بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

Differential Notes
Final revision

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تلخيص المطلوب من كل شابتر في مادة الفاينل (قوانين ، أمثلة ، ملاحظات ...)

1.3

1- verify that y is a solution => substitute y

2- see if it's linear or not / find the order

1.1 generate the differential equation

- Find the behavior :-

1. نساوي المشتقة بالصفر
2. نوجد عندما $y=0$
3. نرسم
4. نأخذ قيمة أكبر من y وقيمة اصغر ونعوض في y'
5. if $y' > 0$: The arrow goes up ↑
   if $y' < 0$: The arrow goes down ↓
6. Find $\lim_{t \to \infty} y(t)$ (i can take it from the plot)
7. اذا كان عندي قيمة أوليه $y(t_0) = y_0$ ، بشوف الفترة يلي فيها y_0 ، ومن ثم بشوف السهم لوين بتروح

$$\text{Behavior: } \lim y(t) = \begin{cases} \infty & , y_0 > X \\ X & , y_0 = X \\ -\infty & , y_0 < X \end{cases}$$

- How to generate the D.E in these questions ?

$$\frac{dy}{dx} = \text{Rate in} - \text{Rate out}$$

$\boxed{2.2}$

① separable

make dy in one side and dt in the other side then integrate

Sometimes you need to assume u or use Partial fraction for integration

② Homogenous

general form : $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$ or $\dfrac{dx}{dy} = F\left(\dfrac{x}{y}\right)$

how to solve ?

① let $y = vx$ ② $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ③ substitute then solve

الهدف هو التخلص من $\frac{y}{x}$ ، نعوض بدالها V حيث $V = \frac{y}{x}$

ومن ثم نكمل الحل ك Linear او Separable ، وبعد التكامل بنرجع y

③ Liner (method of integrating factor)

general form : $y' + P(t)y = g(t)$

how to solve ?

$$y(t) = \frac{1}{\mu_t} \int \mu_t \, g(t)\, dt + C \quad , \quad \mu_t = e^{\int P(t)\,dt}$$

① you have to check if it's linear first

② then you have to make it as the general form

2.3 Tank + Cooling

1- Tank

① Rate in − Rate out

= (flow in * التركيز) − (flow out * التركيز)

* Sometimes you have to build an equation for تركيز

بناءً على كمية المياه الداخلة او الخارجة وسعة الخزانات

* التركيز = $\frac{\text{كمية الملح}}{\text{كمية المياه}}$ * دايما بعوض Q مكان كمية الملح

⇐ التركيز = $\frac{Q}{\text{كمية المياه}}$

2- Newton's Law of Cooling

$$\frac{du}{dt} = -k(u - T)$$

u : The temperature of the object

T : The temperature of the room

k : Positive Constant

1. divide by (u − t) then integrate
2. we usually use seperable method to solve
3. the answer will be ln| something| , give it e
4. then suppose $e^c = A$ and Continue

2.4 See if there is a Unique Solution Without Solving + Bernoulli

What can he ask me?

(1) is there a unique solution for this IVP? (2) find the interval

1- Linear D.E

1. make it like the form $y' + P(t)y = g(t)$ (عشان أصل عل قانون الليني)
2. $g(t)$ و $P(t)$ نوجد أصفار المقام
3. Find the interval that contains the initial value t_0 for $y(t_0) = y_0$
4. if we did all of this $\Rightarrow$ There is a solution and it's unique

2- Non-Linear D.E

1. differentiate y' to get y'' (given y' and $y(t_0) = y_0$)
2. see the interval of y' and check if it contains t_0
3. see the interval of y'' and check if it contains y_0

Both true $\Rightarrow$ it has a unique solution

One at least false $\Rightarrow$ we have to solve the original IVP

3- Bernoulli

general form: $y' + P(t)y = g(t)y^n \quad , n \in \mathbb{R}$

if $n=0$:-

$y' + P(t)y = g(t) \Rightarrow$ Linear

if $n=1$:-

$y' + (g(t) - P(t))y = 0 \Rightarrow$ Separable

if $n \neq 0, 1$:-

1. let $v = y^{1-n}$
2. find y' as "$(1-n)y^{-n}y' = v'$"
3. divide by y^n to remove the other side
4. Solve as Linear in v

2.6 Exact equations

First : the equation is exact

$2x + y^2 + 2xyy' = 0$, $y(1) = 1$

① Find M and N

$N = 2xy$
$M = 2x + y^2$

هو البعيد M , هو القريب N

② Differentiate N By respect to X , and M By respect to y

$N_x = 2y$
$M_y = 2y$

They are equal => exact

③ Choose N or M to integrate "does not matter"

$f = \int f_x = \int M$ OR $f = \int f_y = \int N$ عكس العلية الأولى

$f = \int 2x + y^2 \, dx \Rightarrow f = x^2 + y^2 x + g(y)$

④ differentiate f with respect to y then make it equal to the other variable (M or N depends on g(x) or g(y))

$f' = 2xy + g'(y) = 2xy$

$\int g'(y) = \int 0 \Rightarrow g(y) = C_1$

⑤ Put g(y) in its place then make the eq. equal to C

$f = x^2 + xy^2 + C_1 = C$

$f = x^2 + xy^2 = C$

⑥ Apply the condition $y(x_0) = y_0$

$1^2 + 1 \times 1^2 = C \Rightarrow C = 2$

The final answer :-

$x^2 + xy^2 - 2 = 0$

Second : the equation is not exact

① $\dfrac{M_y - N_x}{N} = f(x) \Rightarrow I = e^{\int f(x)\,dx}$

or

② $\dfrac{M_y - N_x}{M} = f(y) \Rightarrow I = e^{-\int f(y)\,dy}$

then we multiply the equation by I to make it exact
then we start again from the beginning

