

14.4 Chain Rule.

Recall : (Cal. 4)

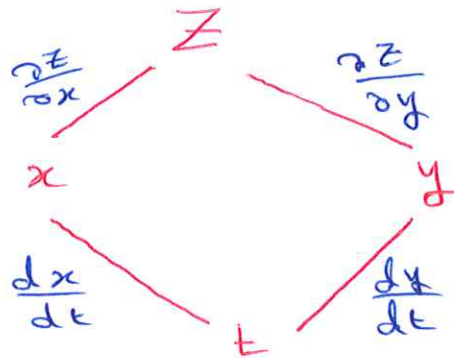
If $y = f(t)$ and $t = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Now : (Cal. 3):

• If $z = f(x, y)$ and $x = x(t)$ & $y = y(t)$

then
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

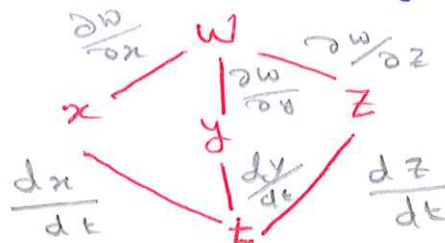


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• If $w = f(x, y, z)$ with $x = x(t)$, $y = y(t)$ & $z = z(t)$

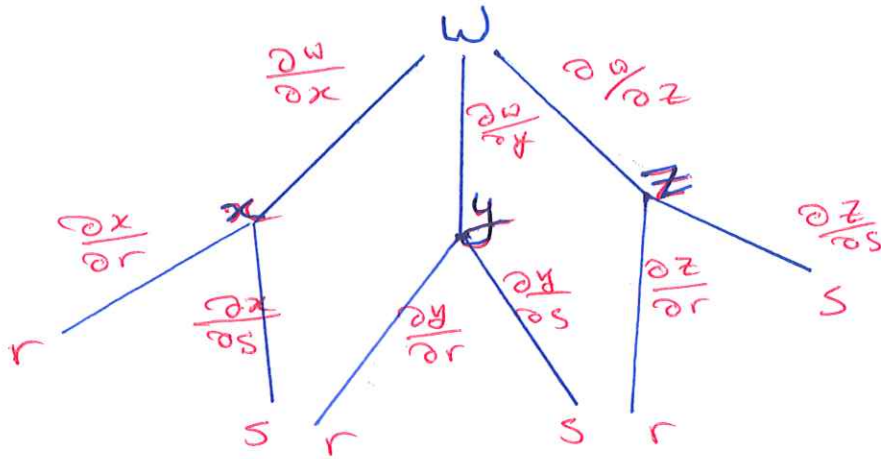
then
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



• If $w = f(x, y, z)$ and

$x = x(r, s)$ and $y = y(r, s)$ & $z = z(r, s)$

then.



$$\Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\text{and } \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Example: Let $w = xy$, $x = \cos t$, $y = \sin t$

Find $\left. \frac{dw}{dt} \right|_{t = \frac{\pi}{2}}$

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$$\begin{aligned} \text{sol: } \frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\ &= -y \sin t + x \cos t \end{aligned}$$

$$\Rightarrow \left. \frac{dw}{dt} \right|_{t=0, x=0, y=1} = -1 \sin \frac{\pi}{2} + 0 \cos \frac{\pi}{2} = \boxed{-1}$$

Example: $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

Find $\frac{dw}{dt}$ at $t = 0$

So! At $t = 0 \Rightarrow x = 1$, $y = 0$, $z = 0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= -y \sin t + x \cos t + 1 \cdot 1$$

$$\Rightarrow \left. \frac{dw}{dt} \right|_{t=0, x=1, y=0, z=0} = 0 + 1 \cos 0 + 1 = \boxed{2}$$

$$t=0, x=1, y=0, z=0$$

Example: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r

and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$

$$z = 2r$$

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$$\text{So! } \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 \cdot \left(\frac{1}{s}\right) + 2 \cdot (2r) + 2\textcircled{z} \cdot (2)$$

$$= \frac{1}{s} + 4r + 4(2r)$$

$$= \frac{1}{s} + 12r$$

(3)

