(t)\,dt = \frac{b-a}{2} \sum_{k=1}^{n} w_k f\left(\frac{a+b}{2} + \frac{b-a}{2} x\right)$$

$$= \frac{b-a}{2}\left(w_1 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_1\right) + w_2 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_2\right) + \dots\right.$$

EXAMPLE

Use $G_3(F)$ to estimate $\int_1^5 \frac{1}{t}\,dt$

$$G_3(F) = 2\left[\frac{5}{9} f\left(3+2\left(-\sqrt{\tfrac{3}{5}}\right)\right) + \frac{8}{9} f\left(3+2(0)\right) + \frac{5}{9} f\left(3+2\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= 1.602694$$

- Gauss legendre Formula are very accurate.

Chapter 9

Numerical solution of 1st order ODE's

1'st order ODE

- $y'(t) = f(t, y(t))$

  $y(t_0) = y_0$

- $y' = \frac{t-y}{2}$

  $y(0) = 1$

- $t^2 y' + \sin t\, y^2 = \cos t$

$$y(t_1) = y(t_0 + h)$$
$$= y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(c)$$

$\longrightarrow$ $y(t_1) \approx y(t_0) + h y'(t_0)$ with error $= \frac{h^2}{2!} y''(c)$

$= y_0 + h f(t_0, y_0)$ (section 9.2)

$y(t_1) \approx y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(c)$ (section 9.4)

9.2 Euler method

consider $y' = f(t, y)$

$y(t_0) = y_0$

we will approximate the solution using set of points (t_k, y_k) where

$$\underbrace{y_k}_{\text{Estimate}} = \underbrace{y(t_k)}_{\text{estimation at } t_k}$$

- we will use n subintervales of $[a, b]$

  $h = \frac{b-a}{n}$, $t_k = a + hk$

  $k = 1, \ldots, n$

- using taylor expansion of $y(t_1)$ at t_0,

$$y(t_1) = y(t_0+h) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(c)$$

$\longrightarrow$ $y_1 = y_0 + h f(t_0, y_0)$ with step error $= \frac{h^2}{2} y''(c)$

notice that $y_1 \approx y(t_1)$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$y_3 = y_2 + h f(t_2, y_2)$$

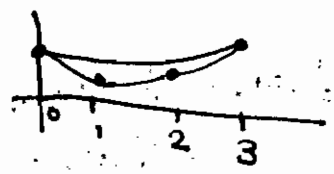
$$\vdots$$

$t_0 =$, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$

$y_{n+1} = y_n + h f(t_n, y_n)$ Euler method, step error $= \frac{h^2}{2} y''(c)$

EXAMPLE

Estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$, on $[0,3]$



$$y_1 = y_0 + h f(t_0, y_0)$$
$$= 1 + 1\, f(0,1) = 1 + (-0.5) = 0.5$$

$$y_2 = y_1 + h f(t_1, y_1)$$
$$= 0.5 + 1\, f(1, 0.5) = 0.5 + \frac{1-0.5}{2} = 0.75$$

$$y_3 = y_2 + h f(t_2, y_2)$$
$$= 0.75 + 1\, f(2, 0.75) = 0.75 + \frac{2-0.75}{2} = 1.375$$

Total error $= E(y(h), h)$

$$= \frac{y''(c_1)h^2}{2} + \frac{y''(c_2)h^2}{2} + \cdots + \frac{y''(c_n)h^2}{2}$$

$$= \frac{h^2}{2}\left(y''(c_1) + y''(c_2) + \cdots + y''(c_n)\right)$$

$$= \frac{h^2}{2}\left(n y''(c)\right)$$

$$= \frac{h^2}{2}\left(\frac{b-a}{h} y''(c)\right) = \frac{(b-a)\, h\, y''(c)}{2} \approx ch$$

$E\left(y(h), \frac{h}{2}\right) = c\left(\frac{h}{2}\right) = \frac{1}{2} ch = \frac{1}{2} E(y(h), h)$ نصف الخطأ السابق

9.4 Taylor method

Derive a formula of total error $O(h^2)$ to solve

$$y' = \frac{t-y}{2} \quad \text{on } [0,3]$$

$$y_0 = 1, \quad h = 1$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f''(t_0, y_0)$$

$$y(t_1) = \underbrace{y_0 + hy'(t_0) + \frac{h^2}{2} y''(t_0)}_{y_1} + \underbrace{\frac{h^3}{3!} y'''(c)}_{\text{Error}}$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} \frac{df(t_0, y_0)}{dt}$$

Step error $= \frac{h^3}{6} y'''(c)$

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''(t_k)$$

Total error $= E(y(h), h) = ch^2 = \frac{y'''(c)(b-a)h^2}{6}$

- Solving the example.