Euler Formula :- $e^{ix} = \cos x + i \sin x$

Exp. rewrite $e^{1-\frac{\pi}{3}i}$

$$= e * e^{i(-\frac{\pi}{3})} = e\left[\cos\left(-\tfrac{\pi}{3}\right) + i \sin\left(-\tfrac{\pi}{3}\right)\right]$$
$$= e\left[\cos\left(\tfrac{\pi}{3}\right) - i \sin\left(\tfrac{\pi}{3}\right)\right]$$
$$= e\left[0.5 - \frac{\sqrt{3}}{2} i\right]$$

$$\begin{array}{ccccc} & 0 & 30 & 45 & 60 & 90 \end{array}$$
$$\begin{array}{c} \cos \\ \sin \end{array} \frac{\sqrt{\begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{array}}}{2}$$

* How to find sin and cos without Calculator

2.8 Picard's Iteration

general form :- $\frac{dy}{dt} = f(t,y) \Rightarrow y = \int f(t,y)\,dt$

$$y(t) = \phi(t) = \int_0^t f(t, \phi(t))\,dt$$

$\phi_0 = 0$

$\phi_1 = \int_0^t f(t,0)\,dt$

$\phi_2 = \int_0^t f(t,\phi_1)\,dt$

$\vdots$

$\phi_n = \int_0^t f(t,\phi_{n-1})\,dt$

Then i take the Lim for ϕ_n

$$\lim_{t\to\infty} \phi_n = \sum_{n=1}^{\infty} \phi_n$$

its usually on a shape like

this $\frac{n\,t^n}{n!}$ or something

2.9 Missing t and Missing y :-

① Missing y :-
$V = y'$, $V' = y''$

② Missing t :-
$V = y'$ $y'' = V'V$

Why to use this method ?

الهدف هو إيجاد y' عن طريق عمل تكامل لـ V' وبالتالي
الناتج يكون V الذي هو y' ، ومن ثم تكامل لـ y' لإيجاد y

How to solve ?

① Use the assumptions above based on the Case
② apply them to the equation then solve for $V = \int V'$
③ after you get $V = y'$, solve for $y = \int y' = \int V$

Remark

$\cosh t = \frac{e^t + e^{-t}}{2}$ $\sinh t = \frac{e^t - e^{-t}}{2}$

$\boxed{3.1}$ + $\boxed{3.3}$

When to use this way "$r^2 + p(t)r + q(t) = g(t)$"

1. coefficients are Constants
2. higher order linear equation

Case 1 : $r_1 \neq r_2$

$y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

Case 2 : $r_1 = r_2 = r$

$y_1 = e^{rt}$, $y_2 = t e^{rt}$

Case 3 : $r = \alpha \pm \omega i$ (Complex roots)

$r_1 = \alpha + \omega i$, $r_2 = \alpha - \omega i$

$y_1 = e^{\alpha t} \cos(\omega t)$, $y_2 = e^{\alpha t} \sin(\omega t)$

Complex : $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -\sqrt{-1}$, $i^4 = 1$

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3.2

1- how to find the largest interval :-

- do it as the form $y'' + p(t)y' + q(t)y = 0$
- نوجد اصفار المقام
- find the largest interval that Contains $y(t_0) = y_0$

2- Principle of SuperPosition :-

- if y_1 and y_2 are solutions for $y'' + p(t)y' + q(t)y = 0$ and $W(y_1, y_2)(t) \neq 0$, then the linear Combination $C_1 y_1 + C_2 y_2$ is also a Solution

- y_1 and y_2 are independent iff $W(y_1, y_2)(t) \neq 0$

3- The Wronskian Solution $W(y_1, y_2)(t)$:-

1) if i have y_1 and y_2 :-

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

2) if i don't have y_1 and y_2 (Abel's Theorem)

$$W(y_1, y_2)(t) = C e^{-\int p(t)\, dt}$$

4- Fundemental set of Solutions :-

Two Conditions Should Occur to Say that $L(y)$ is F.S.S

1. y_1 and y_2 are Solutions to $y'' + p(t)y' + q(t)y = 0$
2. y_1 and y_2 are IndePendent $(W(y_1, y_2)(t) \neq 0)$

5- Finding F.S.S with no y_1 or y_2, with just interval

1. let "$y_1(t_0) = 1$" "$y_1'(t_0) = 0$" "$y_2(t_0) = 0$" "$y_2'(t_0) = 1$" (to get $W \neq 0$)
2. Find the Charastaretrec equation, then generate y
3. Find C_1 and C_2 in the two Cases to get y_3 and y_4

- if he gave me an interval i can choose any t_0

On Point 4

(2) check if $y_1 = x$, $y_2 = \sin x$ form a fundamental set of solutions for the d-e

$$(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi.$$

(1) Verify if y_1 and y_2 are solutions for the equation :-

$y_1 = X$, $y_1' = 1$, $y_1'' = 0$

$(1 - X\cot x) * 0 - X + X = 0 \Rightarrow y_1$ is solution

$y_2 = \sin x$, $y_2' = \cos x$, $y_2'' = -\sin x$

$(1 - X\cot) * -\sin x - X\cos x + \sin x = 0$

$\cancel{-\sin x} + X\sin x\cot x - X\cos x + \cancel{\sin x} = 0$

$\cancel{X}\left(\left(\cancel{\sin x} * \frac{\cos x}{\cancel{\sin x}}\right) - \cos x\right) = \cancel{0}$

$\cos x - \cos x = 0 \Rightarrow y_1$ and y_2 are solutions #

(2) Verify that $W(y_1, y_2) \neq 0$

$W(X, \sin X) = \begin{vmatrix} X & \sin x \\ 1 & \cos x \end{vmatrix} = X\cos X - \sin X \neq 0 \quad \forall\ 0 < X < \pi$

$W(X, \sin X)(\pi/2) \neq 0 \Rightarrow$ they are Lenicrly independent

∴ Since they are solutions and L.E

$\Rightarrow \boxed{\{X, \sin x\}}$ form a fundemental set of solutions #

On Point 5

Ex: Find the fundamental set of solutions specified by thm 3.2.5 for $y'' - y = 0$ using $t_0 = 0$.