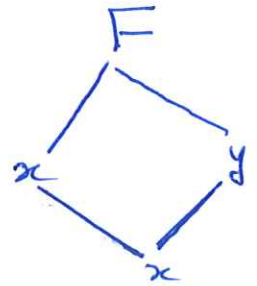
$$\begin{aligned}
 \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} \\
 &= 1 \cdot \left(-\frac{r}{s^2}\right) + 2 \left(\frac{1}{s}\right) + 2z \cdot (0) \\
 &= \frac{-r}{s^2} + \frac{2}{s} = \frac{2s - r}{s^2}
 \end{aligned}$$

Implicit Differentiation Revisited.

If $F(x, y) = 0$. Find $\frac{dy}{dx}$.

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}, \quad F_y \neq 0$$



Theorem 8: A Formula for Implicit Differentiation.

Suppose that $F(x, y)$ is differentiable and that

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the equation $F(x, y) = 0$ defines y as a

differentiable function of x . Then at any point

where $F_y \neq 0$, $\frac{dy}{dx} = \frac{-F_x}{F_y}$.

Example: Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin(xy) = 0$.

Sol: (Cal. 1): $2y \frac{dy}{dx} - 2x - \cos(xy) \left[x \frac{dy}{dx} + y \right] = 0$

$$\Rightarrow \underbrace{2y \frac{dy}{dx}} - 2x - \underbrace{x \cos(xy) \frac{dy}{dx}} - y \cos(xy) = 0$$

$$\Rightarrow [2y - x \cos(xy)] \frac{dy}{dx} = y \cos(xy) + 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y \cos(xy)}{2y - x \cos(xy)}$$

Calc. 3: Using Thm (8):

$$y^2 - x^2 - \sin(xy) = 0$$

$$F(x, y) = y^2 - x^2 - \sin(xy) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$= \frac{2x + y \cos(xy)}{2y - x \cos(xy)}$$

• If $F(x, y, z) = 0$, $z = z(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$ if

$$\underbrace{x^3 + z^2 + y e^{xz} + z \cos y}_{F(x, y, z)} = 0$$

$$\begin{aligned} \left. \frac{\partial z}{\partial x} \right|_{(0,0,0)} &= \left. \frac{-F_x}{F_z} \right|_{(0,0,0)} = \left. \frac{-(3x^2 + yze^{xz})}{2z + xy e^{xz} + \cos y} \right|_{(0,0,0)} \\ &= \frac{-(0+0)}{0+0+1} = \boxed{0} \end{aligned}$$

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$$\begin{aligned} \left. \frac{\partial z}{\partial y} \right|_{(0,0,0)} &= \left. \frac{-F_y}{F_z} \right|_{(0,0,0)} = \left. \frac{-(e^{xz} - z \sin y)}{(2z + xy e^{xz} + \cos y)} \right|_{(0,0,0)} \\ &= \frac{-1+0}{0+0+1} = \boxed{-1} \end{aligned}$$

Example: If $z^3 - xz - y = 0$, where z defines a function of x and y .

Find $\frac{\partial^2 z}{\partial x \partial y}$

Sol: $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(-1)}{(3z^2 - x)} = \frac{1}{(3z^2 - x)}$

Now, $\frac{\partial^2 z}{\partial x \partial y} = \frac{- (1) (6z \frac{\partial z}{\partial x} - 1)}{(3z^2 - x)}$

Q37). If $z = 5 \tan^{-1} x$, $x = e^u + \ln v$

Find $\frac{\partial z}{\partial u}$ at $(u, v) = (\ln 2, 1)$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial u} = \frac{1(5)}{1+x^2} \cdot (e^u)$$

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$$= \frac{5e^u}{1+(e^u + \ln v)^2}$$

$$\Rightarrow \left. \frac{\partial z}{\partial u} \right|_{(\ln 2, 1)} = \frac{5e^{\ln 2}}{1+(e^{\ln 2} + \ln 1)^2} = \frac{5(2)}{1+(2)^2} = \frac{10}{5} = \boxed{2}$$