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} y''(t_0)$$

$$= 1 + 1f(0,1) + \frac{1}{2} y''(0)$$

$$= 1 + \frac{0-1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{(0-1)}{4}\right)$$

$$= 1 - 0.5 + 0.25 + 1/8 = 0.875$$

$$y'(t) = \frac{t-y}{2}$$

$$y''(t) = \frac{1}{2} - \frac{y'}{2}$$

$$= \frac{1}{2} - \frac{(t-y)}{4}$$

$$y_2 = y_1 + hf(t, y_1) + \frac{h^2}{2} y''(t_1)$$

$$= y_1 + hF(t, y_1) + \frac{h^2}{2} \frac{d}{dt}(f(1, 0.875))$$

$$= 0.875 + hf(1, 0.875) + \frac{1}{2}\left(\frac{1}{2} - \frac{1-0.875}{4}\right)$$

$$= 0.875 + 0.0625 + 0.25 - 0.03125$$

$$= 1.15625$$

$y'(t) = f(t, y(t))$
$y(t_0) = y_0$

- Use taylor method of ordor 4 to estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$ on $[0,3]$, $h = 1$

$$y(t_1) = \underbrace{y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)}_{y_1} + \underbrace{\frac{h^5}{5!} y^{(5)}(c)}_{\text{steperror}}$$

- total error $= E(y(t_0), h) = ch^4$

$$y_{k+1} = y_k + h y'(t_k) + \frac{h^2}{2!} y''(t_k) + \frac{h^3}{3!} y'''(t_k) + \frac{h^4}{4!} y^{(4)}(t_k)$$

$$y_1 = y_0 + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)$$

$y'(t) = \frac{t-y}{2}$, $y'(0) = \frac{0-1}{2} = \frac{-1}{2}$ $(y_0 = 1)$

$y''(t) = \frac{1}{2}(1 - y') = \frac{1}{2}\left(1 - \left(\frac{t-y}{2}\right)\right) = \frac{1}{2} - \frac{t-y}{4}$

$y''(0) = \frac{1}{2} - (-\frac{1}{4}) = 0.75$

$y'''(t) = \frac{1}{2}(-y'') = -\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)$

$y'''(0) = -\frac{1}{2}(0.75) = -0.375$

$y''''(t) = -\frac{1}{2} y''' = -\frac{1}{2}\left(-\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)\right) \Rightarrow$

$y''''(0) = -\frac{1}{2}(-0.375) = 0.1875$

$y_1 = 1 + 1(0.5) + \frac{1}{2}(0.75) + \frac{1}{6}(-0.375) + \frac{1}{24}(0.1875)$
$= 0.8203125$

$y_2 = y_1 + h y'(t_1) + \frac{h^2}{2} y''(t_1) + \frac{h^3}{3!} y'''(t_1) + \frac{h^4}{4!} y^{(4)}(t_1)$
$t_1 = 1$, $y_1 = 0.8203125$

$y_2 = 1.1045 \ldots$
$y_3 = 1.670 \ldots$

$E(y(b), h) = ch^4$ taylor $\to$ one evaluation

$E(y(b), h/2) = c(h/2)^4 = ch^4/16$

$E(y(b), 10^{-2}h) = c(10^{-2}h)^4 = c(10^{-8})h^4$

- Modified Euler Method :- (Huen's Method)

$y'(t) = f(t, y(t))$

$y(t_0) = y_0$

1. $\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t, y(t))\,dt \to$ Using trapozoidal