$r^2 - 1 = 0 \Rightarrow r_1 = 1 \quad , \quad r_2 = -1$

$y = c_1 y_1 + c_2 y_2 \Rightarrow \boxed{y = c_1 e^{t} + c_2 e^{-t}}$ ⊛

$\Rightarrow y_1 = e^{t} \quad , \quad y_2 = e^{-t}$

Case 1 : $y_1(0) = 1 \quad , \quad y_1'(0) = 0 \quad , \quad y' = c_1 e^{t} - c_2 e^{-t}$

from ⊛ $c_1 e^0 + c_2 e^0 = 1 \Rightarrow c_1 + c_2 = 1$ ⊛ بعوض $y=0$ و $t=1$ في

$c_1 e^0 - c_2 e^0 = 0 \Rightarrow c_1 - c_2 = 0$ ⊛ بعوض $y'=0$ و $t=0$ في

$\Rightarrow c_1 = c_2 = 1/2$

$y_3 = \frac{e^{t} + e^{-t}}{2} \Rightarrow \boxed{y_3 = \cosh t}$

Case 2 :- $y_2(0) = 0 \quad , \quad y_2'(0) = 1 \quad , \quad y' = c_1 e^{t} - c_2 e^{-t}$

$c_1 e^0 + c_2 e^0 = 0 \Rightarrow c_1 + c_2 = 0$

$c_1 e^0 - c_2 e^0 = 1 \Rightarrow c_1 - c_2 = 1$

$c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$

$y_4 = \frac{e^{t} - e^{-t}}{2} \Rightarrow \boxed{y_4 = \sinh t}$

$\therefore$ $\boxed{\{\cosh t \ , \ \sinh t\}}$ are Fundamental set of Solutions

#

On Abel's theorem

Q34) If y_1 and y_2 are a fundamental set of solutions of $ty'' + 2y' + te^t y = 0$ and if $W(y_1, y_2)(1) = 2$, find $W(y_1, y_2)(5)$.

$$y'' + \frac{2}{t} y' + e^t y = 0 \implies P(t) = \frac{2}{t}$$

$$W(y_1, y_2)(t) = C e^{-\int P(t)dt} = C e^{-\int \frac{2}{t}} = \frac{C}{t^2}$$

$$W(y_1, y_2)(1) = \frac{C}{1} = 2 \implies C = 2$$

$$W(y_1, y_2)(t) = \frac{2}{t^2} \implies \boxed{W(y_1, y_2)(5) = \frac{2}{25}} \quad \#$$

3.4 i have y_1 and want to find y_2

When to use it ?

(1) when he give me y_1 and ask for the other one

1- Reduction of order method :-

(1) let $y = v y_1$ (2) Find y' and y'' (3) عوضهن في y^*

if i still have v of order 2 :-

(4) let $w = v'$ (5) Find w' (6) عوضهن في y^*

(7) Solve it using linear rule as in chapter 2 to get w

(8) then find w from $\int w'$, then find y from $\int v'$

2- Reduction of order formula :-

$$y_2 = y_1 \int \frac{W(y_1, y_2)(t)}{y_1^2}$$

Solve the example below using this method ?

$$W(y_1, y_2)(t) = C e^{-\int P(t)} \Rightarrow W(y_1, y_2)(t) = Ct^6$$

$$y_2 = t^2 \int \frac{ct^{\cancel{6}\,2}}{\cancel{t^4}} \Rightarrow Ct^2 * \frac{t^3}{3}$$

$$\boxed{y_2 = t^5}$$

* don't take the constants it may confuse you in the multiple choices.

Ex. Given that $y_1 = t^2$ is a solution of

$$(*)\quad \boxed{t^2 y'' - 6ty' + 10y = 0,\quad t>0}$$

. Use the method of reduction of order to find a second solution of the given d-e.

$$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 0 \quad , \quad \text{let } y = vt^2$$

$$y' = 2tv + t^2v' \quad , \quad y'' = 2v + 4tv' + t^2v''$$

$$(2v + 4tv' + t^2v'') - \frac{6}{\cancel{t}}(2\cancel{t}v + t^{\cancel{2}}v') + \frac{10}{\cancel{t^2}}(v\cancel{t^2})$$

$$t^2v'' + 4tv' - 6tv' + \cancel{2v} - \cancel{12v} + \cancel{10v}$$

$$t^2v'' - 2tv' = 0 \implies (tv'' - 2v') = 0 \quad \text{let } w = v'$$

$$w' = v''$$

$$tw' - 2w = 0 \implies w' - \frac{2}{t}w = 0$$

$$\mu(t) = e^{-\int \frac{2}{t}} \implies \mu(t) = t^{-2}$$

$$w(t) = t^2\left[\int 0 * t^{-2}\,dt + c\right] \implies w = ct^2 = v'$$

$$v = \int ct^2 \implies v = \frac{c}{3}t^3 + c \implies v = At^3 + C \qquad A = \frac{c}{3}$$

$$y = vy_1 \implies y = (At^3 + C)(t^2)$$

$y = At^5 + Ct^2$ Since this is y_1 then $\boxed{y_2 = t^5}$

ملاحظة :- y صاحب المعادلة الأساسية $c_1y_1 + c_2y_2$
مش y_2 ، لما تترابط

3.5 + 4.3 Non-homogeneous d.e

form : $y'' + P(t)y' + q(t)y = g(t)$

1- $P(t)$, $q(t)$, $g(t)$ are Constants

2- $g(t)$ is : Constant or Poly or exponential or sin or Cos or finite Sum / Product of them

How to Solve ?

① find y_h by Setting the equation to zero (let $g(t) = 0$) then solve it normal as 3.4 ($r^2 + p(t)r + q(t) = 0$)

② find y_p, depending on $g(t)$ form as follows :-

① $g(t) = Ce^{rt} \Longrightarrow y_p = Ae^{rt} * t^s$

② $g(t) = C_n t^n + C_{n-1} t^{n-1} \ldots \Longrightarrow y_p = At^n + Bt^{n-1} \ldots * t^s$

③ $g(t) = C_1 \sin rt$ or $C_2 \cos rt$ or $C_1 \sin rt + C_2 \cos rt$
$\Longrightarrow y_p = A \sin rt + B \cos rt * t^s$

④ $g(t) = C_1 e^{rt} * C_2(\sin, \cos) \Longrightarrow y_p = e^{rt}[A \sin + B \cos] * t^s$

⑤ $g(t) = Ct * (\sin, \cos) \Longrightarrow y_p = [(At+B)\sin + (Ct+D)\cos]$

⑥ $g(t)$ = something + something i have to split them
$y_{p_1} = g_1(t) * t^s$, $y_{p_2} = g_2(t) * t^s \Longrightarrow y_p = y_1 + y_2$

When to multiply by t^s ?

i look to y_h, if there is overlapping between y_h and y_p i multiply by t^s to avoid it, $s = 0, 1, 2 \ldots$

③ find y_p, y_p', y_p'' and substitute in the eq to get $A, B \ldots$

④ The most important : $y = y_h + y_p$

Ex ③. Solve $\boxed{y'' - 3y' - 4y = 2\sin t}$.

① Find y_h :-

$y'' - 3y' - 4y = 0 \Rightarrow r^2 - 3r - 4 = 0 \Rightarrow r = -1 \ , r = 4$

$y_h = C_1 y_1 + C_2 y_2 \Rightarrow \boxed{y_h = C_1 e^{-t} + C_2 e^{4t}}$

② Find y_p :-

$y'' - 3y' - 4y = 2\sin t \Rightarrow y_p = t^s [A\sin t + B\cos t]$

$t^s = 0$ because they are independent $\Rightarrow \boxed{y_p = A\sin t + B\cos t}$

نعوض في المعادلة الأصلية لنجيب A و B

$y_p' = A\cos t - B\sin t \qquad y_p'' = -A\sin t - B\cos t$

$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$

$-5A\sin t - 5B\cos t - 3A\cos t + 3B\sin t = 2\sin t$

$\sin t(-5A + 3B) + \cos t(-5B - 3A) = 2\sin t$

$(-5A + 3B = 2) * 5 \Rightarrow -25A + 15B = 10 \Rightarrow A = -5/17$

$(-5B - 3A = 0) * 3 \qquad -9A - 15B = 0 \qquad B = 3/17$

$$y_p = \frac{-5\sin t}{17} + \frac{3\cos t}{17}$$

$y = y_h + y_p \Rightarrow \boxed{y = C_1 e^{-t} + C_2 e^{4t} - \frac{5\sin t + 3\cos t}{17}}$

Ex⑥. $y'' + y = t(1+\sin t)$.

$y'' + y = t + t\sin t$, $P_h = r^2 + 1 = 0 \Rightarrow r = +i$, $r = -i$

$y_h = C_1 e^{0t}\cos(t) + C_2 e^{0t}\sin(t) \Rightarrow \boxed{y_h = C_1\cos t + C_2\sin t}$

$g_1(t) = t \Rightarrow y_1 = At + B$

$g_2(t) = t\sin t \Rightarrow y_2 = [(Ct+D)\cos t + (Et+F)\sin t] * t^1$

Why $*t$?

Because $D\cos t$ and $F\sin t$ exist on y_h , so i have to multiply by t to make them independent

...

$\Rightarrow y_p = y_1 + y_2 \Rightarrow y_p = At+B + (Ct^2+Dt)\cos t + (Et^2+Ft)\sin$

$y = y_p + y_h$

$y = At+B + (Ct^2+Dt)\cos t + (Et^2+Ft)\sin + C_1\cos t + C_2\sin t$

$\boxed{4.2}$ Same as 3.5 but for higher order

① Find y_h following these steps :-

1. Find the Characteristic equation
2. Find the factors of the smallest derivative
3. Simplify the equation using Long or Synthetic division
4. find the values of r , then generate y_h

The Synthetic division for $Ay^n + By^{n-1} + Cy^{n-2} - - - - = 0$

Since Z is the factor i used as a zero of r $(Z = (r = r_0))$

* if r in ② equals $-3 \Rightarrow Z = -3$ and so on

$$
\begin{array}{c|cccc}
 & r^n & r^{n-1} & r^{n-2} & - - - - \\
 & A & B & C & - - - - \\
Z & * & A*Z & \times Z & \\
\hline
 & A & (B + AZ) & &
\end{array}
$$

The Long division for $Ay^n + By^{n-1} + Cy^{n-2} - - - - = 0$

$$
\begin{array}{r|l}
 & \text{The answer} \\
\hline
(r - r_0) & A r^n + B r^{n-1} + C r^{n-2} - - - -
\end{array}
$$

② Find y_p like 3.5

③ $y = y_h + y_p$

ex. $y^{(3)} + y'' - y' + 2y = e^{-2t} + e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$, Find y