$y(t_1) - y(t_0) = \frac{h}{2}(f(t_0, y_0) + f(t_1, y(t_1)))$ Error $= \frac{-h^3 y''(c)}{12}$

$y(t_1) \cong y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, \underbrace{y(t_1)}_{\text{ندخل قيمة معروفة}}))$

$y(t_1) = y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$

- $y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$

total error for Huen's Method $= ch^2$

- EXAMPLE

Solve Using Huen's Method with $h=1$

$y' = \frac{t-y}{2} = f(t, y)$

$y(0) = 1$

$f(0,1)$

$y_1 = y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$

$= 1 + \frac{1}{2}(f(0,1) + f(1, 1 + (1)f(0,1)))$

$= 1 + \frac{1}{2}(-0.5 + f(1, 1-0.5)) = 1 + \frac{1}{2}(-0.5 + \frac{1-0.5}{2}) = 0.875$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_1 + h f(t_1, y_1))\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1, 0.875) + f(2, 0.875 + (1) f(1, 0.875))\right)$$

$$= 0.875 + \frac{1}{2}\left(\left(\frac{1-0.875}{2}\right) + f(2, 0.875(1 - 0.875/2))\right)$$

$$= 1.171875$$

$$y_3 = 1.732422$$

- if we have a period of $[0, 0.5]$, $h = \frac{1}{4}$

0	0.25	0.5
t_0	t_1	t_2

$y_1 = 1 + 1/2(-0.25) = 0.875$

on $[0, 1]$, $h = \frac{1}{2}$

0	0.5	1
t_0	t_1	t_2

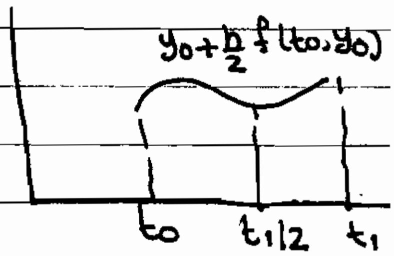
9.5 RK4: Range-Kutta Method of ordor 4

$$y'(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

$$\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t, y(t))\,dt$$

$$y_{n+1} = y_n + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3) \quad \text{where}$$

$$f_0 = f(t_0, y_0)$$
$$f_1 = f(t_0 + h/2,\ y_0 + h/2\, f_0)$$
$$f_2 = f(t_0 + h/2,\ y_0 + h/2\, f_1)$$
$$f_3 = f(\cancel{t_0},\ y_0 + hf_2)$$



$$f_1 = \frac{f_1 + f_2}{2}$$

$$f_2(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f$$
$$f_1(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f(t_0$$

EXAMPLE

Solve $y' = \frac{t - y}{2}$, $y(0) = 1$, $[0, 3]$, $h = \frac{1}{4}$

$$y_1 = y_0 + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3)$$

$$f_0 = f(t_0, y_0) = f(0, 1) = -0.5$$

$$f_1 = f(t_0 + h/2,\ y_0 + h/2\, f_0) = f\left(\tfrac{1}{8},\ 1 + 1/8(-0.5)\right) = -0.40625$$
$$f_2 = f(t_0 + h/2,\ y_0 + h/2\, f_1) = f(1/8,\ 1 + 1/8(-0.40625)) = -0.4121$$
$$f_3 = f(t_1,\ y_0 + hf_2) = f(1/4,\ 1 + 1/4(-0.4121094)) = -0.3234863$$

$$y_1 = 1 + \frac{1/4}{6}((-0.5) + 2(-0.40625) + 2(-0.4121094) + (-0.3234863))$$
$= 0.8974915$ Comparing to the exact 0.8974917

$y_2 = y_1 + h/6 (f_0 + 2f_1 + 2f_2 + f_3).$

$f_0 = f(t_1, y_1) = f(1/4, 0.8974915) = \dots$

$f_1 = f(t_1 + h/2, y_1 + h/2 f_0) = f(3/8, 0.8974915 + 1/8 f_0) = \dots$

$f_2 = f(\frac{3}{8}, 0.8974915 + \frac{1}{8}(f_1)) = \dots$

$f_3 = f(1/2, 0.8974915 + 1/4 (f_2)) = \dots$

$y_2 = \dots$