① Find the Characteristic equation

$r^3 + r^2 - r + 2 = 0$

② look at the smallest derivative $2y^0$ and find its factors

how can i get 2 ? => $-1 * -2$ or $1 * 2$

then ± 1 and ± 2 are the factors

substitute in y and see which one of them is a solution

$r = -2$ is a solution => $(r+2) = 0$

③ Find the full equation Using one of these methods

19/11/25

7/10/23

Synthetic division

القسمة التركيبية

	r^3	r^2	r	r^0
	1	1	-1	2
-2		-2	2	-2
	1	-1	1	0 الباقي

$r^2 - r + 1$

Long division

القسمة الطويلة

$$
\begin{array}{r|l}
 & r^2 - r + 1 \\
\hline
r+2 & r^3 + r^2 - r + 2 \\
 & -(r^3 + 2r^2) \\
\hline
 & 0 - r^2 - r + 2 \\
 & -(-r^2 - 2r) \\
\hline
 & 0 + r + 2 \\
 & -(r + 2) \\
\hline
 & 0
\end{array}
$$

④ The final equation is $(r+2)(r^2 - r + 1)$, find r

$$= \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$r = -2 \ , \ \frac{1}{2} + \frac{\sqrt{3}}{2} i \ , \ \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$y_h = C_1 e^{-2t} + C_2 e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right) + C_3 e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right)$

$$g(t) = e^{-2t} + e^{\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t \Longrightarrow \begin{array}{l} y_{p_1} \text{ takes } e^{-2t} \\ y_{p_2} \text{ takes } e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t \end{array}$$

$$y_{p_1} = A e^{-2t} * t \Longrightarrow y_{p_1} = A t e^{-2t}$$

$$y_{p_2} = \left[B e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + D e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right] * t$$

$$y_{p_2} = \left[B t e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + D t e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right]$$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_p = A t e^{-2t} + \left[B t e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + D t e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right]$$

Finally

$$y = y_p + y_h$$

$$\boxed{\begin{aligned} y = {} & A t e^{-2t} + \left[B t e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + D t e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right] \\ & + C_1 e^{-2t} + C_2 e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) + C_3 e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) \end{aligned}}$$

ex : $r = 2, 2, 2, 3, 2 \pm 4i, 2 \pm 4i, 2 \pm 4i$

$$y = C_1 e^{2t} + C_2 t e^{2t} + C_3 t^2 e^{2t} + C_4 e^{3t}$$
$$+ C_5 e^{2t} \cos(4t) + C_6 e^{2t} \sin(4t)$$
$$+ C_7 e^{2t} t \cos(4t) + C_8 e^{2t} t \sin(4t)$$
$$+ C_9 e^{2t} t^2 \cos(4t) + C_{10} e^{2t} t^2 \cos(4t)$$

مهم :- طرق حل المعادلات ذات الدرجة العالية

1. grouping : بشوف إذا في أقواس عوامل مشتركة
2. قسمة طويلة : بقسم المعادلة على " r + the zero "
3. قسمة تركيبية : نفس فوق

$\boxed{3.6}$ Non-homogenious

general form : $y'' + P(t)y' + q(t)y = g(t)$

When to use it ?

① if $g(t)$ is any function in the world

② if $P(t)$ and $q(t)$ are Constants or functions

1- $P(t)$ and $q(t)$ are Constants :-

i find y_h and Solve normal

2- $P(t)$ and $q(t)$ are NOT Constants :-

he Should give me y or y_1 and y_2

then i find the Solutions directly by V_1 and V_2

or i may use Euler to solve it

① find y_h

as in 3.5 by the Char. equation

② find y_p

$y_p : V_1 y_1 + V_2 y_2$, where

$$V_1 = -\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)} \quad , \quad V_2 = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)}$$

* i have to make sure that the wronskian $\neq 0$

③ $y = y_h + y_p$

ex. $x^2y'' - 3xy' + 4y = x^2\ln x, \quad x > 0.$

Euler: $m^2 + (-3-1)m + 4 \Rightarrow m^2 - 4m + 4 = 0$

$(m-2)(m-2) \Rightarrow m = 2$

$y_h = C_1 e^{2\ln x} + C_2 \ln x \, e^{2\ln x} \Rightarrow y_h = C_1 x^2 + C_2 x^2 \ln x$

$y_1 = x^2 \quad , \quad y_2 = x^2 \ln x$

$$W(y_1, y_2)(t) = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix} = x^3 \neq 0$$

* rearrange the equation to general form: $y'' - \frac{3}{x}y' + \frac{4}{x}y = \ln x$

$$V_1 = -\int \frac{y_2(t)\,g(t)}{W(y_1, y_2)} \Rightarrow V_1 = -\int \frac{x^2 \ln^2 x}{x^3} = -\int \frac{\ln^2 x}{x}$$

$$V_1 = -\int u^2\,du = \frac{-u^3}{3} \Rightarrow \boxed{V_1 = \frac{-\ln^3 x}{3}}$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$V_2 = \int \frac{y_1(t)\,g(t)}{W(y_1, y_2)} \Rightarrow V_2 = \int \frac{x^2 \ln x}{x^3} = \int \frac{\ln x}{x}$$

$$V_2 = \int u\,du = \frac{u^2}{2} \Rightarrow \boxed{V_2 = \frac{\ln^2 x}{2}}$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$y_p = V_1 y_1 + V_2 y_2 \Rightarrow y_p = \frac{x^2(\ln x)^3}{6}$$

$$y = y_p + y_h \Rightarrow \boxed{y = \frac{x^2(\ln x)^3}{6} + C_1 x^2 + C_2 x^2 \ln x}$$

$\boxed{5.2}$ Solving D.E Using Series

When to use it ?

- when the coefficients are Polynomial functions
- and also it should be homog.

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

if $P(x) \neq 0 \Rightarrow x$ is ordinary Point

if $P(x) = 0 \Rightarrow x$ is Singular Point

What can he ask me ?

① it may ask to find the ordinary and singular Points

$$(x^2+4)y'' + xy = 0$$

$P(x) = x^2+4 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4} \Rightarrow x = \pm 2i$

Singular Points : $x = 2i$

ordinary Points : all other Points

② to find the solution using Series (100% في سؤال)

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$ القانون العام للحل

ملاحظات :-

① لما أشتق بنقص 1 من الأندكس

② لازم أحولهن كلهن لـ x^n (اذا زدت فوق بنقص تحت والعكس)

③ بحولهن كلهن لأكبر اندكس

Ex. consider the d.e

$$(x^2+1)y'' - 4x\,y' + 6y = 0 \quad , \quad -\infty < x < \infty .$$

Find two series solutions y_1 and y_2 near an ordinary point $\underline{\underline{x_0 = 0}}$. Show that y_1 and y_2 form a fundamental set of solutions.

① let $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \Rightarrow y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ لما أشتق $\sum_{n=}^{\infty}$ بنقص 1 من الأندكس

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

② substitute y, y', y' on the equation

$$(x^2+1) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n$$

③ we enter the X's and number inside

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n$$

④ make X^n all alike in all equations

$x^{n-2} \Rightarrow x^n$, $n-2 \longrightarrow n$, add 2 above , sub 2 in the index

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n$$

5) Make all the indices alike

Since the biggest index is 2 ⟹ send them all to 2

$$2a_2 + 6a_3 X + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} X^n + \sum_{n=2}^{\infty} n(n-1) a_n X^n -$$

$$4Xa_1 + \sum_{n=2}^{\infty} 4n\, a_n X^n + 6(a_0 + a_1 X) + \sum_{n=2} 6a_n X^n$$

6) make them together (smth X^0 + smth X^1 ---- .)

$$(2a_2 + 6a_0) X^0 + (6a_3 + 2a_1) X +$$

$$\sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n^2 - 5n + 6) a_n \right] X^n$$

7) Make everything equal to zero

$2a_2 + 6a_0 = 0 \implies 3a_0 + a_2 = 0 \implies a_2 = -3a_0$

$6a_3 + 2a_1 = 0 \implies a_1 + 3a_3 = 0 \implies a_3 = -\frac{1}{3} a_1$

$$a_{n+2} = \frac{-(n^2 - 5n + 6) a_n}{(n+2)(n+1)} \qquad n = 2, 3, 4 \ldots$$

$n=2 \quad a_4 = -\dfrac{((n-3)*0*a_2)}{\text{-----}} \implies a_4 = 0$

$n=3 \quad a_5 = -\dfrac{(0*(n-2)*a_3)}{\text{-----}} \implies a_5 = 0$

$n=4 \quad a_6 = -\dfrac{((n-3)(n-2)*0)}{\text{-----}} \implies a_6 = 0$

$a_n = 0 \quad \forall n = 4, 5, \ldots$

$$y = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \ldots$$

$$y = a_0 + a_1 X - 3a_0 X^2 - \frac{1}{3} a_1 X^3 \ldots$$

$$y = a_0 \underbrace{\left[1 - 3X^2 \ldots\right]}_{y_1} + a_1 \underbrace{\left[X - \frac{X^3}{3} \ldots\right]}_{y_2}$$

⑧ Check the wronskian to see if its L.I

$$W(y_1, y_2) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$$

it's linearly independent #

$\boxed{5.3}$ Another way to solve D.E, radios of Conv.

When to use it ?

- When the coefficients are Not Polynomial functions
- When he asks me to find the first non-zero terms

The Rule : $a_m = \dfrac{y^{(m)}(x_0)}{m!}$

Note : $a_0 = y(x_0)$, $a_1 = y'(x_0)$, given in the Qs

Exp Given the IVP :

$$(x+1)y'' - \ln(e+x^2)y' - 2y = 0 , \quad y(0)=1 , y'(0)=1$$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series solution of this IVP, find the first four terms.

$a_0 = y(0) = 1$, $a_1 = y'(0) = 1$

$a_2 = \dfrac{y''(0)}{2}$, Find $y''(0)$: $y''(0) - \ln e \, y'(0) - 2y(0) = 0$

$y''(0) - 1 - 2 = 0 \Rightarrow y''(0) = 3$

$a_2 = \dfrac{3}{2}$

$a_3 = \dfrac{y'''(0)}{6}$, diff the eq. : $\left((x+1)y''' + y''\right) - \left(\ln(e+x^2)y'' + \dfrac{2xy'}{e+x^2}\right) - 2y' = 0$

$a_3 = \dfrac{1}{3}$ Find $y'''(0)$: $y'''(0) + 3 - 3 - 0 - 2 = 0$

$y'''(0) = 2$

the first four terms : $y = 1 + x + \frac{3}{2}x^2 + \frac{1}{3}x^3$

he also may ask me to find the radios of convergence

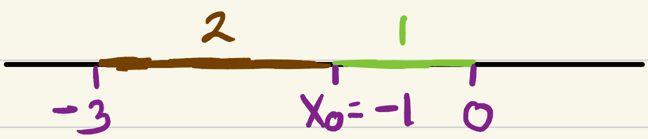
$\rho = \min\{\rho_1, \rho_2\}$, ρ_1 : radios of $p(x)$
ρ_2 : radios of $q(x)$

Example- $(x^2+3x)y'' + y' + y = 0$ about $x_0 = -1$

$P(x) = x^2+3x$
$Q(x) = 1$
$R(x) = 1$

$\Longrightarrow$ $p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{1}{x(x+3)}$ | $q(x) = \dfrac{R(x)}{P(x)} = \dfrac{1}{x(x+3)}$

Find the singular points : $x(x+3)=0 \Rightarrow x=0$ or $x=-3$
Since $p(x)$ equals $q(x)$ we can draw once



we take the smallest : $\rho_1 = \rho_2 = 1$

$\boxed{\rho = \min\{1, 1\} \Rightarrow \rho = 1}$

Example- $(1+x^2)y'' + (1+x^2)y' + y = 0$ about $x_0 = 0$

$P(x) = 1+x^2$
$Q(x) = 1+x^2$
$R(x) = 1$

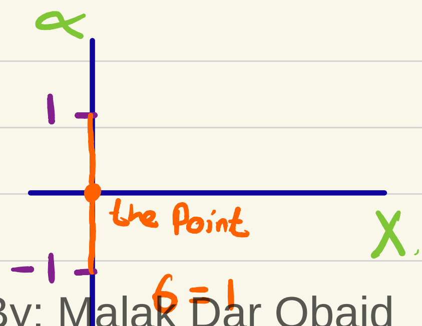
, $p(x) = \dfrac{Q(x)}{P(x)} = 1$, $q(x) = \dfrac{R(x)}{P(x)} = \dfrac{1}{x^2+1}$

$\rho_1 = \infty$

$p(x)$: $\rho = \infty$ because there is no singular points

$q(x)$: $x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x = \pm i$
Comparing to $z = x \mp \alpha i$

من السؤال $\boxed{x=0 \quad , \quad \alpha = \pm 1}$ البعد عن محور X و α



$\boxed{\rho = \min\{1, 1\} \Rightarrow \rho = 1}$

Example - $(1+x^2)y'' + 2xy' + 4x^2 y = 0$ about $x_0 = 0$, $x_0 = \frac{1}{2}$

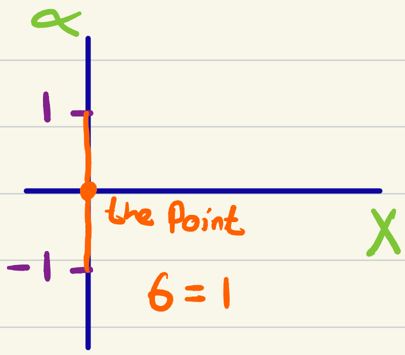
$P(x) = (1+x^2)$, $p(x) = \frac{Q(x)}{P(x)} = \frac{2x}{1+x^2}$, $q(x) = \frac{R(x)}{P(x)} = \frac{4x^2}{1+x^2}$

$Q(x) = 2x$

$R(x) = 4x^2$

for p and q : $x = \pm i$

$\boxed{x = 0}$

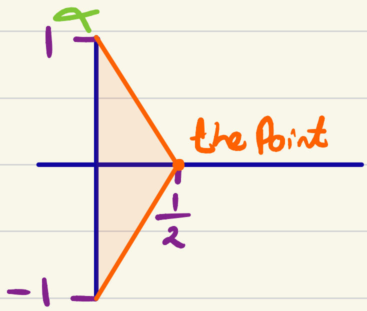


they have the same plot

so, when $x = 0$, $6 = \min\{1, 1\}$

when $x = 0$, $6 = 1$

$\boxed{x = \frac{1}{2}}$



$6_1 = 6_2 = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$

hence, when $x = \frac{1}{2}$, $6 = \min\{6_1, 6_2\}$

when $x = \frac{1}{2}$, $6 = \frac{\sqrt{5}}{2}$

Note :

when $x = 0$: فيعوض X ، بيصير بمقدار 0 ، ثم بوجد المسافة بين Xo

when $x = 1$: فيعوض X ، بيصير بمقدار 1 ، ثم بوجد المسافة بين Xo

ملاحظة في السؤال السابق

X تساوي صفر من تحليل المعادلة ، وانا بدي النقطة $\frac{1}{2}$ ، بجيب المسافة بينهم تساوي $\frac{1}{2}$

5.4 Euler and Regular Singular Points (RSP)

Euler formula : $ax^2y'' + bxy' + cy = 0$

The Solution: $am^2 + (b-a)m + c = 0$

Cases :-

① if $m_1 \neq m_2$

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

② if $m_1 = m_2$

$$y = C_1 x^{m} + C_2 x^{m} \ln x$$

③ if m is Complex

$$y = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$$

Example- $4x^2y'' + 17y = 0, \quad x > 0.$

$$4m^2 + (0-4)m + 17 = 0 \implies 4m^2 - 4m + 17 = 0$$

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{4 \pm \sqrt{16 - 16*17}}{8} = \frac{1}{2} \pm \frac{\sqrt{16(-16)}}{8}$$

$$= \frac{1}{2} \pm \frac{16\sqrt{-1}}{8} = \frac{1}{2} \pm 2i$$

$$y = C_1 x^{\frac{1}{2}} \cos(2\ln x) + C_2 x^{\frac{1}{2}} \sin(2\ln x)$$

Note : when it's not homog. $\Rightarrow$ $y = y_h + y_p$

y_h : from normal Euler rules. y_p : from variation of parameters

RSP and IRSP

if P, Q, R are Polynomial:

if $\lim_{x \to x_0} (x - x_0) p(x) < \infty$ and $\lim_{x \to x_0} (x - x_0)^2 q(x) < \infty$

then $x = x_0$ is Regular Singular Point (RSP)

else if P, Q, R are not Polynomials:

if $(x - x_0) p(x) < \infty$ and $(x - x_0)^2 q(x) < \infty$

then $x = x_0$ is Regular Singular Point (RSP)

if not, i have to find the taylor series to check

Example- $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$.

$P(x) = 2x(x-2)^2$ $\quad p(x) = \frac{Q}{P} = \frac{3}{2(x-2)^2}$ $\quad q(x) = \frac{R}{P} = \frac{1}{2x(x-2)}$

$Q(x) = 3x$

$R(x) = x-2$

$$x = 0, 2$$

$\boxed{x=0}$

$$\lim (x-0) * \frac{3}{2(x-2)^2} = \frac{3x}{2(x-2)^2} < \infty$$

$$\lim (x-0)^2 * \frac{3}{2(x-2)^2} = \frac{3x^2}{2(x-2)^2} < \infty$$ hence $x=0$ is RSP

$\boxed{x=2}$

$$\lim_{x \to 2} \frac{3(x-2)}{2(x-2)^2} = \frac{3}{2} \lim_{x \to 2} \frac{1}{x-2} = \infty$$

hence, $x=2$ is IRSP

6.1 Laplace

general form : $L\{f(t)\} = \int_0^\infty f(t) \cdot e^{-st}\, dt$

① $L\{K\} = \frac{1}{s}$

② $L\{e^{\alpha t}\} = \frac{1}{s-\alpha}$

③ $L\{t^n\} = \frac{n!}{s^{n-1}}$

④ $L\{\sin kt\} = \frac{k}{s^2+k^2}$

⑤ $L\{\cos kt\} = \frac{s}{s^2+k^2}$

⑥ $L\{\sinh kt\} = \frac{k}{s^2-k^2}$

⑦ $L\{\cosh kt\} = \frac{s}{s^2-k^2}$

⑧ $L\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$

⑨ $L\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$

⑩ $L\{e^{at}\, t^n\} = \frac{n!}{(s-a)^{n+1}}$

⑪ $L\{t^n f(t)\} = (-1)^n \frac{d}{dt^n} L\{f(t)\}$

⑫ $L\{y^{(n)}(t)\} = s^n y - s^{n-1} y(0) - s^{n-2} y'(0) --- - s^0 y^{(n-1)}(0)$

Important Notes :

- $\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$

- $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$

- $\sin(nt+k) = \sin(nt)\cos(k) + \cos(nt)\sin(k)$

- $\cos(nt+k) = \cos(nt)\cos(k) - \sin(nt)\sin(k)$

- $L\{e^{nt} * \text{anything}\} = L\{\text{anything}\}$ with shifting s by n

ex. $L\{e^t t^2\} = \frac{2!}{(s-1)^3}$ ex. $L\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+4^2}$

$\boxed{6.2}$ Laplace inverse + Solving IVP

ex. $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+4)}\right\}$

$$= \frac{A}{s+2} + \frac{Bs+D}{s^2+4} \Longrightarrow A(s^2+4) + (Bs+D)(s+2) = s$$

$$As^2 + 4A + Bs^2 + 2Bs + Ds + 2D = s$$

when the Power is small i solve by Comparing the Coefficients of s^n, s^{n-1} - -

s^2 : $A + B = 0$ ---------- ①

s^1 : $2B + D = 1$ ---------- ②

s^0 : $2A + D = 0$ ---------- ③

② − ③ : $(2B+D) - (2A+D) \Longrightarrow B - A = \frac{1}{2}$ -------- ④

① + ④ : $2B = \frac{1}{2} \Longrightarrow \boxed{B = \frac{1}{4}}$

Substitute in ① : $A + \frac{1}{4} = 0 \Longrightarrow \boxed{A = -\frac{1}{4}}$

〃 〃 ③ : $-\frac{1}{2} + D = 0 \Longrightarrow \boxed{D = \frac{1}{2}}$

$$L\left\{\frac{s}{(s+1)(s^2+4)}\right\} = -\frac{1}{4}L\left\{\frac{1}{s+2}\right\} + L\left\{\frac{\frac{1}{4}s + \frac{1}{2}}{s^2+2^2}\right\}$$

$$= \frac{-e^{-2t}}{4} + \frac{1}{4}L\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{2}L\left\{\frac{1}{s^2+2^2}\right\}$$

$$\frac{-e^{-2t}}{4} + \frac{\cos 2t}{4} + \frac{\sin 2t}{4}$$

$$= \frac{\cos 2t + \sin 2t - e^{-2t}}{4}$$

ex. $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = -1$

① Take Laplace for each one of them

$$L\{y''\} + 2L\{y'\} + L\{y\} = \underline{4L\{e^{-t}\}}$$

$$[s^2Y - sy(0) - y'(0)] + 2[sY - y(0)] + Y = \frac{4}{s+1}$$

$$= s^2Y - 2s + 1 + 2sY - 4 + Y = \frac{4}{s+1}$$

② Put Y coeffesiants together , then make Y alone

$$(s^2 + 2s + 1)Y - 2s - 3 = \frac{4}{s+1}$$

$$(s+1)^2 Y = \frac{(2s+3)(s+1) + 4}{(s+1)} \Rightarrow Y = \frac{2s^2 + 5s + 7}{(s+1)^3}$$

③ find $L^{-1}\{Y\}$ using anyway

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$A(s+1)^2 + B(s+1) + C = 2s^2 + 5s + 7$$

when the Power is high we diff. to find A, B, C

نشتق لأنه اسهل من مقارنة المعاملات

$s = -1$: $\boxed{C = 4}$

1'st diff : $2A(s+1) + B = 4s + 5$

$s = -1$: $\boxed{B = 1}$

2'nd diff : $2A = 4 \Rightarrow \boxed{A = 2}$

$$L^{-1}\{Y\} = 2L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= 2e^{-t} + e^{-t}t + 2e^{-t}t^2 = \boxed{e^{-t}[2t^2 + t + 2]}$$